SLE and CFT

Mitsuhiro Kato @ QFT2005

1. Introduction

Critical phenomena

- Conformal Field Theory (CFT)

  Algebraic approach, field theory, BPZ(1984)

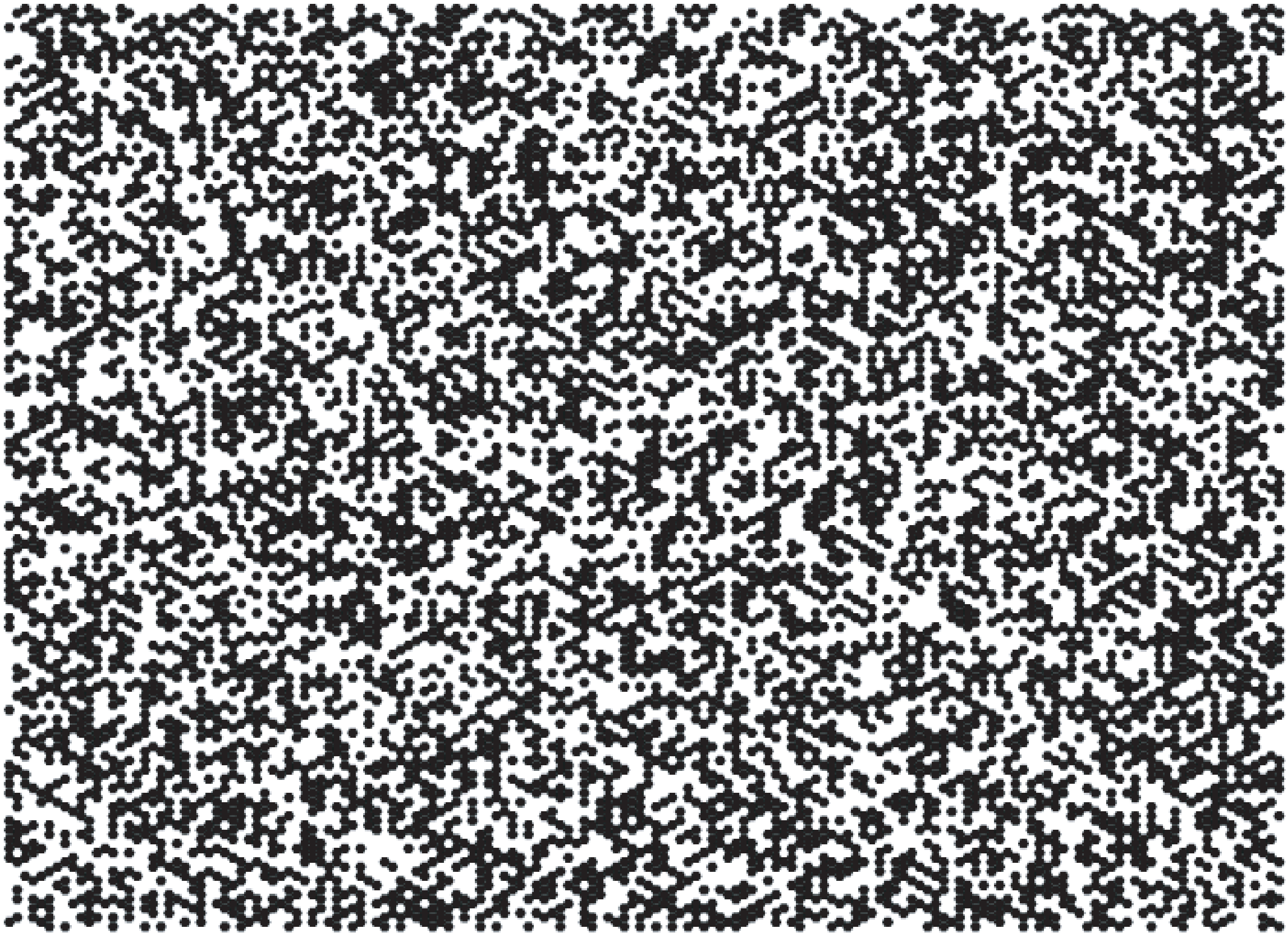
- Stochastic Loewner Evolution (SLE)

  Geometrical approach, stochastic process, Schramm(2000)

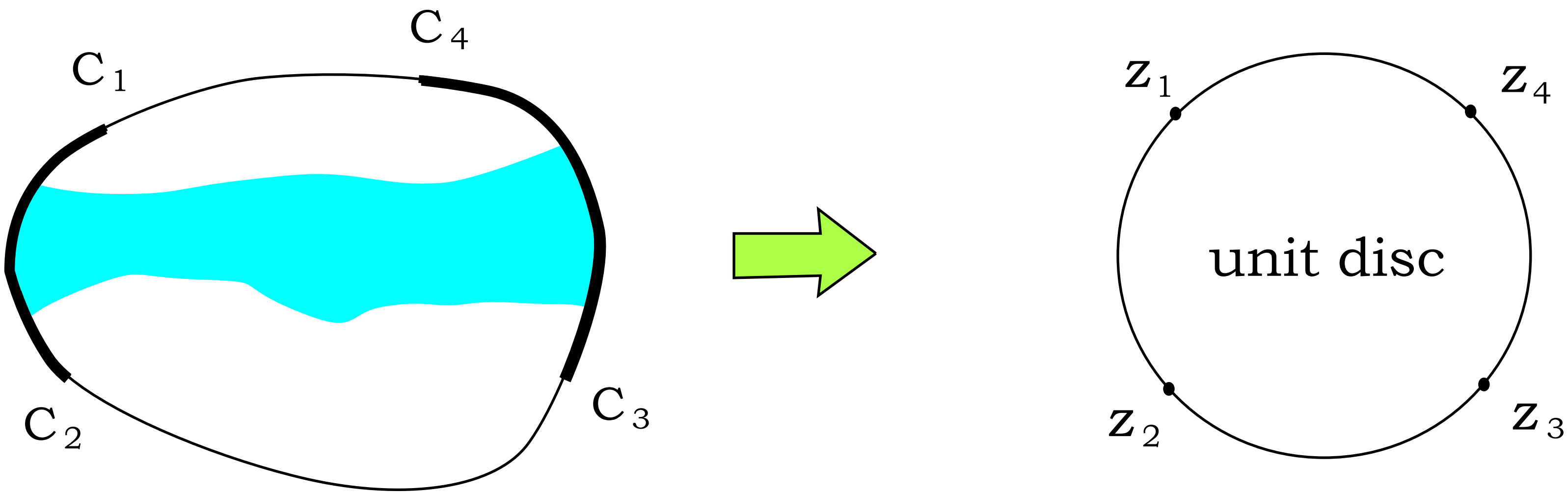
Percolation

Consider triangular lattice whose site is colored black with probability p or white with probabilty $1-p$.
Convenient to consider dual lattice whose face (hexagon) is colored accordingly.
Study clustering property of colored faces.

$p \to p_c$: percolation threshold at which mean cluster size diverge



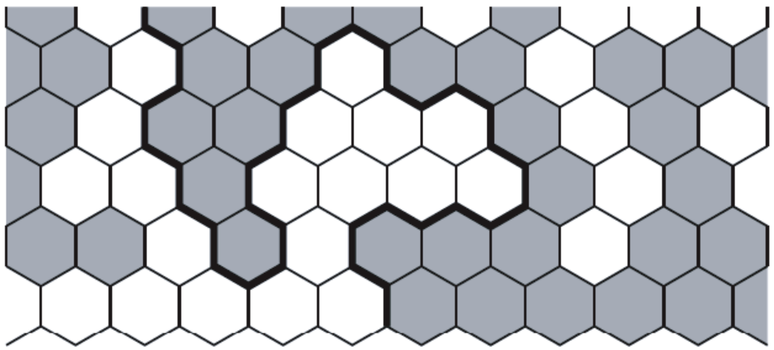
Crossing probability



$P(\gamma_1, \gamma_2)$ is a function only of cross-ratio $\eta = \frac{(z_1 - z_2)(z_3 - z_4)}{(z_1 - z_3)(z_2 - z_4)}$

$$P = \frac{\Gamma(\frac{2}{3})}{\Gamma(\frac{4}{3})\Gamma(\frac{1}{3})}\eta^{1/3}\,{}_2F_1(\frac{1}{3}, \frac{2}{3}, \frac{4}{3}; \eta)$$

SLE treats cluster boundary via stochastic process.



Plan of the talk

1. Introduction

2. SLE

3. Critical models

4. Relation to CFT

5. Remarks

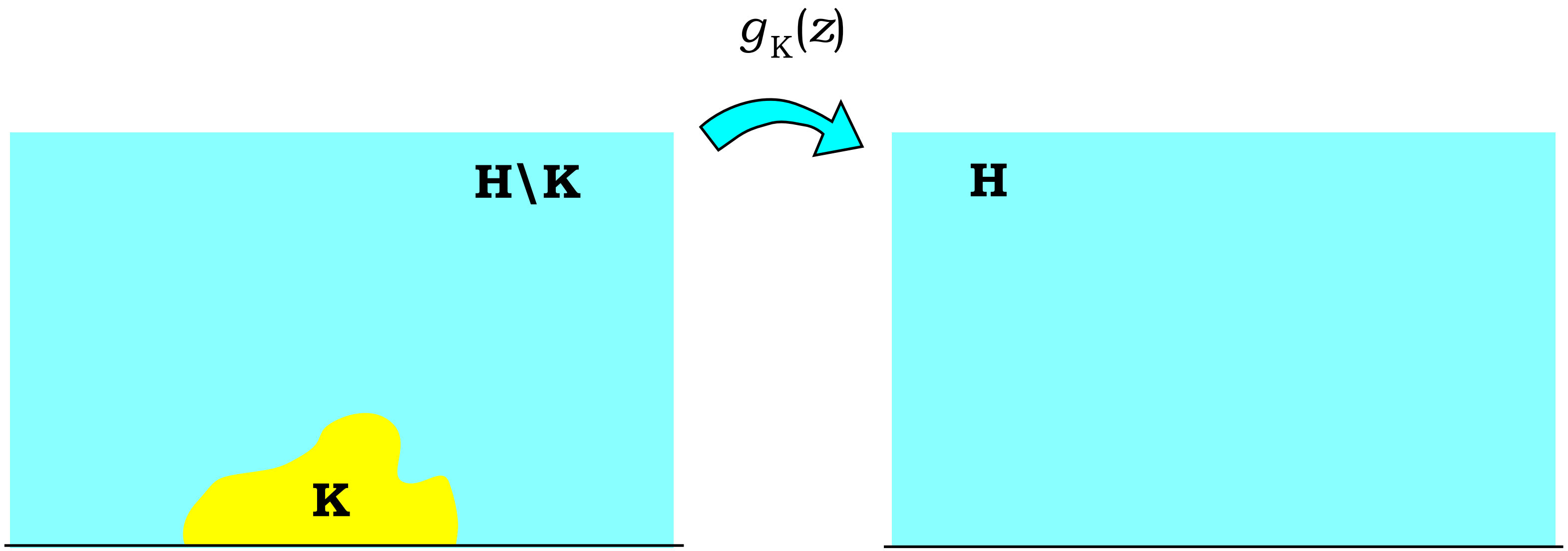
2. SLE

Hull

A compact subset $\mathbf{K}$ in $\mathbf{H}$ s.t.

$$\mathbf{H} \setminus \mathbf{K} \text{ is simply connected}, \qquad \mathbf{K} = \overline{\mathbf{K} \cap \mathbf{H}}$$

is called a **hull**.



Conformal map

For any hull $\mathbf{K}$, there exists a unique conformal map

$$g_{\mathsf{K}} \; : \; \mathbf{H} \setminus \mathbf{K} \to \mathbf{H}$$

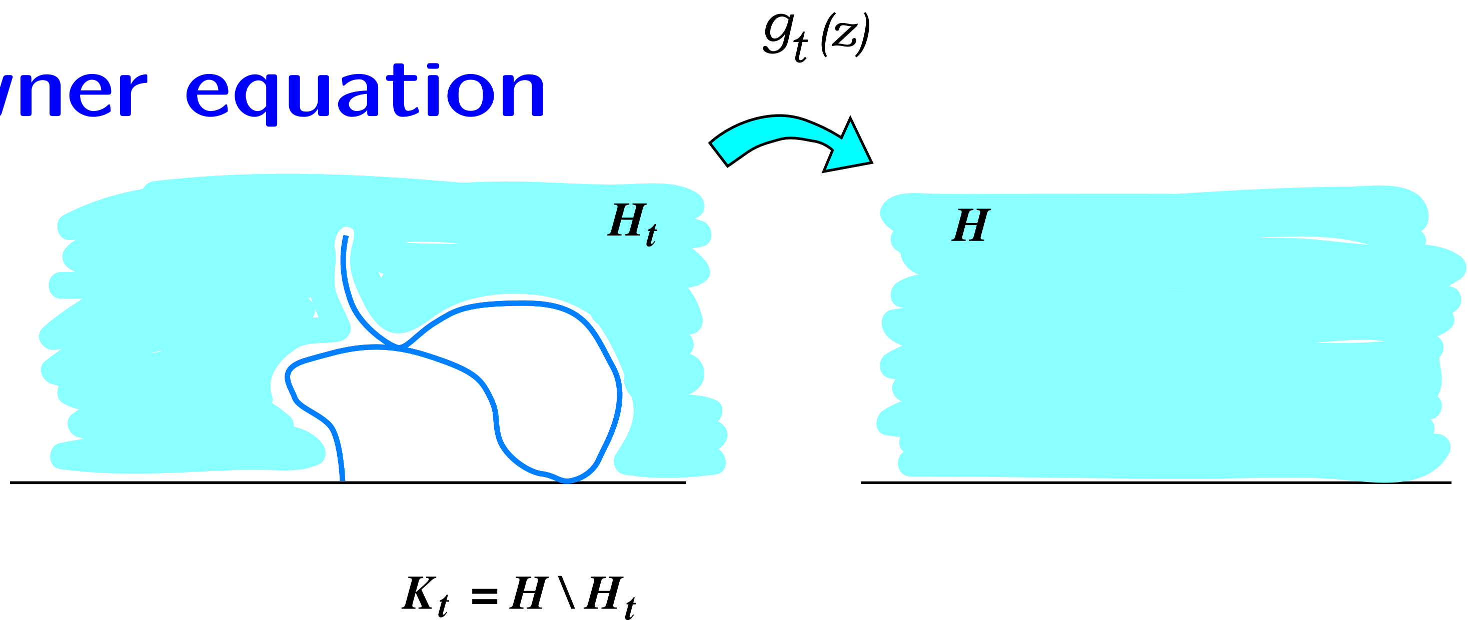
$$\lim_{z \to \infty} (g_{\mathsf{K}}(z) - z) = 0$$

This map has an expansion for $z \to \infty$

$$g_{\mathsf{K}}(z) = z + \frac{a_1}{z} + \cdots + \frac{a_n}{z^n} + \cdots$$

$a_1 = a_1(\mathbf{K})$ is called **capacity** of the hull $\mathbf{K}$.

Loewner equation



$$U_t = g_t(\gamma(t))$$

Let $\gamma(t)$ be parametrized s.t. $a_1(K_t) = 2t$. Then

$$\frac{\partial}{\partial t} g_t(z) = \frac{2}{g_t(z) - U_t}, \qquad g_0(z) = z$$

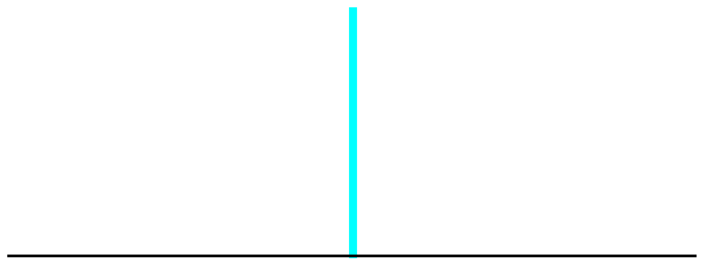
example

$U_t = 0$ case

$$\frac{\partial}{\partial t} g_t(z) = \frac{2}{g_t(z)}, \qquad g_0(z) = z$$

$$g_t(z) = \sqrt{z^2 + 4t}$$

$$\gamma(t) = 2i\sqrt{t}$$



SLE

$$\frac{\partial}{\partial t} g_t(z) = \frac{2}{g_t(z) - \sqrt{\kappa} B_t}, \qquad g_0(z) = z$$

where B_t is standard Brawnian motion on $\mathbf{R}$,

κ is a real parameter.

Alternatively, for $\widehat{g}_t(z) = g_t(z) - \sqrt{\kappa} B_t$

$$d\widehat{g}_t(z) = \frac{2}{\widehat{g}_t(z)} dt - \sqrt{\kappa} dB_t$$

Brawnian motion

For $U_t = \sqrt{\kappa} B_t$

$$\langle\langle U_t \rangle\rangle = 0, \qquad \langle\langle U_{t_1} U_{t_2} \rangle\rangle = \kappa |t_1 - t_2|$$

Thus

$$\langle\langle dU_t dU_t \rangle\rangle = \kappa dt$$

Itô formula

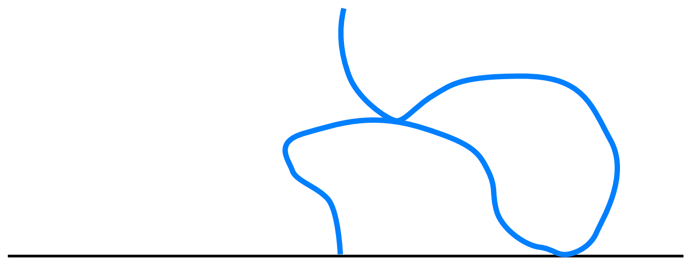
Suppose X_t satisfies stochastic differential eq.

$$dX_t = a(X_t, t)dt + b(X_t, t)dB_t$$

Then for a function $f(X_t)$

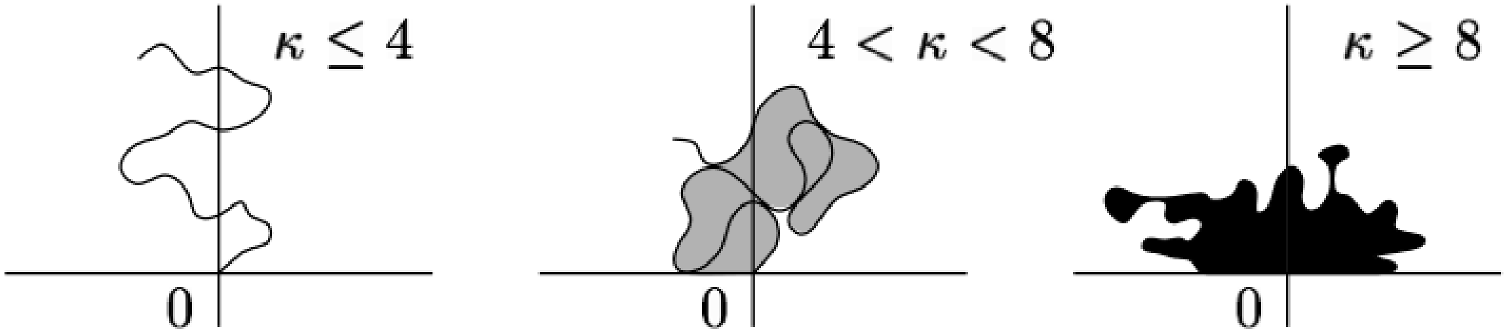
$$df = (af' + \frac{1}{2}b^2 f'')dt + bf'dB_t$$

SLE trace



$$\gamma(t) := \lim_{z \to 0} g_t^{-1}(z + \sqrt{\kappa} B_t)$$

Phases of SLE



simple curve double points space-filling

duality conjecture

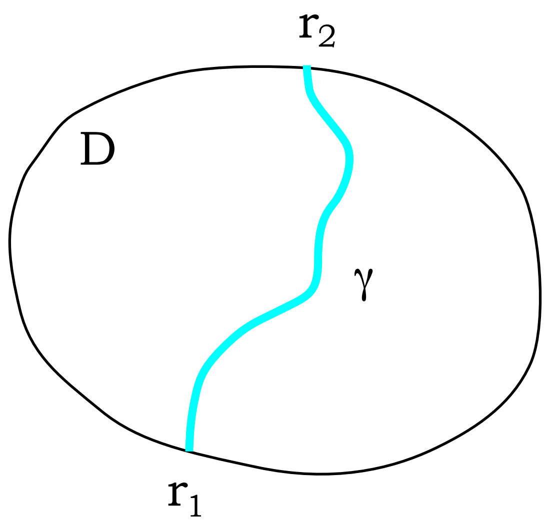
$\partial \mathbf{K}_t$ for $\kappa > 4$ $\Leftrightarrow$ SLE trace for $\hat{\kappa} = 16/\kappa$ < 4

Hausdorff dimensions

$$d_H = \begin{cases} 1 + \kappa/8 & (\kappa < 8) \\ 2 & (\kappa > 8) \end{cases}$$

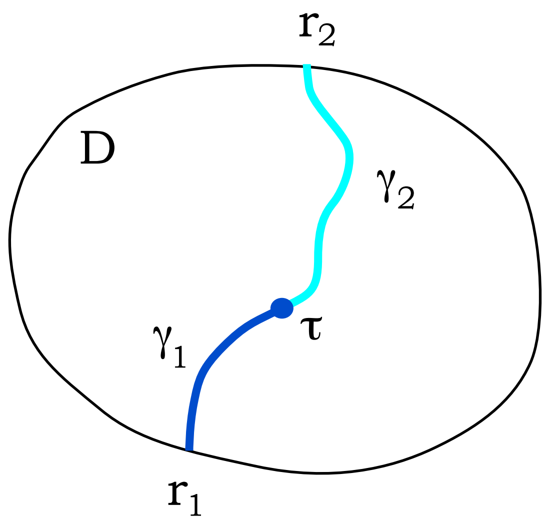
Basic properties

Denote measure $\mu(\gamma; D, r_1, r_2)$ for



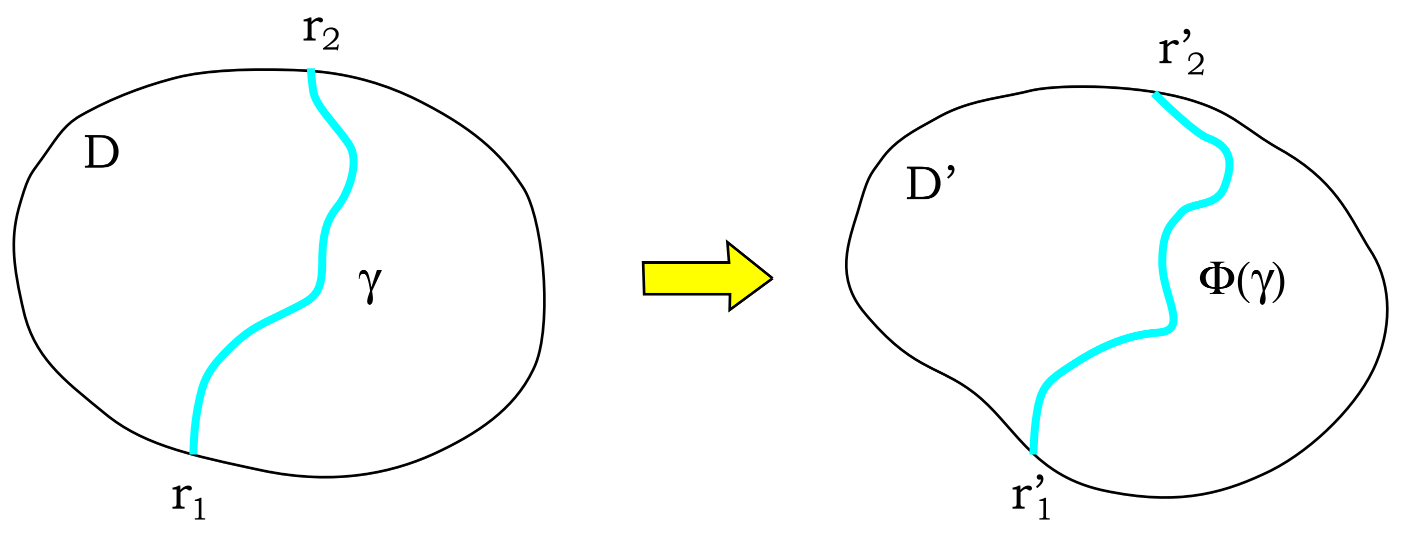
Property 1 (Martingale)

$$\mu(\gamma_2|\gamma_1; D, r_1, r_2) = \mu(\gamma_2; D \setminus \gamma_1, \tau, r_2)$$



Property 2 Conformal invariance

$$(\Phi * \mu)(\gamma; D, r_1, r_2) = \mu(\Phi(\gamma); D', r_1', r_2')$$

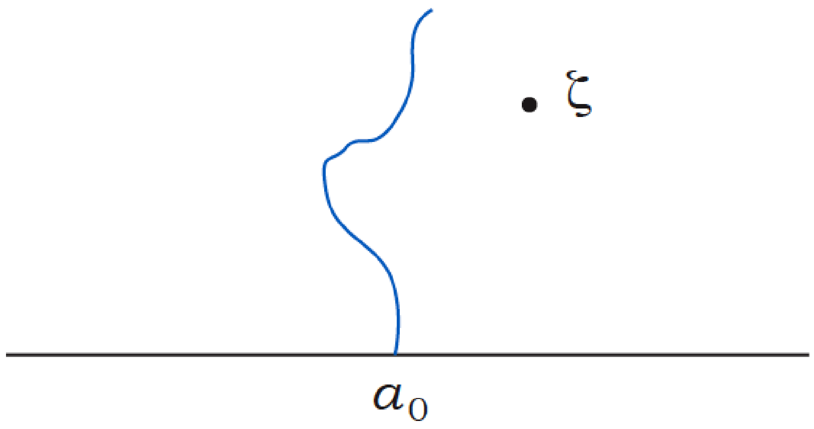


Example calculation with SLE

Schramm's formula

Probability that γ passes to the left of a given point

$P(\zeta, \bar{\zeta}; a_0)$



For infinitesimal dt,

g_{dt} : $\{$remainder of $\gamma\} \to \gamma'$

By prop. 1 and 2, it has same measure as SLE started from $a_{dt} = a_0 + \sqrt{\kappa} dB_t$

$$\zeta \to g_{dt}(\zeta) = \zeta + \frac{2dt}{\zeta - a_0}$$

γ' lies to the left of ζ' iff γ does of ζ

$$P(\zeta, \bar{\zeta}; a_0) = \left\langle\!\!\left\langle \, P\left(\zeta + \frac{2dt}{\zeta - a_0}, \bar{\zeta} + \frac{2dt}{\bar{\zeta} - a_0}; a_0 + \sqrt{\kappa} dB_t\right) \right\rangle\!\!\right\rangle$$

$\uparrow$

over Brownian motion dB_t up to time dt

Using $\langle\langle dB_t \rangle\rangle = 0$ and $\langle\langle (dB_t)^2 \rangle\rangle = dt$, one obtains

$$\left(\frac{2}{\zeta - a_0} \frac{\partial}{\partial \zeta} + \frac{2}{\bar{\zeta} - a_0} \frac{\partial}{\partial \bar{\zeta}} + \frac{\kappa}{2} \frac{\partial^2}{\partial a_0^2} \right) P(\zeta, \bar{\zeta}; a_0) = 0$$

By scale inv., P depends only on $\theta = \arg(\zeta - a_0)$

$\rightarrow$ linear 2nd-order ordinary diff. eq. (hypergeometric)

With b.c. $P(\theta = \pi) = 0$, $P(\theta = 0) = 1$

$$\rightarrow P = \frac{1}{2} + \frac{\Gamma(2/3)}{\sqrt{\pi}\Gamma(1/6)} (\cot\theta)\ {}_2F_1\left(\frac{1}{2}, \frac{2}{3}, \frac{3}{2}; -\cot^2\theta \right)$$

3. Critical Models

$\kappa = 2$ loop-erased random walk

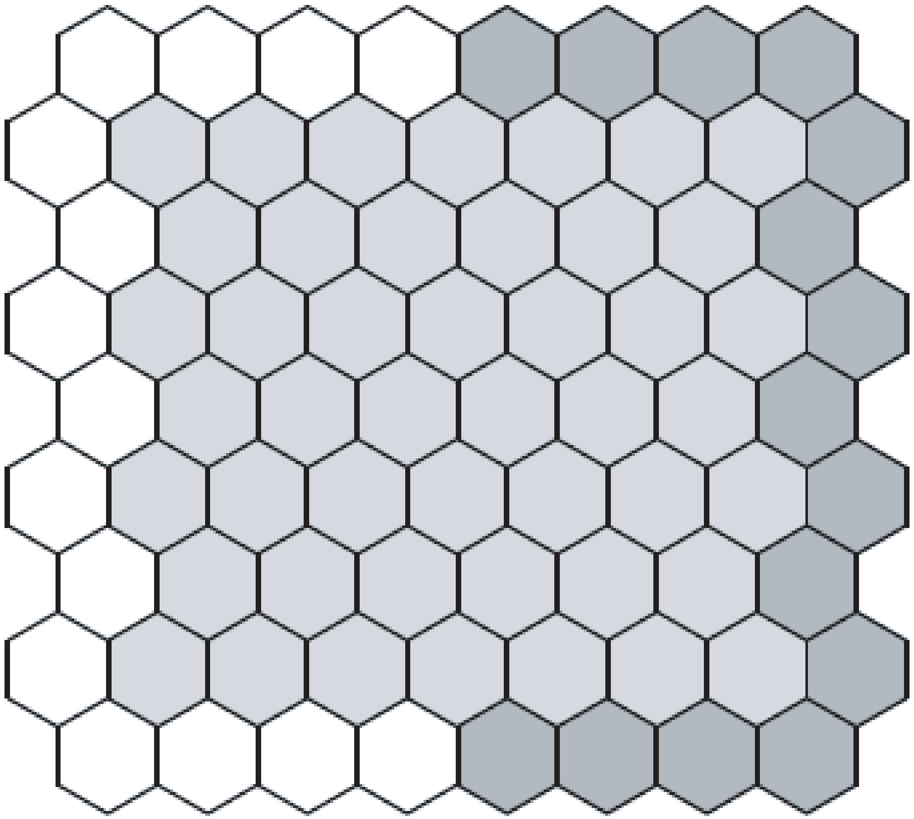
$\kappa = 8/3$ self-avoiding walk*

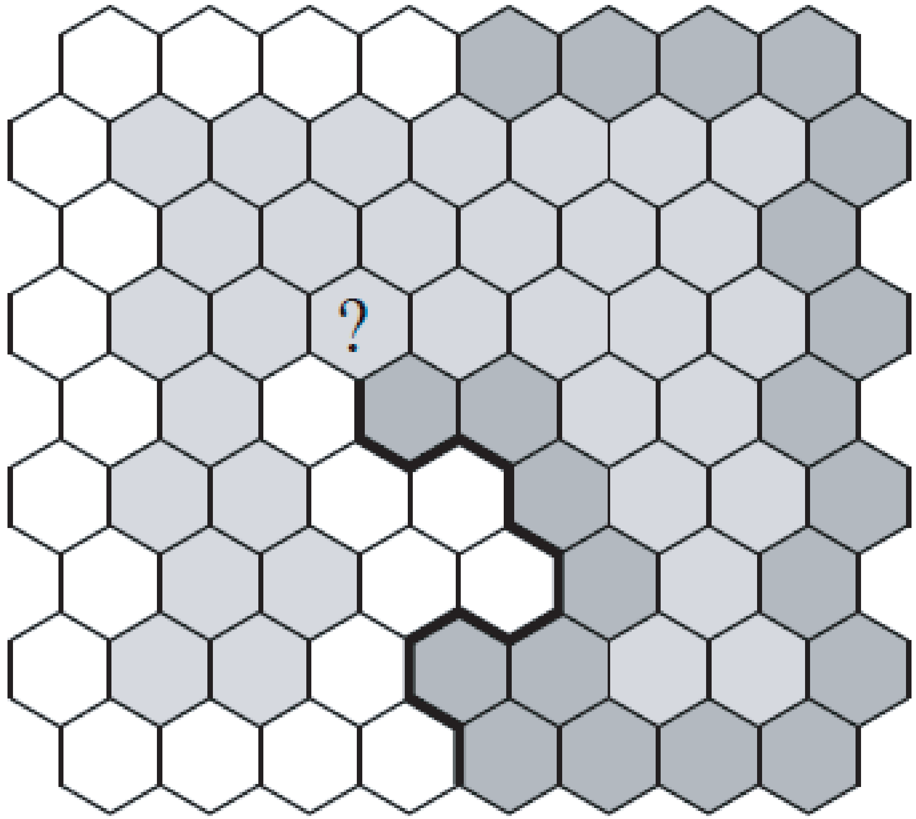
$\kappa = 3$ cluster boundary in Ising model*

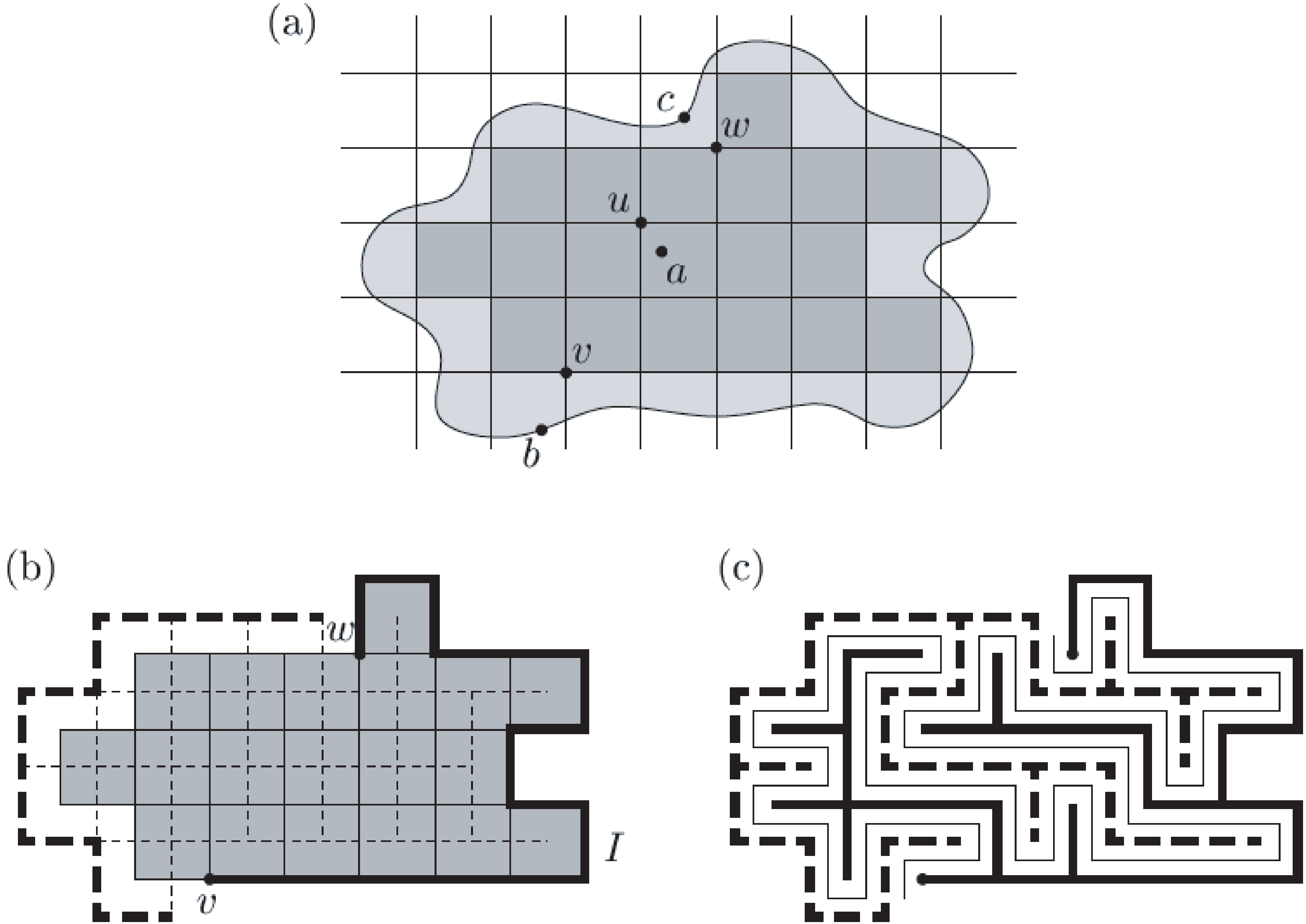
$\kappa = 4$ BCSOS model of roughening transition* (4-state Potts), harmonic explorer, dual to the KT transition in XY model*

$\kappa = 6$ cluster boundary in critical percolation

$\kappa = 8$ Peano curve associated with uniform spanning tree



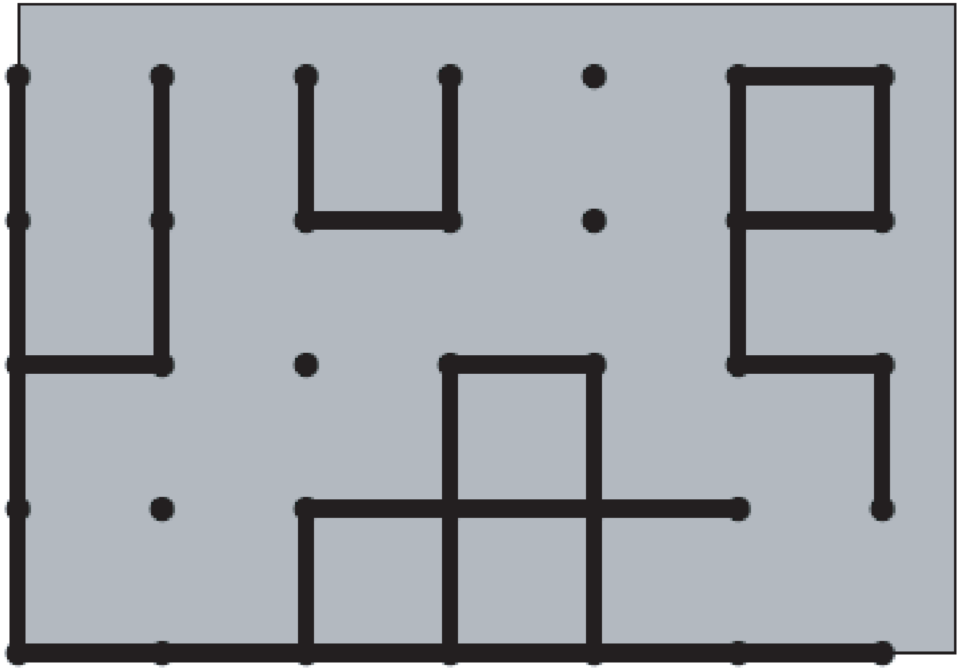


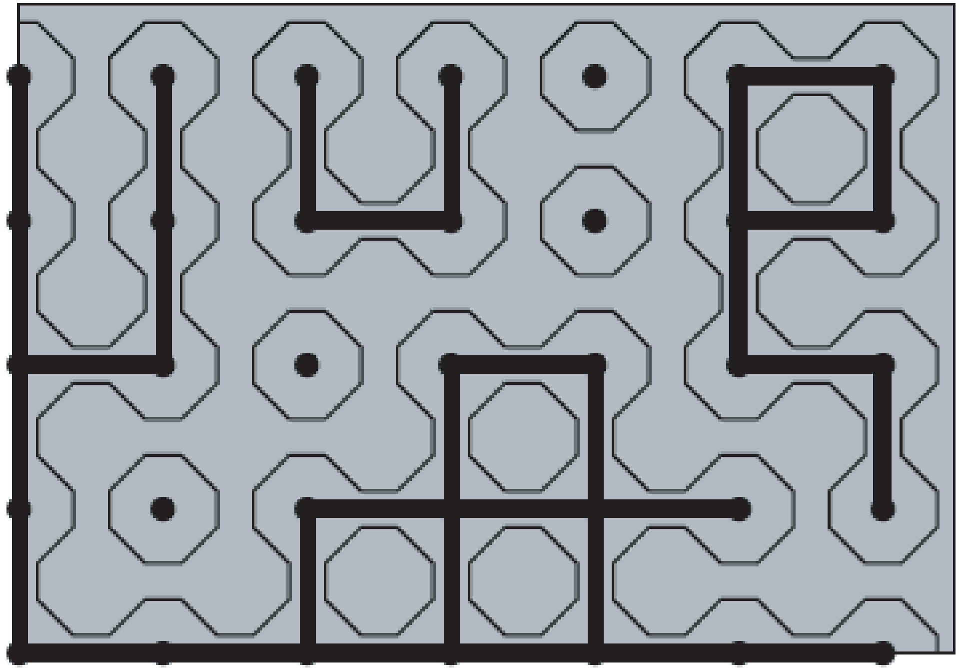

(a)
c
w
u
a
v
b
(b)
w
I
v
(c)

q-states Potts model

$$
\begin{aligned}
Z &= \sum_{\{s\}} \exp\left(\beta \sum_{\langle j,k\rangle} \delta_{s_j,s_k}\right) \\
&= \sum_{\{s\}} \prod_{\langle j,k\rangle} \left(1 + (e^\beta - 1)\delta_{s_j,s_k}\right) \\
&= \sum_{\text{graphs}} (e^\beta - 1)^b q^c
\end{aligned}
$$

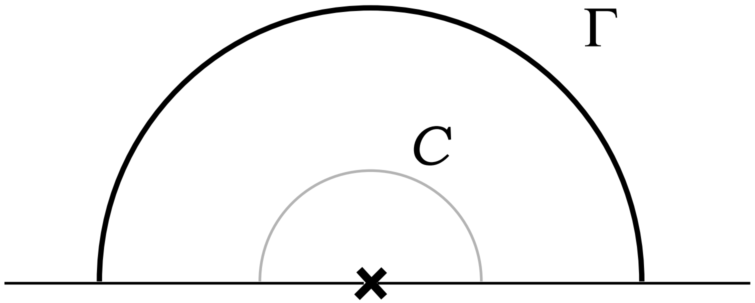
$$
q = 2 + 2\cos(8\pi/\kappa)
$$





4. Relation to CFT

BCFT



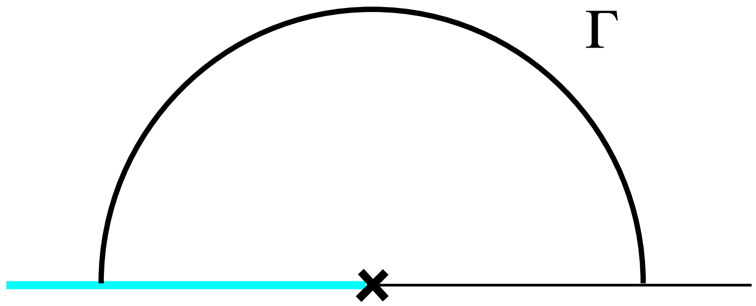
Hilbert space of BCFT $= \{\psi_\Gamma\}$ on Γ

$$|0\rangle = \int [d\psi'_\Gamma] \int_{\psi_\Gamma=\psi'_\Gamma} [d\psi] e^{-S[\psi]} |\psi'_\Gamma\rangle$$

$$|\phi\rangle = \int [d\psi'_\Gamma] \int_{\psi_\Gamma=\psi'_\Gamma} [d\psi] \phi(0) e^{-S[\psi]} |\psi'_\Gamma\rangle$$

$$L_n|\phi\rangle = \int [d\psi'_\Gamma] \int_{\psi_\Gamma=\psi'_\Gamma} [d\psi] \int_C \frac{dz}{2\pi i} z^{n+1} T(z) \phi(0) e^{-S[\psi]} |\psi'_\Gamma\rangle$$

Insertion of a boundary condition changing operator



$$|h\rangle = |h_t\rangle = \int d\mu(\gamma_t)|\gamma_t\rangle$$

$$|\gamma_t\rangle = \int [d\psi'_\Gamma] \int_{\psi_\Gamma = \psi'_\Gamma;\gamma_t} [d\psi] e^{-S[\psi]} |\psi'_\Gamma\rangle$$

$d\mu(\gamma_t)$ is given by the path-integral in **H**.

$|h\rangle$ is independent of t.

Measure is also determined by SLE

$$d\widehat{g}_t = \frac{2dt}{\widehat{g}_t} - \sqrt{\kappa}\, dB_t$$

This is an infinitesimal conformal mapping which corresponds to the insertion $(1/2\pi i) \int (2dt/z - \sqrt{\kappa}\, dB_t) T(z)$.

Thus for $t_1 < t$

$$|g_{t_1}(\gamma_t)\rangle = \mathsf{T} \exp\left(\int_0^{t_1} (2L_{-2}dt' - L_{-1}\sqrt{\kappa}\, dB_{t'}) \right) |\gamma_t\rangle$$

$$
\begin{aligned}
&(\text{measure on } \gamma_t) \\
&\quad = (\text{measure on } \gamma_t \setminus \gamma_{t_1}, \text{ conditioned on } \gamma_{t_1}) \\
&\qquad \times (\text{measure on } \gamma_{t_1}) \\
&\quad = (\text{measure on } g_{t_1}(\gamma_t)) \times (\text{measure on } \gamma_{t_1})
\end{aligned}
$$

$$|h_t\rangle =$$

$$\int d\mu(g_{t_1}(\gamma_t)) \int d\mu(\sqrt{\kappa} B_{t' \in [0,t_1]}) \mathsf{T} e^{\int_{t_1}^{0} (2L_{-2} dt' - L_{-1} \sqrt{\kappa} dB_{t'})} |g_{t_1}(\gamma_t)\rangle$$

$$\downarrow$$

$$|h_t\rangle = \exp\left(-(2L_{-2} - \frac{\kappa}{2} L_{-1}^2) t_1 \right) |h_{t-t_1}\rangle$$

However, $|h_t\rangle$ is independent of t.

Thus

$$(2L_{-2} - \frac{\kappa}{2} L_{-1}^2) |h\rangle = 0$$

$$\Downarrow$$

$$h = h_{2,1} = \frac{6-\kappa}{2\kappa}$$

$$c = 13 - 6(\frac{\kappa}{2} + \frac{2}{\kappa})$$

$$P(\zeta; a_0) = \frac{\langle \phi_{2,1}(a_0)\, O(\zeta)\, \phi_{2,1}(\infty)\rangle}{\langle \phi_{2,1}(a_0)\, \phi_{2,1}(\infty)\rangle}$$

5. Remarks

A generalization

SLE(κ, $\vec{\rho}$) : a minimal generalization of SLE which retains self-similarity $\sigma^{-1} g_{\sigma^2 t}(\sigma z)$

$$dW_t = \sqrt{\kappa}\, dB_t - \sum_{j=1}^{n} \frac{\rho_j dt}{X_t^{(j)}}$$

$$dX_t^{(j)} = \frac{2dt}{X_t^{(j)}} - dW_t$$

This is a special case of

$$dW_t = \sqrt{\kappa} dB_t - J_t^x(0) dt$$

$$(2L_{-2} - \frac{\kappa}{2}L_{-1}^2 - J_{-1}L_{-1})|h\rangle = 0 \qquad (J_{-1} = J_0^x(0))$$

$$J^\mu \propto \epsilon^{\mu\nu}\partial_\nu\phi$$

$\longrightarrow$

$\kappa = 4$ case free field with piecewise constant Dirichlet b.c.

$\kappa \neq 4$ case Coulomb gas representation.

Review articles

G.F.Lawler; An introduction to the stochastic Loewner evolution, http://www.math.duke.edu/~jose/papers.html, 2001.

W.Kager, B.Nienhuis; A guide to stochastic Löwner evolution and its applications, math-ph/0312056.

J.Cardy; SLE for theoretical physicists, cond-mat/0503313.