tree-level Yang-Mills

Twistors and ~~Perturbative QCD~~

A new method of computing scattering amplitudes

String Theory and Quantum Field Theory
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Plan of this talk

1. Twistor space (1960's~70's)
2. Scattering amplitudes (1970's~80's)
3. Twistor amplitudes (2003~04)
4. MHV diagrams (2004~05)

1. Twistor space

A brief introduction to twistor theory

References:

R. Penrose
Twistor Algebra
J.Math.Phys. 8 (1967) 345

R. Penrose
Twistor theory: An approach to the quantization of fields and space-time
Phys.Rept. C6 (1972) 241

Textbooks:

R.S.Ward, R.O.Wells,Jr
Twistor Geometry and Field Theory
Cambridge University Press

S. A. Huggett, K. P. Tod
An Introduction to Twistor Theory
Cambridge University Press

1.1 Plane waves

One vector index = a pair of undotted and dotted spinor indices

$$p_\mu \to p_{a\dot{a}} \equiv p_\mu(\sigma^\mu)_{a\dot{a}}$$

Spin s massless fields have $2s$ symmetric spinor indices.

Irreducible decomposition

Weyl fermion $\psi_a, \quad \psi_{\dot{a}}$

Maxwel field strength $F_{\mu\nu} \to F_{a\dot{a}b\dot{b}} = F_{ab}\epsilon_{\dot{a}\dot{b}} + F_{\dot{a}\dot{b}}\epsilon_{ab}$

Gravitino field strength

$$\psi_{a\mu\nu} \to \psi_{ab\dot{b}c\dot{c}} = \psi_{abc}\epsilon_{\dot{b}\dot{c}}$$

$$\psi_{\dot{a}\mu\nu} \to \psi_{\dot{a}b\dot{b}c\dot{c}} = \psi_{\dot{a}\dot{b}\dot{c}}\epsilon_{bc}$$

Weyl tensor $W_{\mu\nu\rho\sigma} \to W_{a\dot{a}b\dot{b}c\dot{c}d\dot{d}} = W_{abcd}\epsilon_{\dot{a}\dot{b}}\epsilon_{\dot{c}\dot{d}} + W_{\dot{a}\dot{b}\dot{c}\dot{d}}\epsilon_{ab}\epsilon_{cd}$

positive helicity ($h = +s$)
$\to$ dotted indices

negative helicity $h = -s$
$\to$ undotted indices

Equation of motion

$$\partial^{a\dot{a}_1}\psi_{\dot{a}_1\dot{a}_2\cdots\dot{a}_{2s}}(x) = 0 \quad (h=+s) \qquad \partial^{\dot{a}a_1}\psi_{a_1a_2\cdots a_{2s}}(x) = 0 \quad (h=-s)$$

These can be solved as follows

(1) Decompose the wave function into a spin part (polarization) and orbital part like $\psi_{ab\cdots c}(x) = \zeta_{ab\cdots c}f(x)$

(2) Take a plane wave $f(x) = e^{ipx}$ for the orbital part.

(3) Represent the null vector p as a product of two spinors.

$$p_{a\dot{a}} = \lambda_a\widetilde{\lambda}_{\dot{a}}$$

$\left(\begin{array}{l} \lambda \text{ and } \widetilde{\lambda} \text{ are bosonic spinors.} \\ \lambda_a\lambda^a \equiv \lambda^a\epsilon_{ab}\lambda^b = 0 \end{array}\right)$

(4) Solutions of the equations of motion are

$$\zeta_{a_1\cdots a_{2s}} = \lambda_{a_1}\lambda_{a_2}\cdots\lambda_{a_{2s}} \qquad \zeta_{\dot{a}_1\cdots\dot{a}_{2s}} = \widetilde{\lambda}_{\dot{a}_1}\widetilde{\lambda}_{\dot{a}_2}\cdots\widetilde{\lambda}_{\dot{a}_{2s}}$$

Another way to solve the equation

Let's change the order in solving equations of motion

(1) Decompose the wave function into a spin part (polarization) and orbital part like $\psi_{ab\cdots c}(x) = \zeta_{ab\cdots c} f(x)$

(2) Instead of taking plane wave for orbital part, we fix the spin part first as follows.

$$\psi_{a_1\cdots a_{2s+1}}(x) = \lambda_{a_1}\lambda_{a_2}\cdots\lambda_{a_{2s}} f(x)$$

(3) The equation of motion (transversality equation) gives

$$\lambda^a \partial_{a\dot{a}} f(x) = 0$$

(4) This can be solved by $f(x) = g(\lambda_a x^{a\dot{a}})$

$\longrightarrow$ expanded by functions $f_{(\lambda,\mu)}(x) \equiv \delta^2(\lambda_a x^{a\dot{a}} + \mu^{\dot{a}})$

1.2 Twistor space

Expansion with
$f_p(x) = e^{ipx}$

$\to$ momentum space $\{p_\mu\}$

Expansion with
$f_{(\lambda,\mu)}(x) = \delta^2(\lambda_a x^{a\dot{a}} + \mu^{\dot{a}})$

$\to$ Twistor space $\{(\lambda^a, \mu^{\dot{a}})\}$

The rescaling $(\lambda, \mu) \to (\alpha\lambda, \alpha\mu)$ does not give independent functions.

$\downarrow$

The twistor space is a projective space $\mathbf{CP}^3$.
λ^a and $\mu^{\dot{a}}$ are homogeneous coordinates.

The transformation between the coordinate space x^μ and the twistor space (λ, μ) is an integral transformation with the kernel $\delta^2(\lambda x + \mu)$.

$$\psi^{ab\cdots c}(x) = \int_{\mathbf{CP}^3} \Omega \delta^2(\lambda_a x^{a\dot{a}} + \mu^{\dot{a}}) \lambda^a \lambda^b \cdots \lambda^c \varphi(\lambda, \mu)$$

Ω is the invariant measure in the projective space $\mathbf{CP}^3$

momentum p^μ is transformed as $\quad p_{a\dot{b}} = \lambda_a \tilde{\lambda}_{\dot{b}} = \dfrac{\partial}{\partial x^{a\dot{b}}} \quad \rightarrow \quad \lambda^a \dfrac{\partial}{\partial \mu^{\dot{b}}}$

$$\text{Twistor space} \xleftrightarrow[\text{Fourier tr.}]{\mu \leftrightarrow \tilde{\lambda}} \text{Momentum space}$$

Comments

- We treat momenta as complex variables.
  λ^a and $\widetilde{\lambda}^{\dot{a}}$ are independent complex variables.
- Twistors are powerful tool to construct multi instanton solutions.
  Actually, the method known as "ADHM construction" is a byproduct of twistors.
- The (complexified) conformal symmetry $SL(4, \mathbf{C})$ is realized as the isometry of the twistor space $\mathbf{CP}^3$.
- Super(conformal) symmetries can easily be incorporated with twistors. $\mathcal{N}$ fermionic coordinates are added to the twistor coordinates (λ, μ). The supertwistor space is a supermanifold $\mathbf{CP}^{3|\mathcal{N}}$.

2. Scattering amplitudes

A holomorphic structure in tree level gluon scattering amplitudes

References:

S. J. Parke, T. R. Taylor
Perturbative QCD utilizing extended supersymmetry
PLB157(1985)81

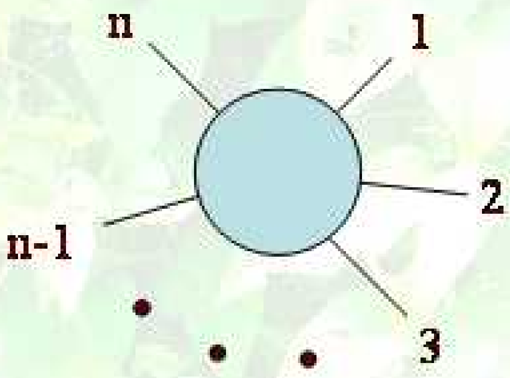
M. T. Grisaru, H. N. Pendelton
Some properties of scattering amplitudes in supersymmetric theories
NPB124(1977)81

S. J. Parke, T. R. Taylor
Amplitude for n-gluon scattering
PRL56(1986)2459

2.1 Color ordering

We consider tree level scattering amplitudes of U(N) adjoint particles.

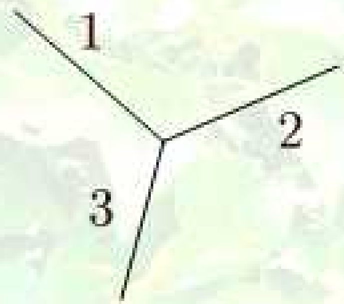
External lines are labeled in color-ordering



$$= \quad g_{\rm YM}^{n-2}\,\mathrm{tr}(T_{a_1}T_{a_2}\cdots T_{a_n})\,\delta^4\left(\textstyle\sum_{i=1}^{n} p_i\right)A(\lambda_i,\widetilde{\lambda}_i)$$

In what follows, we focus on only the $\delta^4(\sum p)A(\lambda_i,\widetilde{\lambda}_i)$ part.

2.2 3 and 4-particle amplitudes



Due to the momentum conservation,

$$\langle\lambda_1,\lambda_3\rangle[\widetilde{\lambda}_3,\widetilde{\lambda}_2] = \lambda_1^a\left(\sum_{i=1}^{3} p_{ia\dot{a}}\right)\widetilde{\lambda}_2^{\dot{a}} = 0$$

notations

$$\langle\lambda,\chi\rangle \equiv \lambda^a\chi_a$$

$$[\mu,\xi] \equiv \mu^{\dot{a}}\xi_{\dot{a}}$$

$\longrightarrow$ $\langle\lambda_1,\lambda_3\rangle = 0$ or $[\widetilde{\lambda}_3,\widetilde{\lambda}_2] = 0$

If $\langle\lambda_1,\lambda_3\rangle = 0$,

$$\langle\lambda_i,\lambda_j\rangle = 0$$

$A(\lambda,\widetilde{\lambda})$ cannot depend on λ_i

(``anti-holomorphic'')

Amp.$= \delta^4(\sum p_i)A(\widetilde{\lambda}_i)$

If $[\widetilde{\lambda}_3,\widetilde{\lambda}_2] = 0$,

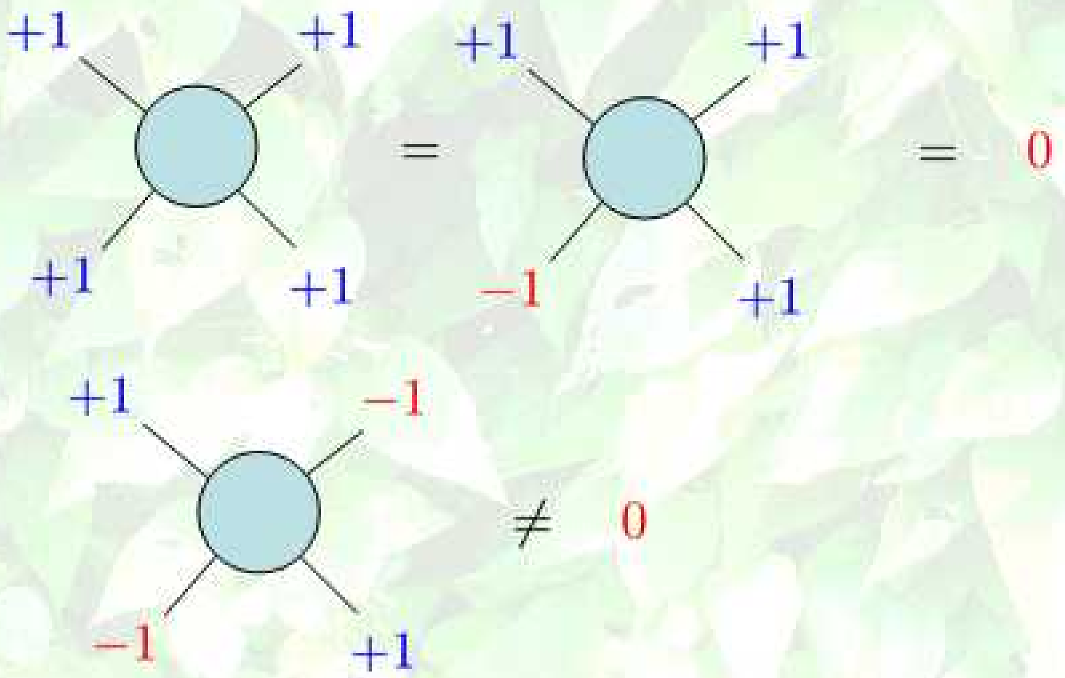
$$[\widetilde{\lambda}_i,\widetilde{\lambda}_j] = 0$$

$A(\lambda,\widetilde{\lambda})$ cannot depend on $\widetilde{\lambda}_i$

(``holomorphic'')

Amp.$= \delta^4(\sum p_i)A(\lambda_i)$

4-particle amplitudes



Amplitudes vanish unless total helicity = 0 (helicity conservation)

(Problem 17.3 (b) in Peskin & Schroeder)

2.3 Holomorphy

In general, the holomorphy of an n-particle amplitude depends on its total helicity $h_{\text{total}} = h_1 + \cdots + h_n$,

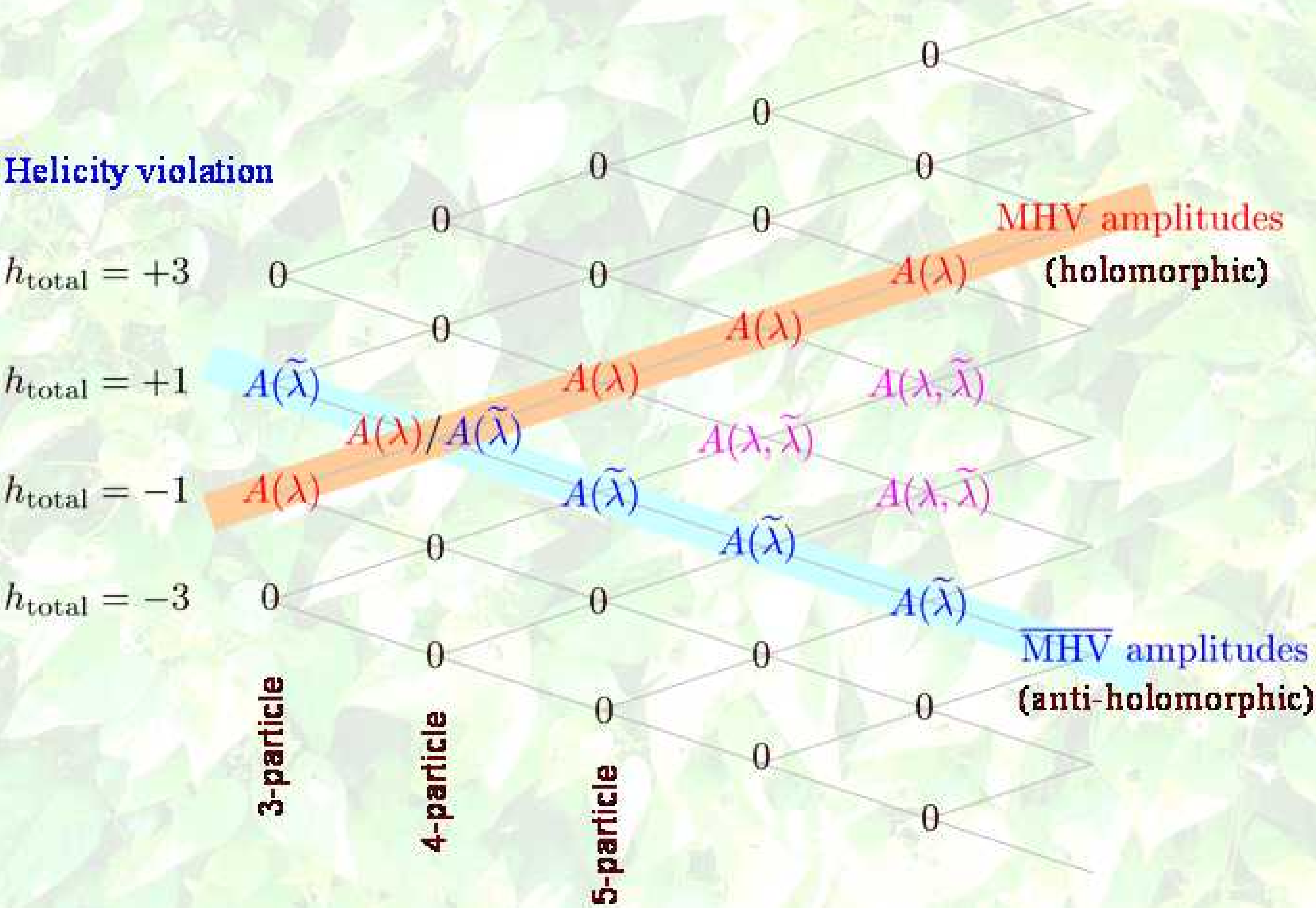
h_{total} is often referred to as "helicity violation".

$A(\lambda, \tilde{\lambda})$ is "holomorphic" when $h_{\text{total}} = n - 4$. (two $h = -1$)

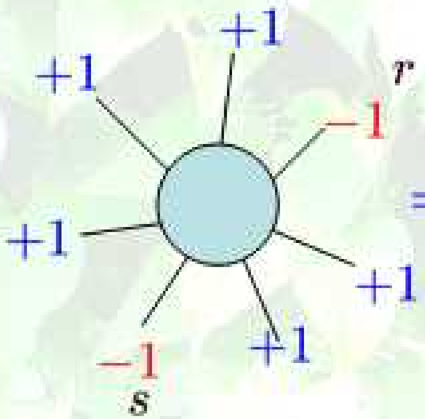
Such amplitudes are called Maximally Helicity Violating amplitudes.

If all or all but one helicities are the same, the amplitude vanishes.
(The three particle amplitudes are exceptions.)

Helicity violation

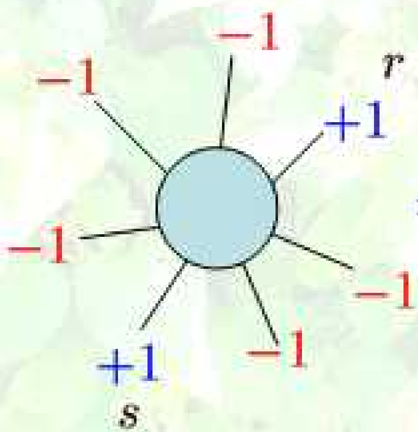


Explicit form of **MHV amplitudes** is known.



$$= \delta^4\Big(\sum_i \lambda_i^a \widetilde{\lambda}_i^b\Big) \frac{\langle \lambda_r, \lambda_s \rangle^4}{\prod_{i=1}^n \langle \lambda_i, \lambda_{i+1} \rangle}$$

$\overline{\text{MHV}}$ amplitudes are the “conjugate” of the MHV amplitudes.



$$= \delta^4\Big(\sum_i \lambda_i^a \widetilde{\lambda}_i^b\Big) \frac{[\widetilde{\lambda}_r, \widetilde{\lambda}_s]^4}{\prod_{i=1}^n [\widetilde{\lambda}_i, \widetilde{\lambda}_{i+1}]}$$

3. Twistor amplitudes

Duality between supersymmetric Yang-Mills and string theory in the twistor space

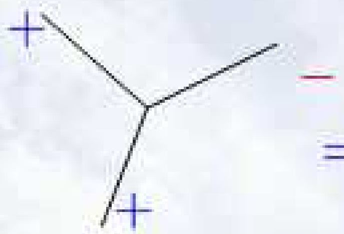
Reference:

E. Witten
Perturbative gauge theory as a string theory in twistor space
Commun.Math.Phys. 252 (2004) 189-258, hep-th/0312171

3.1 Fourier transformation

Twistor amplitudes are obtained from spinor amplitudes by the Fourier tr. $\tilde{\lambda} \to \mu$.

Let us start by three-particle interaction with $h_{\text{total}} = +1$



$$= \quad \delta^4(\sum p) A(\tilde{\lambda}_i)$$

In this case, $\langle \lambda_i, \lambda_j \rangle = 0$ and $A(\tilde{\lambda}_i)$ depend only on $\tilde{\lambda}_i$,

$\longrightarrow$ Three λ_i are the same (up to rescaling).

$$\delta^4(\sum p) A(\tilde{\lambda}_i) \sim \delta(\lambda_1 - \lambda_2)\delta(\lambda_2 - \lambda_3)\delta^2(\sum_i \tilde{\lambda}_i) A(\tilde{\lambda})$$

With this in mind, let us carry out the Fourier tr.

$$
\begin{aligned}
\text{Amp.} &= \int d^6\widetilde{\lambda} e^{i\sum_{i=1}^{3}[\mu_i,\widetilde{\lambda}_i]} \delta^4\Big(\sum_{i=1}^{3} \lambda_i^a \widetilde{\lambda}_i^{\dot{a}}\Big) A(\widetilde{\lambda}_i) \\
&= \int d^4x \int d^6\widetilde{\lambda} e^{i\sum_i \mu_i^{\dot{a}} \widetilde{\lambda}_{i\dot{a}}} e^{i\sum_i \lambda_{ia} x^{a\dot{a}} \widetilde{\lambda}_{i\dot{a}}} A(\widetilde{\lambda}_i) \\
&= \int d^4x \int d^6\widetilde{\lambda} A\left(\frac{\partial}{\partial \mu_i}\right) e^{i\sum_i (\lambda_a x^{a\dot{a}} + \mu_i^{\dot{a}})\widetilde{\lambda}_{i\dot{a}}} \\
&= \int d^4x A\left(\frac{\partial}{\partial \mu_i}\right) \prod_{i=1}^{3} \delta^2(\lambda_{ia} x^{a\dot{a}} + \mu_i^{\dot{a}})
\end{aligned}
$$

$\delta(\lambda x + \mu)$ $\rightarrow$ three points $(\lambda_i, \mu_i) \in \mathbf{CP}^3$ are on the same line.

λ_i are the same $\rightarrow$ three points $(\lambda_i, \mu_i) \in \mathbf{CP}^3$ coincide

$\downarrow$

This is a local interaction in $\mathbf{CP}^3$

(Even though $A(\widetilde{\lambda})$ is a rational function and $A(\partial/\partial\mu)$ is non-local operator, the result is correct for 3-particle amplitudes.)

Yang-Mills with only interactions with $h_{\text{total}} = +1$ (Self-dual Yang-Mills) $\longleftrightarrow$ A local theory in the twistor space (holomorphic Chern-Simons)

duality

Full Yang-Mills $\longleftrightarrow$??? in the twistor space

Fourier tr. of the MHV amplitude

$$\text{Amp.} = \int d^{2n}\widetilde{\lambda} e^{i\sum[\mu_i,\widetilde{\lambda}_i]}\delta^4(\sum_i \lambda_i^a \widetilde{\lambda}_i^{\dot{b}})\frac{\langle\lambda_r,\lambda_s\rangle^4}{\prod_{i=1}^n\langle\lambda_i,\lambda_{i+1}\rangle}$$

$$= \int d^{2n}\widetilde{\lambda} e^{i\sum[\mu_i,\widetilde{\lambda}_i]}\int d^4x \exp(i\sum_i \lambda_i^a x_{a\dot{a}}\widetilde{\lambda}_i^{\dot{a}})\frac{\langle\lambda_r,\lambda_s\rangle^4}{\prod_{i=1}^n\langle\lambda_i,\lambda_{i+1}\rangle}$$

$$= \int d^4x \int d^{2n}\widetilde{\lambda} \exp(i\sum_i(\lambda_{ai}x^{a\dot{a}} + \mu_i^{\dot{a}})\widetilde{\lambda}_{\dot{a}i})\frac{\langle\lambda_r,\lambda_s\rangle^4}{\prod_{i=1}^n\langle\lambda_i,\lambda_{i+1}\rangle}$$

$$= \int d^4x \delta^{2n}(\lambda_{ai}x^{a\dot{a}} + \mu_i^{\dot{a}})\frac{\langle\lambda_r,\lambda_s\rangle^4}{\prod_{i=1}^n\langle\lambda_i,\lambda_{i+1}\rangle}$$

This is a non-local interaction in the twistor space.

All the points (λ_i, μ_i) are on the same line.
(not at the same point)

Witten proposed an interpretation of this amplitude.

3.2 Witten's proposal (Full YM = B-model in the twistor space)

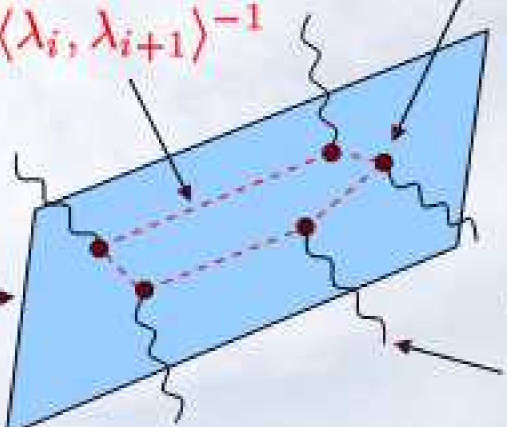
$$\int d^4x\,\delta^{2n}(\lambda_{ai}x^{a\dot{a}} + \mu_i^{\dot{a}})\frac{\langle\lambda_r,\lambda_s\rangle^4}{\prod_{i=1}^n\langle\lambda_i,\lambda_{i+1}\rangle}$$

Moduli integral

D1-brane

Corr. func. on D5

target space = twistor space filled by D5



Comments

- The B-model is defined only in Calabi-Yau spaces.
  $\mathbf{CP}^{3|\mathcal{N}}$ is CY only when $\mathcal{N} = 4$.
  $\rightarrow \quad \mathcal{N} = 4$ SYM.
- It seems impossible to decouple the conformal gravity,
  which arises in the closed string sector of the B-model
  because the conformal gravity has dimensionless coupling $\sim g_{\mathrm{YM}}$.
- It is not known how to choose integration contour of the moduli integral.

4. MHV diagrams

A new method for computation of scattering amplitudes

References:

F. Cachazo, P. Svrcek, E. Witten
MHV Vertices And Tree Amplitudes In Gauge Theory
JHEP 0409 (2004) 006, hep-th/0403047

R. Britto, F. Cachazo, B. Feng
New Recursion Relations for Tree Amplitudes of Gluons
Nucl.Phys. B715 (2005) 499-522, hep-th/0412308

R. Britto, F. Cachazo, B. Feng, E. Witten
Direct Proof Of Tree-Level Recursion Relation In Yang-Mills Theory
Phys.Rev.Lett. 94 (2005) 181602, hep-th/0501052

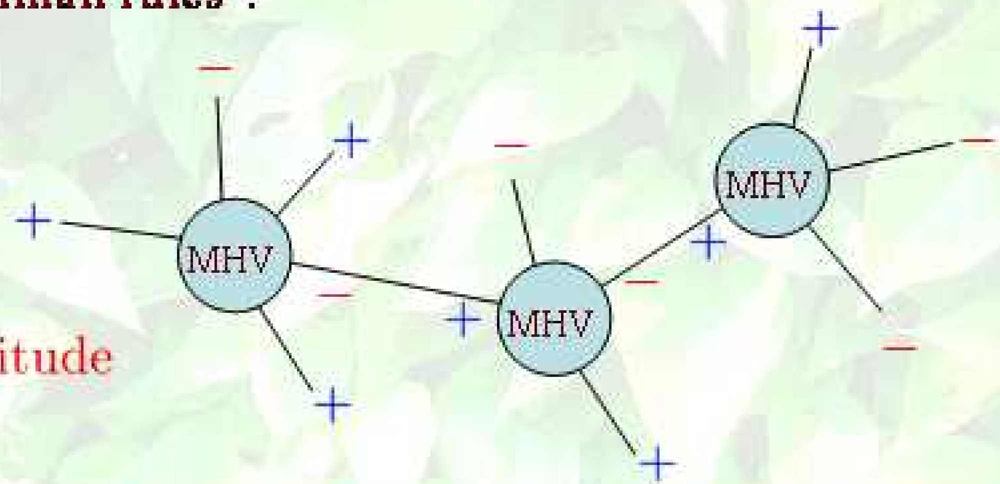
4.1 MHV diagrams

If $\delta(\lambda x + \mu)$ represents a physical D1-brane,
there must also be multi-D-brane contributions to amplitudes.

This gives new ``Feynman rules".

propagator: $1/p^2$

vertex: MHV amplitude



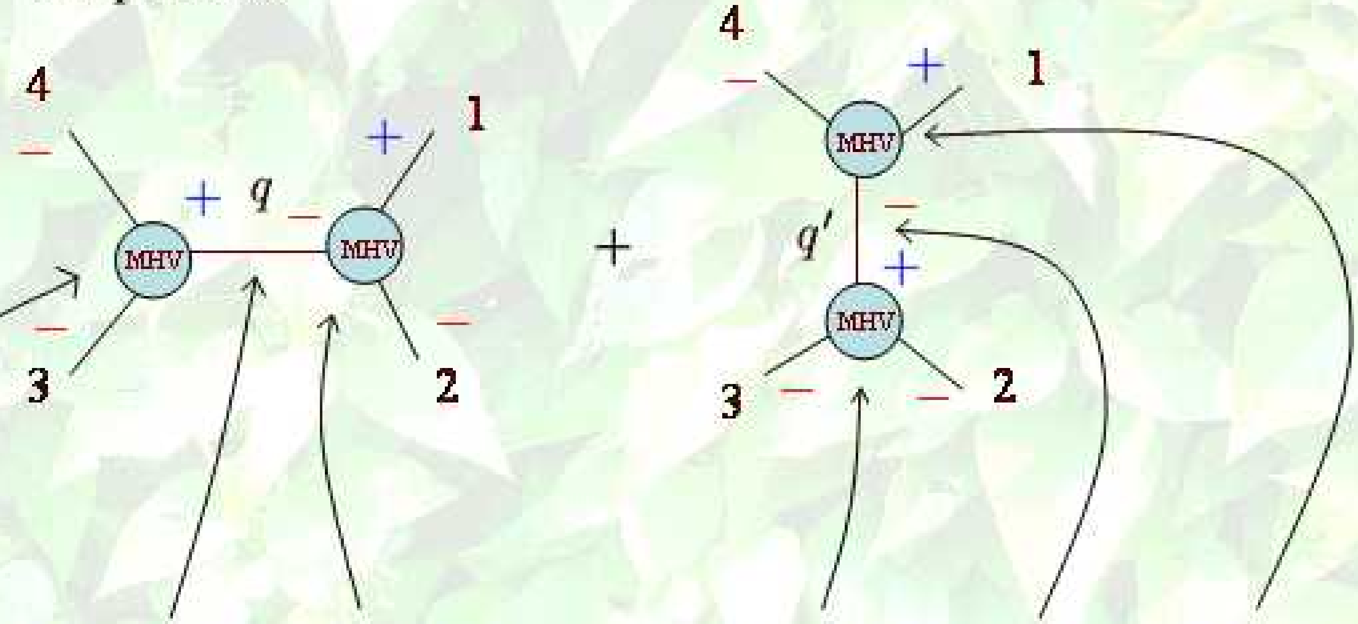
In order to use the MHV amplitudes as vertices, we have to define a rule to give spinor variables for arbitrary (off-shell) momenta.

$\lambda_a = p_{a\dot{a}}\eta$ gives correct amplitudes.
(η is an arbitrary spinor)

Example

$\#$ of vertices $= \#$ of "$-$" -1

$+ - - -$ amplitude



$$= \frac{\langle \lambda_3, \lambda_4 \rangle^3}{\langle \lambda_q, \lambda_3 \rangle \langle \lambda_4, \lambda_q \rangle} \frac{1}{q^2} \frac{\langle \lambda_2, \lambda_q \rangle^3}{\langle \lambda_1, \lambda_2 \rangle \langle \lambda_q, \lambda_1 \rangle} + \frac{\langle \lambda_3, \lambda_2 \rangle^3}{\langle \lambda_{q'}, \lambda_3 \rangle \langle \lambda_2, \lambda_{q'} \rangle} \frac{1}{q'^2} \frac{\langle \lambda_4, \lambda_{q'} \rangle^3}{\langle \lambda_1, \lambda_4 \rangle \langle \lambda_{q'}, \lambda_1 \rangle}$$

$= 0 \quad \cdots \quad$ correct answer

The # of MHV diagrams contributing an amplitude is much smaller than the # of Feynman diagrams.

# of Feynman diagrams $\sim e^{cn}$

# of MHV diagrams $\propto n^{2k-4}$

n # of external lines

k # of negative helicities

MHV diagrams are efficient especially for small k.
Even for k~n/2, the number of MHV diag, is much smaller than that of Feynman diag.

MHV diagrams drastically simplify the computation of scattering amplitudes.

4.2 pole structure

An amplitude becomes singular when a propagator in a Feynman diagram becomes on-shell.

$$p_{ij} = p_i + p_{i+1} + \cdots + p_{j-1} + p_j \quad \rightarrow \quad 0$$

Feynman diagram

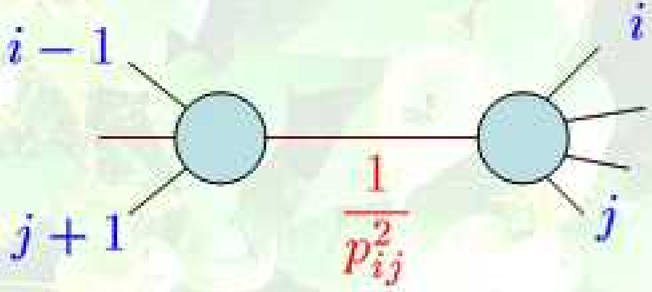
The residue for the pole is the product of two amplitudes connected by the on-shell propagator.

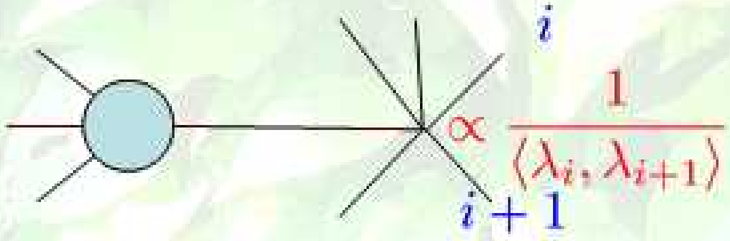
$$A(p_1, \ldots, p_n) \sim A''(p_{j+1}, \ldots, p_{i-1}, p_{ij}) \frac{1}{p_{ij}^2} A'(p_i, \ldots, p_j, -p_{ij})$$

Factorization formula

This structure is reproduced by the MHV diagrams correctly.

There are two kinds of singularities in MHV diagrams.





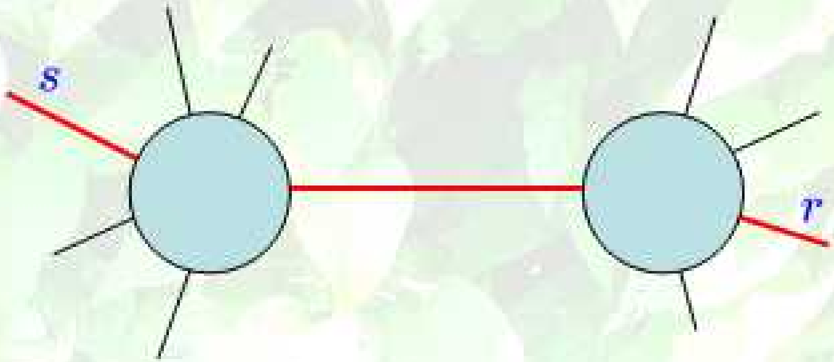
Singularities in MHV propagators

Singularities in MHV vertices

These singularities correctly reproduce the physical singularities in Feynman diagrams.

4.3 BCF recursion relation

Choose two external lines r and s



(Feynman diag.)

and shift their momenta by $\pm z\lambda_r^a \widetilde{\lambda}_s^{\dot{a}}$.

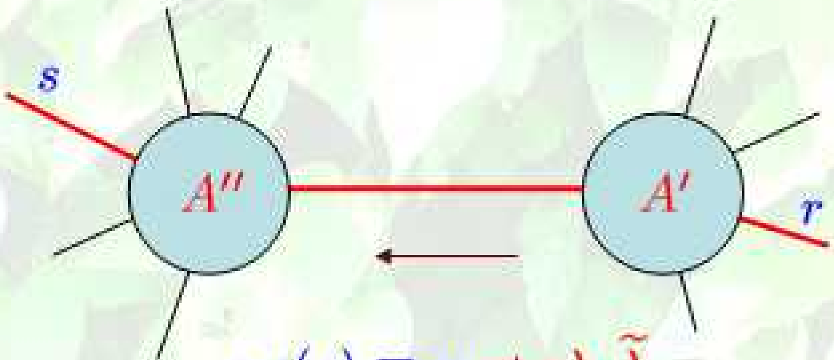
$$p_r = \lambda_r \widetilde{\lambda}_r \to p_r(z) = p_r + z\lambda_r \widetilde{\lambda}_s = \lambda_r(\widetilde{\lambda}_r + z\widetilde{\lambda}_s)$$

$$p_s = \lambda_s \widetilde{\lambda}_s \to p_s(z) = p_s - z\lambda_r \widetilde{\lambda}_s = (\lambda_s - z\lambda_r)\widetilde{\lambda}_r$$

Regardress of the variable z, the momenta are on-shell.

The amplitude $A(z)$ is a rational function of z.

pole of $A(z)$ $\leftrightarrow$ pole of propagator



(Feynman diag.)

$$p_{ij}(z) \equiv p_{ij} + z\lambda_r \widetilde{\lambda}_s$$

All residues can be determined by the factorization formula

$$A(z) \sim A''(p_{j+1}, \ldots, p_{i-1}, p_{ij}(z_{ij})) \frac{1}{p_{ij}(z_{ij})^2} A'(p_i, \ldots, p_j, -p_{ij}(z))$$

near $z = z_{ij}$, where z_{ij} is the solution of $p_{ij}(z) = 0$.

We can also show that $\lim_{z \to \infty} A(z) = 0$

Given all the residue and the asymptotic value of $A(z)$, we can uniquely determine the function $A(z)$.

BCF recursive relation:

$$A(z) = \sum_{i,j} A''(p_{j+1}, \ldots, p_{i-1}, p_{ij}(z_{ij})) \frac{1}{p_{ij}^2(z)} A'(p_i, \ldots, p_j, -p_{ij}(z_{ij}))$$

If we have 3-pt amplitudes, we can construct an arbitrary tree amplitude using this relation recursively.

4.4 Proof of the MHV formula

BCF recursion relation shows that
the on-shell amplitudes are uniquely determined by the pole structure.

In order to prove the MHV formula, we have only to show that
the MHV formula gives the correct pole structure.

We have already shown that MHV diagrams correctly reproduce the pole structure of tree-level amplitudes.

$$\downarrow$$

MHV formula is proven!

5. Conclusions

Although the twistor string theory itself has not been established, it inspired the new method to compute scattering amplitudes.

MHV diagrams drastically simplify the computation of scattering amplitudes.

At the tree level, it was proven that the MHV diagrams give correct scattering amplitudes.

The proof is based on the BCF recursion relation, which does not depend on string theory.

Generalization

At the tree level, MHV rules for $\mathcal{N}=4$ SYM can be diverted to amplitudes in other theories with different field contents.

J. Bedford, A. Brandhuber, B. Spence, G. Travaglini
A recursion relation for gravity amplitudes
Nucl.Phys. B721 (2005) 98-110, hep-th/0502146

The BCF recursion relation is generalized for graviton and charged scalar fields.

J.-B. Wu, C.-J. Zhu
MHV Vertices and Scattering Amplitudes in Gauge Theory
JHEP 0407 (2004) 032, hep-th/0406085

MHV diagrams correctly give one-loop MHV amplitudes.

A. Brandhuber, B. Spence, G. Travaglini
One-Loop Gauge Theory Amplitudes in N=4 Super Yang-Mills from MHV Vertices
Nucl.Phys. B706 (2005) 150-180, hep-th/0407214