"Topological Strings and Nekrasov's formulas"

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§ Nekrasov's formula and TST

SU(2) gauge theory

$$Z_{inst} = \sum_{k \geq 1} Z_k \Lambda^{4k}$$

$$Z_k = \sum_{\substack{\ell_{R_1} + \ell_{R_2} \\ = k}} \prod_{l,m=1,2} \prod_{i,j=1}^{\infty} \frac{\sinh\beta(a_{lm} + \hbar(\mu_{l,i} - \mu_{m,j} + j - i))}{\sinh\beta(a_{lm} + \hbar(j-i))}$$

sum over Young tables : random partitions

$\beta, \hbar, \Lambda$

$\beta \to 0$

$\hbar \to 0$ $\quad Z = e^{\frac{1}{\hbar^2}F_0 + \cdots}$

$F_0 = F_{SW}$

agrees with Seiberg-Witten prepotential

We consider Nekrasov's formula with

$\beta \neq 0, \quad \hbar \neq 0$

Physical identification

$\beta = R$ radius of S^1 of 5-th dimension
5 dimensional gauge theory on $\mathbb{R}^4 \times S^1$

$q = e^{-2\beta\hbar}$

$= e^{i g_s}$ g_s: string coupling constant

$\hbar \neq 0$ coupling to gravity

① $\beta = 0, \ \hbar = 0$

4 dim. gauge theory IIA/CY_3

② $\beta \neq 0, \ \hbar = 0$

5 dim. gauge theory M/CY_3

counts genus $= 0$ curves in CY_3

③ $\beta \neq 0,\ \hbar \neq 0$

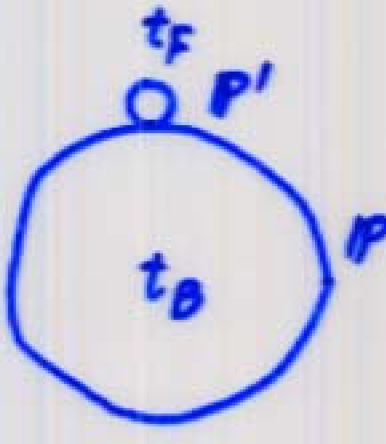
topological string on $K_{\mathbb{F}_0}$

$\mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$ $\quad K_{\mathbb{F}_0}$: canonical bundle on $\mathbb{F}_0$

$t_B,\ t_F$

$t_F = 4\beta a \to 0$

$Q_B = e^{-t_B} = (\beta\Lambda)^4, \quad t_B = \frac{1}{4}\log\frac{1}{\beta\Lambda} \to \infty$



A_1 singularity fibered over $\mathbb{P}^1$

We can compute $Z^{K_{\mathbb{F}_0}}_{str}$ using Chern-Simons theory.

We claim

$$Z^{K_{\mathbb{F}_0}}_{str} = Z^{SU(2)}_{Nek}$$

$$Z^{K_{F_0}}_{str} = \exp\Big[\sum_{n,m} \sum_g \sum_k \frac{1}{k} \frac{N^g_{m,n}}{\left(\sin\frac{k g_s}{2}\right)^{2-2g}} e^{-k(m t_B + n t_F)} \Big]$$

$N^g_{m,n}$: Gopakumar-Vafa invariants
counts genus g curves
winding m times base $\mathbb{P}^1$
n times fiber $\mathbb{P}^1$

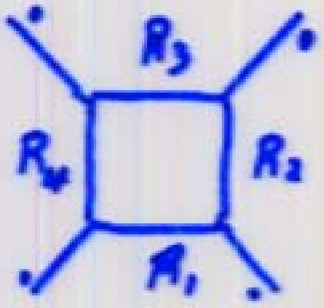
Nekrasov's formula encodes complete information on # of curves of all genus and all degrees.

Similar formulas may be obtained also for local del Pezzo surfaces etc.

§ Chern-Simons calculation

local $\mathbb{F}_0$

toric diagram



$= W_{R_1 R_2}(q)$

$$Z_{str}^{K_{\mathbb{F}_0}} = \sum_{R_1,\cdots,R_4} W_{R_4 R_1} W_{R_1 R_2} W_{R_2 R_3} W_{R_3 R_4} \times e^{-t_F(l_{R_1}+l_{R_3})} e^{-t_B(l_{R_2}+l_{R_4})}$$

$W_{R_1 R_2}$: Hopf link inv. of CS theory

$\Big\langle$  $\Big\rangle_{CS}$

Wilson lines of reps. R_1 and R_2

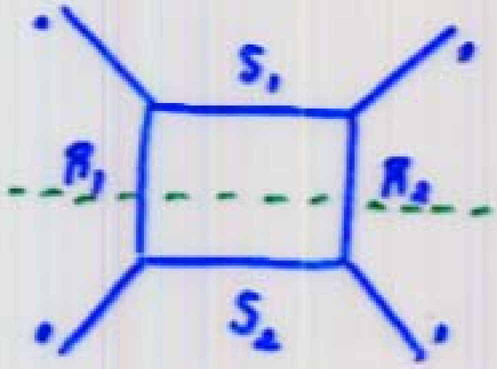
$q = e^{\frac{2\pi i}{N+k}}$ $\qquad N \to \infty$, q : fixed

$$W_{R_1 R_2}(q) = W_{R_1}(q)\, S_{R_2}\big(x_i = q^{\mu_i^{R_1} - i + \frac{1}{2}}\big)$$

Schur polynomial $\qquad q^{\mu^{R_1}+\rho}$

$$W_R(q) = \dim_q R = S_R(q^\rho)$$

Method of computation



$$Q = e^{-t}$$

Define $$K_{R_1 R_2}(Q) = \sum_S Q^{l_S} W_{R_1 S}(q) W_{S R_2}(q)$$

$$Z_{str}^{K_{F_0}} = \cdots \sum_{R_1, R_2} \left(K_{R_1 R_2}(Q_F)\right)^2 e^{-t_B(l_{R_1} + l_{R_2})}$$

proposition 1

$$K_{R_1 R_2}(Q) = W_{R_1}(q) W_{R_2}(q) \exp\left[\sum_{n=1} \frac{\tilde{f}_{R_1 R_2}(q^n)}{n} Q^n\right]$$

$$\tilde{f}_{R_1 R_2} = \frac{(q-1)^2}{q} \bar{f}_{R_1}(q) \bar{f}_{R_2}(q)$$

$$\bar{f}_R(q) = f_R(q) + \frac{q}{(q-1)^2}$$

$$f_R(q) = \frac{q}{q-1} \sum_{i=0} (q^{\mu_i - i} - q^{-i})$$

$$\tilde{f}_{R_1R_2} = f_{R_1R_2} + \frac{q}{(q-1)^2}$$

$$f_{R_1R_2} = \frac{(q-1)^2}{q} f_{R_1} f_{R_2} + f_{R_1} + f_{R_2}$$

$$= \sum_k C_k(R_1, R_2) q^k$$

$$K_{R_1R_2}(Q_F) = W_{R_1}(q) W_{R_2}(q) \exp\left[\sum_{n=1} \frac{q^n}{(q^n-1)^2} \frac{Q_F^n}{n}\right]$$

$$\times \prod_k (1 - q^k Q_F)^{-C_k(R_1, R_2)}$$

$$\prod_k (1 - q^k Q_F)^{-C_k(R_1, R_2^t)}$$

$$= (4Q_F)^{-\frac{1}{2}(\ell_{R_1} + \ell_{R_2})} q^{-\frac{1}{4}(\kappa_{R_1} + \kappa_{R_2})} \prod_k \frac{1}{\sinh R(2a + \hbar k)^{C_k(R_1, R_2^t)}}$$

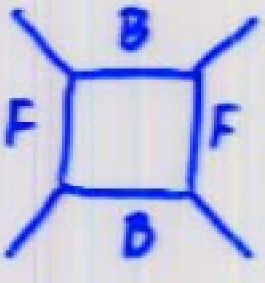
proposition 2

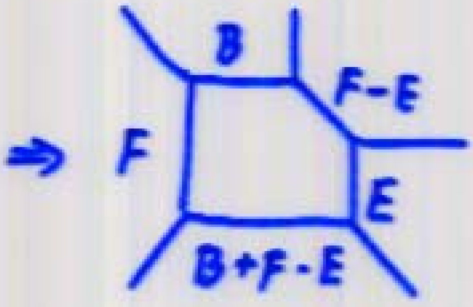
$$\prod_k \frac{1}{\sinh R(2a + \hbar k)^{C_k(R_1, R_2^t)}}$$

$$= \prod_{i,j=1}^{\infty} \frac{\sinh R(2a + \hbar(\mu_{1,i} - \mu_{2,j} + j - i))}{\sinh R(2a + \hbar(j-i))}$$

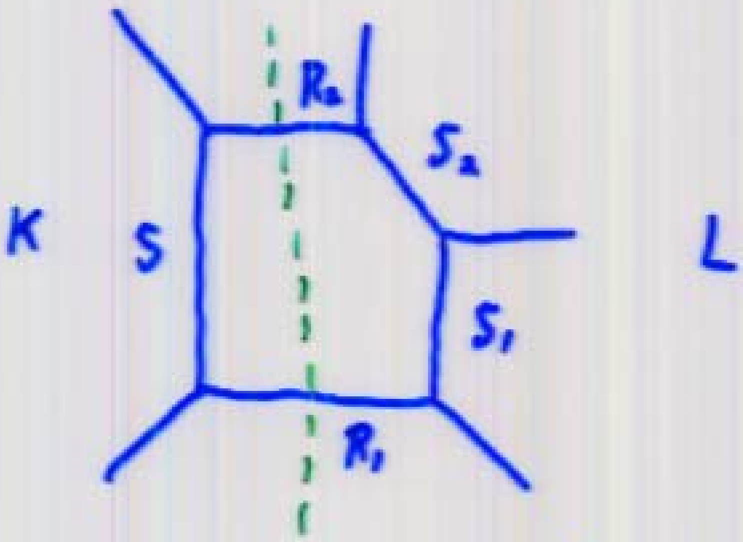
$$\therefore \quad Z_{str}^{K_{\mathbb{F}_0}} = Z_{Nek}^{SU(2)}$$

§ Adding matter





$\mathbb{F}_0$ $\qquad\qquad$ dP_2



$$K_{R_1R_2}(Q) = \sum_S Q^{\ell_S} W_{R_1S} W_{SR_2}$$

$$L_{R_1R_2}(Q_1,Q_2) = \sum_{S_1,S_2} Q_1^{\ell_{S_1}} Q_2^{\ell_{S_2}} W_{R_1S_1} W_{S_1S_2} W_{S_2R_2}$$

$$Z_{str}^{dP_2} = \sum_{R_1,R_2} Q_B^{\ell_{R_1}+\ell_{R_2}} K_{R_1R_2}(Q_F)\, L_{R_1R_2}(Q_F Q_E^{-1}, Q_E)$$

proporition 3

$$L_{R_1,R_2}(Q_1,Q_2)$$

$$= W_{R_1}(q)W_{R_2}(q)\exp\Big[-\sum_{n=1}\tilde{f}_{R_1}\frac{(q^n)}{n}Q_1^n-\sum_{n=1}\tilde{f}_{R_2}\frac{(q^n)}{n}Q_2^n + \sum_{n=1}\hat{f}_{R_1R_2}\frac{(q^n)}{n}(Q_1Q_2)^n\Big]$$

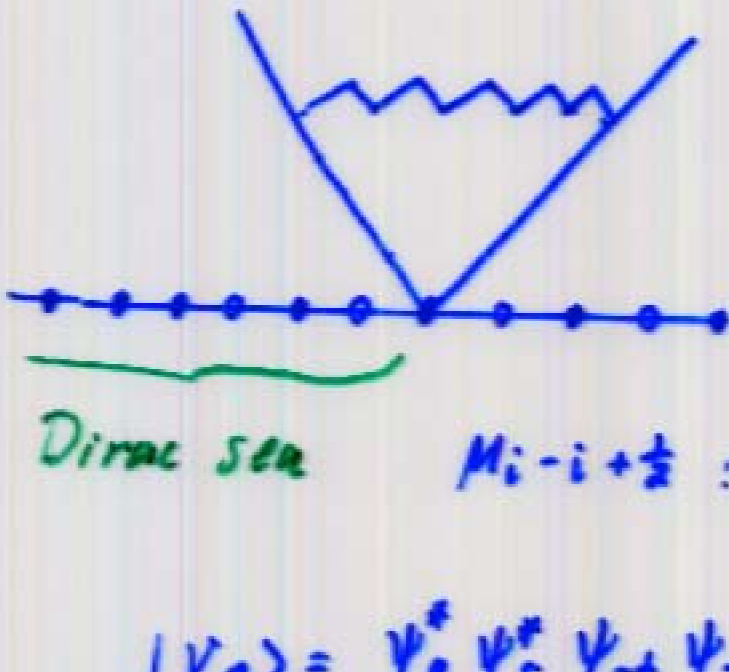
§ free fermions, vertex operators

proof of proposition 1, 3 :

Schur function calculus

We recall

Young diagrams $\Leftrightarrow$ free fermions



$\mu_1 = 5,\ \mu_2 = 4$

$\mu_3 = 3,\ \mu_4 = 2$

$\mu_5 = 1$

Dirac sea

$\mu_i - i + \frac{1}{2}$: $\frac{9}{2}, \frac{5}{2}, \frac{1}{2}, -\frac{3}{2}, -\frac{7}{2}, -\frac{9}{2}, -\frac{11}{2}, \cdots$

$$|V_R\rangle = \psi^*_{\frac{9}{2}}\psi^*_{\frac{5}{2}}\psi_{-\frac{1}{2}}\psi_{-\frac{5}{2}}|\mathrm{sea}\rangle$$

$$|\mathrm{sea}\rangle \equiv |0\rangle$$

$$S_R(x_i) = \langle V_R|\exp\sum_{n=1} \frac{P_n\alpha_{-n}}{n}|0\rangle$$

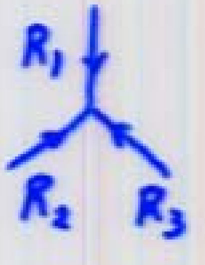
$$P_n \equiv \sum_{i=1} x_i^n$$

Skew Schur functions

$$S_{R/Q}(x_i) = \langle v_R | e^{\sum_{n=1} \frac{p_n a_{-n}}{n}} | v_Q \rangle$$

$$R \supset Q$$

topological vertex



$$C_{R_1 R_2 R_3} = q^{(\kappa_{R_2} + \kappa_{R_3})/2} S_{R_2^t}(q^\rho) \times \sum_{Q_3} S_{R_1/Q_3}(q^{\mu^{R_2^t}+\rho}) S_{R_3^t/Q_3}(q^{\mu^{R_2}+\rho})$$

$$\kappa_R = l_R + \sum_{j=1}^{d_R} \mu_j(\mu_j - 2j)$$

sum over partitions

= sum over fermion Fock space

⇒ vertex operator calculus

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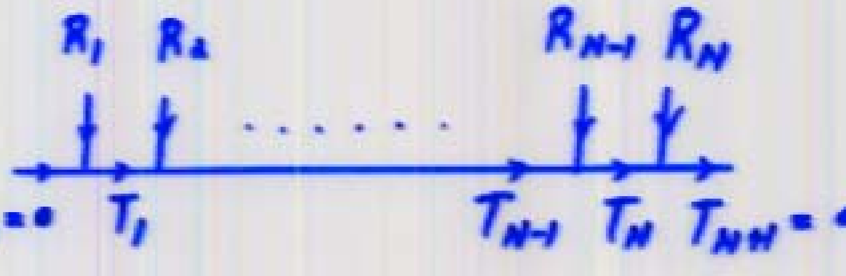
SU(N) theory



$$C_{T_{k-1} R_k T_k^t} (-1)^{l_{T_k}} = (-1)^{l_{T_k}} q^{-\kappa_{T_k}/2} S_{R_k}(q^{\rho}) \cdot \sum_{U_k} S_{T_{k-1}/U_k}(q^{\mu_{R_k^t}+\rho}) \cdot S_{T_k/U_k}(q^{\mu_{R_k}+\rho})$$

$$Q_k = e^{-2R(a_k - a_{k-1})}$$

$$K^{SU(N)}_{\{R_i\}} = \prod_{k=1}^{N} S_{R_k}(q^{\rho}) \sum_{T_1 \cdots T_{N-1}} \prod_{k=1}^{N} Q_k^{l_{T_k}} \sum_{U_k} S_{T_{k-1}/U_k}(q^{\mu_{R_k^t}+\rho}) \cdot S_{T_k/U_k}(q^{\mu_{R_k}+\rho})$$

vertex operator

$$V_{\pm}^{[R]}(q) \equiv V_{\pm}(x_i = q^{\mu_{R_i} - i + \frac{1}{2}})$$

$$Q_k^{l_{T_k}} \sum_{U_k} S_{T_{k-1}/U_k}(q^{\mu_{R_k^t}+\rho}) S_{T_k/U_k}(q^{\mu_{R_k}+\rho})$$

$$= Q_k^{l_{T_k}} \sum_{U_k} \langle T_{k-1} | V_{-}^{[R_k^t]}(q) | U_k \rangle \langle U_k | V_{+}^{[R_k]}(q) | T_k \rangle$$

$$= \langle T_{k-1} | V_{-}^{[R_k^t]}(q) V_{+}^{[R_k]}(q) Q^{L_0} | T_k \rangle$$

$$K^{SU(N)}_{\{R_i\}} = \prod_{k=1}^{N} S_{R_k}(q^{\rho}) \sum_{T_1 \cdots T_{N-1}} \prod_{k=1}^{N-1} \langle T_{k-1} | V_-^{[R_k^t]}(q) V_+^{[R_k]}(q) Q^{L_0} | T_k \rangle$$

$$= \prod_{k=1}^{N} S_{R_k}(q^{\rho}) \langle 0 | \prod_{k=1}^{N} V_-^{[R_k^t]}(q) V_+^{[R_k]}(q) Q^{L_0} | 0 \rangle$$

Veneziano-type amplitude

$$Q^{L_0} V_\pm(x_i) = V_\pm(Q^{\mp} x_i) Q^{L_0}$$

$$\langle 0 | V_+(x_i) V_-(y_j) | 0 \rangle$$

$$= \prod_{i,j \geq 1} (1 - x_i y_j)^{-1}$$

$$\therefore \quad K^{SU(N)}_{\{R_i\}} = \prod_{k=1}^{N} \dim_q R_k$$

$$\times \prod_{1 \leq m < \ell \leq N} \prod_{k=1} \Big(1 - \prod_{n=m}^{\ell-1} Q_n q^k\Big)^{-k}$$

$$\times \prod_{k=1} \Big(1 - \prod_{n=m}^{\ell-1} Q_n q^k\Big)^{-C_k(R_m, R_\ell^t)}$$