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Integrability in Super Yang-Mills and Strings

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Thanks for collaborations to:
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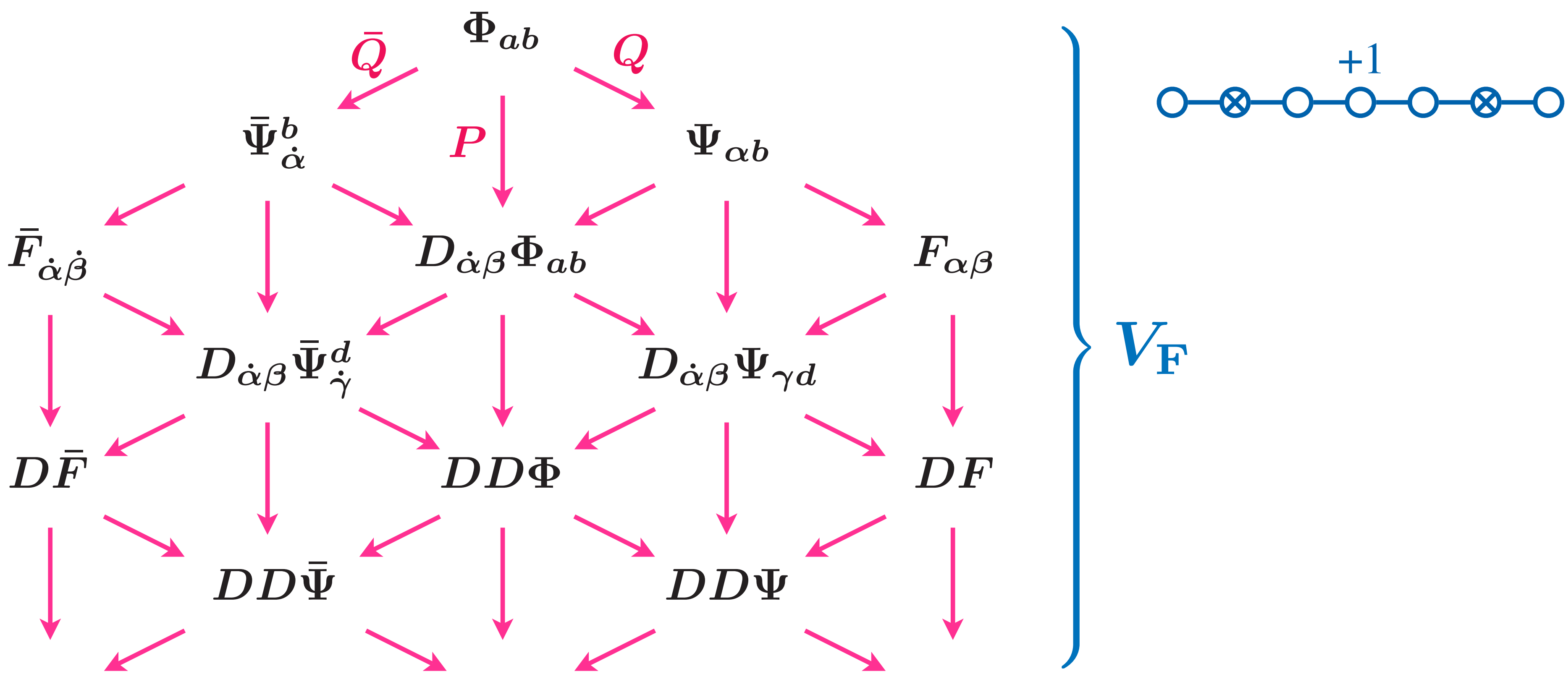
Contents

$\mathcal{N}=4$ U(N) Super Yang-Mills

$$\mathcal{L} = -\frac{1}{4g_{\mathrm{YM}}^2}\mathrm{Tr}\Big((F_{\mu\nu})^2 + 2(D_\mu\Phi_i)^2 - ([\Phi_i,\Phi_j])^2 + 2i\bar{\Psi}\not{D}\Psi - 2\bar{\Psi}\Gamma_i[\Phi_i,\Psi]\Big)$$

Global symmetry: $SO(4,2)\times SU(4) \in PSU(2,2|4)$



Conformal Field Theory

- Correlation function of local operators

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\rangle = \delta_{D_1 D_2} \frac{B_{12}}{|x_{12}|^{D_1+D_2}}$$

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\mathcal{O}_3(x_3)\rangle = \frac{C_{123}}{|x_{12}|^{D_1+D_2-D_3}|x_{23}|^{D_2+D_3-D_1}|x_{31}|^{D_3+D_1-D_2}}$$

D_i : scaling dimension of the local operator $\mathcal{O}_i$

- Single trace operator

$$\mathcal{O} = \mathrm{Tr}[W_{A_1} W_{A_2} \cdots W_{A_J}]$$

$$W_A \in \{D^k \Phi, D^k \Psi, D^k \bar{\Psi}, D^k F\}$$

(Beisert '03)

  - dominant in the large N limit

- Scaling dimension:

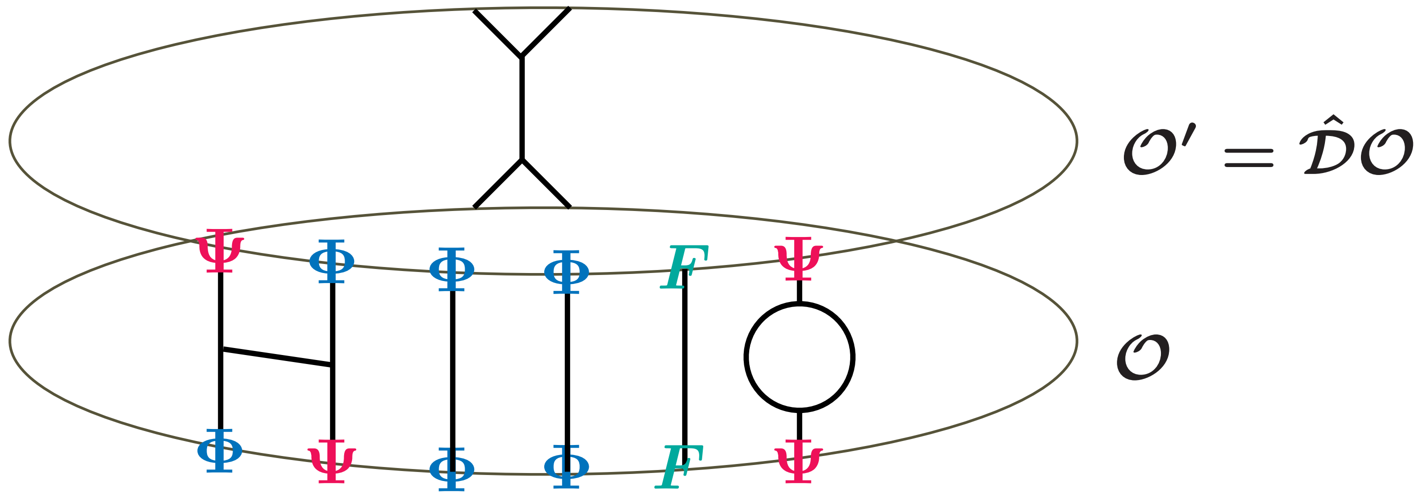
  eigenvalue of the Dilatation operator $\hat{\mathcal{D}}$

- At tree level:

$$\hat{\mathcal{D}}_0 \mathcal{O} = \dim(\mathcal{O}) \mathcal{O}$$

$$[\Phi] = 1, \quad [\Psi] = \frac{3}{2}, \quad [F] = 2, \quad [D] = 1$$

- Quantum correction: operator mixing



Diagonalize $\hat{\mathcal{D}}$!

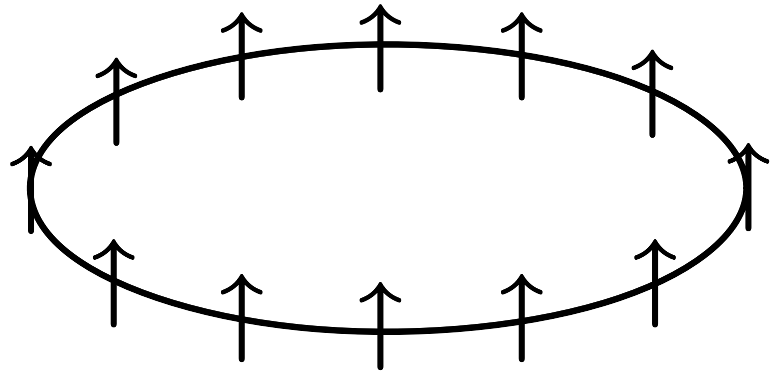
$$\hat{\mathcal{D}} = \sum_{n=0}^{\infty} \lambda^n \hat{\mathcal{D}}_n \qquad \lambda = g_{\rm YM}^2 N \quad \text{('t Hooft coupling)}$$

$\hat{\mathcal{D}}_1 \Leftrightarrow$ Hamiltonian of $\mathfrak{su}(2,2|4)$ spin chain

(Minahan-Zarembo '02) (Beisert-Staudacher '03)

- SU(2) subsector

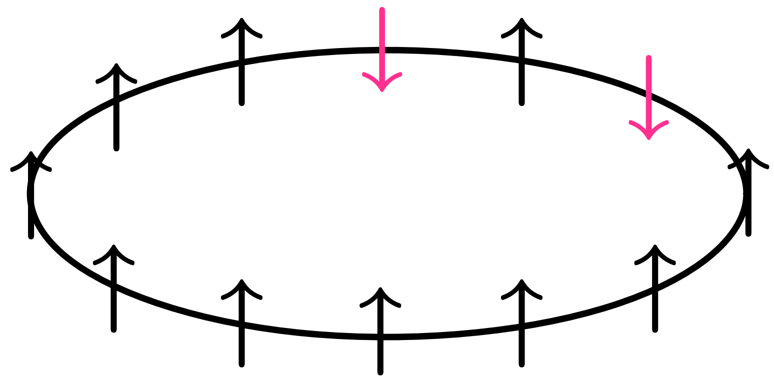
$$\mathrm{Tr}\, Z^L \quad \Leftrightarrow$$



ferromagnetic vacuum

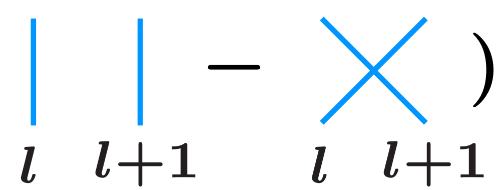
$$X = \Phi_1 + i\Phi_2$$
$$Z = \Phi_5 + i\Phi_6$$

$$\mathrm{Tr}(ZZZXZXZZ\cdots) \quad \Leftrightarrow$$



XXX Heisenberg Spin chain

$$H = \sum_{l=1}^{L} (\quad)$$



Bethe Ansatz Equation

One-magnon States

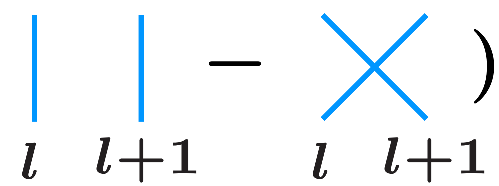
$$|\Psi(p)\rangle = \sum_{l=1}^{L} \psi(l)|\uparrow \cdots \uparrow \underset{}{\overset{l}{\downarrow}} \uparrow \cdots \uparrow\rangle$$

$$\psi(l) = e^{ipl}$$

Schrödinger Eq.

$$H|\Psi\rangle = E|\Psi\rangle$$

$$H = \sum_{l=1}^{L} (\quad - \quad)$$



$$\begin{aligned} E &= 2 - e^{ip} - e^{-ip} \\ &= 4\sin^2\frac{p}{2} \end{aligned}$$

: Dispersion Relation

Two-magnon States

$$|\Psi(p_1,p_2)\rangle = \sum_{1\le l_1<l_2\le L} \psi(l_1,l_2)|\uparrow\cdots\uparrow\underset{l_1}{\downarrow}\uparrow\cdots\uparrow\underset{l_2}{\downarrow}\uparrow\cdots\uparrow\rangle$$

Schrödinger Eq. $\quad H|\Psi\rangle = E|\Psi\rangle$

$$E = \sum_{k=1}^{2} 4\sin^2\frac{p_k}{2}$$

(Dispersion)

$$\psi(l_1,l_2) = e^{ip_1 l_1 + ip_2 l_2} + S(p_2,p_1)e^{ip_1 l_2 + ip_2 l_1}$$

(Bethe's Ansatz)

$$S(p_1,p_2) = -\frac{e^{ip_1+ip_2} - e^{2ip_1} + 1}{e^{ip_1+ip_2} - e^{2ip_2} + 1}$$

S-matrix

Factorized scattering

$$\psi(p_2, p_1, p_3, \ldots, p_J) = S(p_1, p_2)\psi(p_1, p_2, p_3, \ldots, p_J)$$

Periodic boundary condition

$$\psi(p_2, \ldots, p_J, p_1) = e^{-ip_1 L}\psi(p_1, \ldots, p_J)$$

Yang equations

$$e^{ip_k L} = \prod_{l \neq k}^{J} S(p_k, p_l)$$

rapidity variable $u = \frac{1}{2}\cot\frac{p}{2}$

Bethe Ansatz Equations

$$\left(\frac{u_k + \frac{i}{2}}{u_k - \frac{i}{2}}\right)^L = \prod_{l \neq k}^{J} \frac{u_k - u_l + i}{u_k - u_l - i} \qquad (k = 1, \ldots, J)$$

Local Charges

Momentum

$$P = Q_1 = \sum_k \frac{1}{i} \ln \frac{u_k + \frac{i}{2}}{u_k - \frac{i}{2}}$$

Energy

$$E = Q_2 = \sum_k \left(\frac{i}{u_k + \frac{i}{2}} - \frac{i}{u_k - \frac{i}{2}} \right)$$

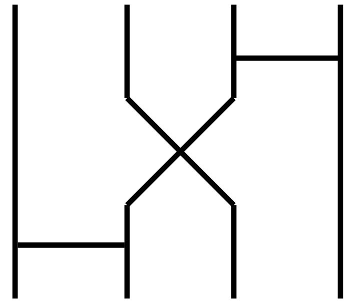
Higher charges

$$Q_r = \sum_k \frac{i}{r-1} \left(\frac{i}{(u_k + \frac{i}{2})^{r-1}} - \frac{i}{(u_k - \frac{i}{2})^{r-1}} \right)$$

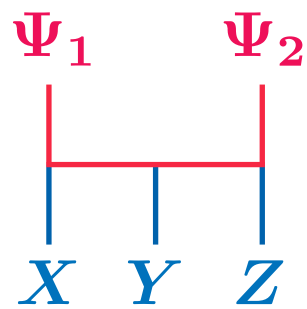
Higher loops

Difficulties:

- long-range interaction



- fluctuation in length



Attempts:

- closed subsector without change of length

  $\mathfrak{su}(2)$ (Beisert-Dippel-Staudacher '04)

  $\mathfrak{su}(1|1)$ (Staudacher '04) etc.

- relationship with known models

  Inozemtsev chain (Serban-Staudacher '04) (start deviating at 3 loops)

  Hubbard model (Rej-Serban-Staudacher '05)

Particle model (Excitation picture)

(Berenstein-Maldacena-Nastase '02)

(Staudacher '04) (Beisert '05)

- Vacuum

$$|\mathbf{0}\rangle^{\mathbf{I}} := |\cdots ZZZZZZZZZZZZZZZZZZ\cdots\rangle$$

- Asymptotic state

$$|\boldsymbol{X_1 X_2}\rangle^{\mathbf{I}}$$

$$:= \sum_{\boldsymbol{n_1} \ll \boldsymbol{n_2}} e^{ip_1 \boldsymbol{n_1} + ip_2 \boldsymbol{n_2}} |\cdots ZZZ\underset{\uparrow n_1}{\boldsymbol{X_1}}ZZZ\cdots ZZZ\underset{\uparrow n_2}{\boldsymbol{X_2}}ZZZ\cdots\rangle$$

- Physical state: total momentum $\sum_j p_j = 0$

$$\left(\begin{aligned} \mathrm{Tr}|X\rangle^{\mathrm{I}} &= \mathrm{Tr}\sum_n e^{ipn}|\cdots ZZ\overset{n}{\downarrow}{X}ZZ\cdots\rangle \\ &= \sum_n e^{ipn}\mathrm{Tr}|XZZ\cdots\rangle = \delta(p)\mathrm{Tr}|XZZ\cdots\rangle \end{aligned}\right)$$

- One particle state

$$|X\rangle^{\mathrm{I}} = \sum_n e^{ipn}|\cdots ZZ\overset{n}{\downarrow}{X}ZZ\cdots\rangle$$

$$|Z^{+}X\rangle^{\mathrm{I}} = \sum_n e^{ipn}|\cdots ZZZ\overset{n+1}{\downarrow}{X}ZZ\cdots\rangle$$

$$|XZ^{+}\rangle^{\mathrm{I}} = \sum_n e^{ipn}|\cdots ZZ\overset{n}{\downarrow}{X}ZZZ\cdots\rangle$$

$$\boxed{|Z^{\pm}X\rangle^{\mathrm{I}} = e^{\mp ip}|XZ^{\pm}\rangle^{\mathrm{I}}}$$

- One particle state: 8 bosons + 8 fermions

$$\Phi_{i\ (i=1,\dots,4)},\quad D_i Z_{\ (i=1,\dots,4)},\quad \Psi_{\alpha\dot{a}}, \Psi_{a\dot{\alpha}\ (a,\alpha=1,\dots,2)}$$: single excitation of Z

$\bar{Z}, F_{\alpha\beta}, D_i\Phi_j, \dots$: multiple excitation

- Vacuum breaks the global symmetry

$$PSU(2,2|4) \rightarrow PSU(2|2) \times PSU(2|2) \ltimes \mathbb{R}$$
$$(8|8) = (2|2) \times (2|2)$$

Z	ϕ_1	ϕ_2	ψ_1	ψ_2
$\bar{\phi}_1$	Φ_{11}	Φ_{12}	Ψ_{11}	Ψ_{12}
$\bar{\phi}_2$	Φ_{21}	Φ_{22}	Ψ_{21}	Ψ_{22}
$\bar{\psi}_1$	$\dot{\Psi}_{11}$	$\dot{\Psi}_{12}$	$D_{11}Z$	$D_{12}Z$
$\bar{\psi}_2$	$\dot{\Psi}_{21}$	$\dot{\Psi}_{22}$	$D_{21}Z$	$D_{22}Z$

- $\mathfrak{su}(2|2)$ algebra

$$
\begin{aligned}
[R^a{}_b, J^c] &= \delta^c_b J^a - \tfrac{1}{2}\delta^a_b J^c \\
[L^\alpha{}_\beta, J^\gamma] &= \delta^\gamma_\beta J^\alpha - \tfrac{1}{2}\delta^\alpha_\beta J^\gamma \\
\{Q^\alpha{}_a, S^b{}_\beta\} &= \delta^b_a L^\alpha{}_\beta + \delta^\alpha_\beta R^b{}_a + \delta^b_a \delta^\alpha_\beta C
\end{aligned}
$$

- Transformation of one-particle states: $(2|2)$ rep.

$$
\begin{aligned}
Q^\alpha{}_a |\phi^b\rangle^{\mathrm{I}} &= a\, \delta^b_a |\psi^\alpha\rangle^{\mathrm{I}} \\
Q^\alpha{}_a |\psi^\beta\rangle^{\mathrm{I}} &= b\, \epsilon^{\alpha\beta}\epsilon_{ab} |\phi^b Z^+\rangle^{\mathrm{I}} \\
S^a{}_\alpha |\phi^b\rangle^{\mathrm{I}} &= c\, \epsilon^{ab}\epsilon_{\alpha\beta} |\psi^\beta Z^-\rangle^{\mathrm{I}} \\
S^a{}_\alpha |\psi^\beta\rangle^{\mathrm{I}} &= d\, \delta^\beta_\alpha |\phi^a Z^-\rangle^{\mathrm{I}}
\end{aligned}
$$

$\Rightarrow$

$$
\boxed{C^2 = \frac{1}{4}}
$$

$$
C = \tfrac{1}{2} n_{\text{particle}} + \tfrac{1}{2} E
$$

- Central extension of $\mathfrak{su}(2|2)$ algebra

$$
\begin{aligned}
[R^a{}_b, J^c] &= \delta^c_b J^a - \tfrac{1}{2}\delta^a_b J^c \\
[L^\alpha{}_\beta, J^\gamma] &= \delta^\gamma_\beta J^\alpha - \tfrac{1}{2}\delta^\alpha_\beta J^\gamma \\
\{Q^\alpha{}_a, S^b{}_\beta\} &= \delta^b_a L^\alpha{}_\beta + \delta^\alpha_\beta R^b{}_a + \delta^b_a \delta^\alpha_\beta C \\
\{Q^\alpha{}_a, Q^\beta{}_b\} &= \epsilon^{\alpha\beta}\epsilon_{ab} P \\
\{S^a{}_\alpha, S^b{}_\beta\} &= \epsilon^{ab}\epsilon_{\alpha\beta} K
\end{aligned}
$$

- Transformation of one-particle states: $(2|2)$ rep.

$$
\begin{aligned}
Q^\alpha{}_a |\phi^b\rangle^{\mathrm{I}} &= a\, \delta^b_a |\psi^\alpha\rangle^{\mathrm{I}} \\
Q^\alpha{}_a |\psi^\beta\rangle^{\mathrm{I}} &= b\, \epsilon^{\alpha\beta}\epsilon_{ab} |\phi^b Z^+\rangle^{\mathrm{I}} \\
S^a{}_\alpha |\phi^b\rangle^{\mathrm{I}} &= c\, \epsilon^{ab}\epsilon_{\alpha\beta} |\psi^\beta Z^-\rangle^{\mathrm{I}} \\
S^a{}_\alpha |\psi^\beta\rangle^{\mathrm{I}} &= d\, \delta^\beta_\alpha |\phi^a Z^-\rangle^{\mathrm{I}}
\end{aligned}
$$

$\Rightarrow$ $$\boxed{C^2 - PK = \frac{1}{4}}$$

$$C = \tfrac{1}{2} n_{\text{particle}} + \tfrac{1}{2} E$$

$$P|X\rangle^{\mathrm{I}} = \alpha|Z^+X\rangle^{\mathrm{I}} - \alpha|XZ^+\rangle^{\mathrm{I}} = \alpha(e^{-ip}-1)|XZ^+\rangle^{\mathrm{I}}$$
$$K|X\rangle^{\mathrm{I}} = \beta|Z^-X\rangle^{\mathrm{I}} - \beta|XZ^-\rangle^{\mathrm{I}} = \beta(e^{+ip}-1)|XZ^-\rangle^{\mathrm{I}}$$

$$E = \sqrt{1 + 16\alpha\beta\sin^2(\tfrac{p}{2})} - 1$$

$\alpha\beta$: some function of λ

- From perturbative computation up to 3 loops (small λ) (Minahan-Zarembo '02) (Beisert-Staudacher '03) etc. and BMN energy formula (large λ), (Berenstein-Maldacena-Nastase '02)

$$E = \sqrt{1 + \frac{\lambda}{\pi^2}\sin^2\left(\frac{p}{2}\right)} - 1$$

(dispersion relation for a 'magnon')

(Beisert-Dippel-Staudacher '04)

$\mathfrak{su}(2|2)$ S-matrix

(Beisert '05)

- most general ansatz:

$$
\begin{aligned}
\mathcal{S}_{12}|\phi_1^a\phi_2^b\rangle^{\mathrm{I}} &= A_{12}|\phi_2^{\{a}\phi_1^{b\}}\rangle^{\mathrm{I}} + B_{12}|\phi_2^{[a}\phi_1^{b]}\rangle^{\mathrm{I}} + \tfrac{1}{2}C_{12}\epsilon^{ab}\epsilon_{\alpha\beta}|\psi_2^\alpha\psi_1^\beta Z^-\rangle^{\mathrm{I}} \\
\mathcal{S}_{12}|\psi_1^\alpha\psi_2^\beta\rangle^{\mathrm{I}} &= D_{12}|\psi_2^{\{\alpha}\psi_1^{\beta\}}\rangle^{\mathrm{I}} + E_{12}|\psi_2^{[\alpha}\psi_1^{\beta]}\rangle^{\mathrm{I}} + \tfrac{1}{2}F_{12}\epsilon^{\alpha\beta}\epsilon_{ab}|\phi_2^a\phi_1^b Z^+\rangle^{\mathrm{I}} \\
\mathcal{S}_{12}|\phi_1^a\psi_2^\beta\rangle^{\mathrm{I}} &= G_{12}|\psi_2^\beta\phi_1^a\rangle^{\mathrm{I}} + H_{12}|\phi_2^a\psi_1^\beta\rangle^{\mathrm{I}} \\
\mathcal{S}_{12}|\psi_1^\alpha\phi_2^b\rangle^{\mathrm{I}} &= K_{12}|\psi_2^\alpha\phi_1^b\rangle^{\mathrm{I}} + L_{12}|\phi_2^b\psi_1^\alpha\rangle^{\mathrm{I}}
\end{aligned}
$$

- $PSU(2|2) \ltimes \mathbb{R}^3$ symmetry

$\Longrightarrow$ Coefficients $A_{12}(p_1, p_2), \ldots, L_{12}(p_1, p_2)$

are uniquely determined up to an overall factor

Properties of the S-matrix

- Unitarity

$$\mathcal{S}_{21}\mathcal{S}_{21} = \mathcal{I}$$

- Associativity (Yang-Baxter equation)

$$\mathcal{S}_{12}\mathcal{S}_{13}\mathcal{S}_{23} = \mathcal{S}_{23}\mathcal{S}_{13}\mathcal{S}_{12}$$

- Not of difference form

$$\mathcal{S}_{12}(\cancel{u_1 - u_2})$$

- Similarity to the Shastry's R-matrix

  for the Hubbard model

- The whole S-matrix

$$\mathcal{S}_{\mathfrak{psu}(2,2|4)} = S_0 \left[\mathcal{S}_{\mathfrak{su}(2|2)} \otimes \mathcal{S}_{\mathfrak{su}(2|2)}\right]$$

with an overall scalar factor

$$S_0 = 1 + \mathcal{O}(\lambda^3) \qquad \text{(for the gauge theory)}$$

- Asymptotic Bethe ansatz equations (finite 'length' J)

Impose periodic boundary condition

Yang equations: $$e^{ip_j J} = \prod_{k \neq j}^{K} \mathcal{S}(p_j, p_k)$$

Diagonalize these equations by the nested Bethe ansatz

The S-matrix reproduce long range $\mathfrak{psu}(2,2|4)$ Bethe ansäze

(Beisert-Staudacher '05)

Several checks up to 3 loops, at most valid up to $\mathcal{O}(\lambda^{J-2})$

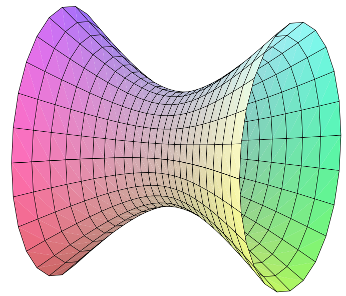
(Beisert-Kristjansen-Staudacher) (Eden-Jarczak-Sokatchev)

AdS/CFT Correspondence

$\mathcal{N} = 4 \ \mathrm{U}(N)$

Super Yang-Mills

IIB Superstrings on



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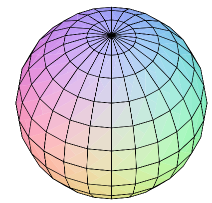


AdS_5 $\quad$ S^5

$$SO(4,2) \times SO(6)$$

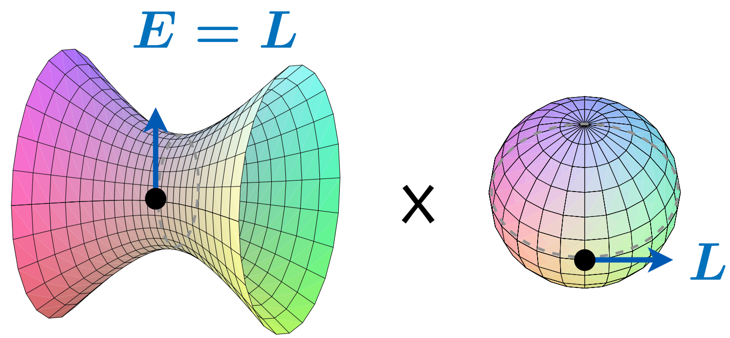
$$\lambda = g_{YM}^2 N \qquad R^4 = 4\pi g_s \alpha'^2 N$$

$$g_{YM}^2 = g_s$$

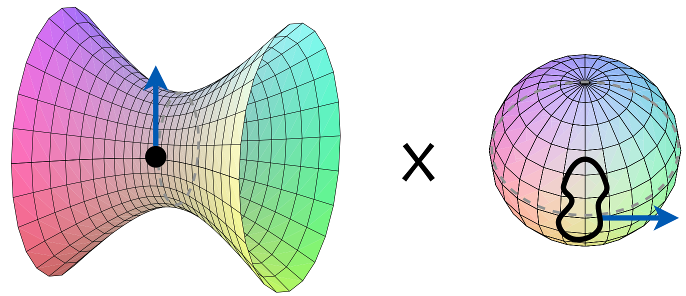
$$N \to \infty$$

$$4\pi\lambda = \frac{R^4}{\alpha'^2}$$

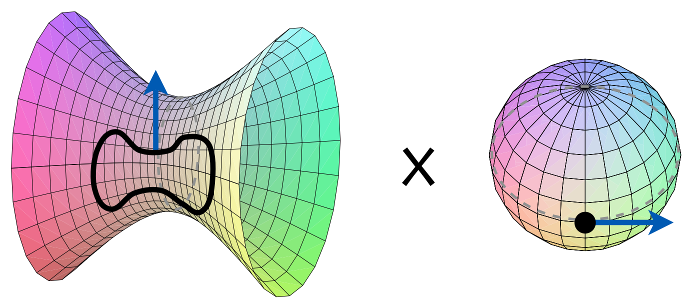
$$\mathcal{O} = \mathrm{Tr}(\overbrace{ZZZ\cdots ZZZ}^{L})$$



$$\mathcal{O} = \mathrm{Tr}(Z\cdots X\cdots \bar{Y}\cdots Z) + \cdots$$



$$\mathcal{O} = \mathrm{Tr}(Z\cdots \nabla^{s}Z\cdots \nabla^{s'}Z\cdots Z) + \cdots$$



Sigma model on $\mathbb{R} \times S^3$

$$S = \frac{\sqrt{\lambda}}{4\pi} \int d\sigma d\tau \left[-\partial_a X_0 \partial^a X_0 + \partial_a X_i \partial^a X_i + \Lambda \left(X_i X_i - 1 \right) \right]$$

$$(i = 1, \ldots, 4)$$

Equations of Motion

$$\partial_+ \partial_- X_i + (\partial_+ X_j \partial_- X_j) X_i = 0, \quad \partial_+ \partial_- X_0 = 0$$

Gauge: $X_0 = \kappa \tau$

$$\kappa = \frac{\Delta}{\sqrt{\lambda}}$$

Δ: energy of the string

$$\left(\Delta = \frac{\sqrt{\lambda}}{2\pi} \int_0^{2\pi} d\sigma \partial_\tau X_0 = \sqrt{\lambda} \kappa \right)$$

Virasoro Constraints

$$(\partial_\pm X_i)^2 = (\partial_\pm X_0)^2 = \kappa^2$$

SU(2) Principal Chiral Field Model

$$g \in \mathrm{SU}(2) \quad \leftrightarrow \quad \vec{X} \in S^3$$

$$g = \begin{pmatrix} X_1 + iX_2 & X_3 + iX_4 \\ -X_3 + iX_4 & X_1 - iX_2 \end{pmatrix}$$

Right current

$$j = -g^{-1}dg$$

$$d\,j - j \wedge j = 0, \qquad d * j = 0$$

Virasoro constraints

$$\frac{1}{2}\mathrm{Tr} j_{\pm}^2 = -\kappa^2$$

Lax Connection

$$a(x) = \frac{1}{1-x^2} j + \frac{x}{1-x^2} * j$$

x : spectral parameter

$$d\,j - j \wedge j = 0$$
$$d * j = 0$$

$$\Leftrightarrow \quad d\,a(x) - a(x) \wedge a(x) = 0$$

$$\Leftrightarrow \quad [\mathcal{L}(x), \mathcal{M}(x)] = 0$$

Lax pair

$$\mathcal{L}(x) = \partial_\sigma - a_\sigma(x) = \partial_\sigma - \frac{1}{2}\left(\frac{j_+}{1-x} - \frac{j_-}{1+x}\right)$$
$$\mathcal{M}(x) = \partial_\tau - a_\tau(x) = \partial_\tau - \frac{1}{2}\left(\frac{j_+}{1-x} + \frac{j_-}{1+x}\right)$$

Auxiliary Linear Problem

$$\begin{cases} \mathcal{L}(x)\Psi(x;\tau,\sigma) = 0 \\ \mathcal{M}(x)\Psi(x;\tau,\sigma) = 0 \end{cases} \qquad \begin{cases} \partial_\sigma \Psi = a_\sigma \Psi \\ \partial_\tau \Psi = a_\tau \Psi \end{cases}$$

$$\Psi(x;\tau,\sigma) = \mathrm{P}\exp \int_0^\sigma a_\sigma d\sigma$$

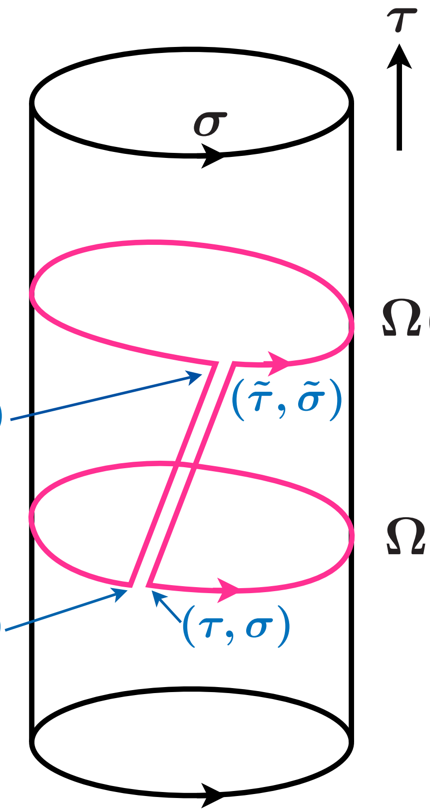
Monodromy matrix

$$\Psi(x;\tau,\sigma+2\pi) = \Omega(x;\tau,\sigma)\Psi(x;\tau,\sigma)$$

$$\Omega(x;\tau,\sigma) = \mathrm{P}\exp \int_0^{2\pi} a_\sigma d\sigma$$

Monodromy Matrix

$$\Omega(x;\tilde{\tau},\tilde{\sigma}) = U^{-1}\Omega(x;\tau,\sigma)U$$



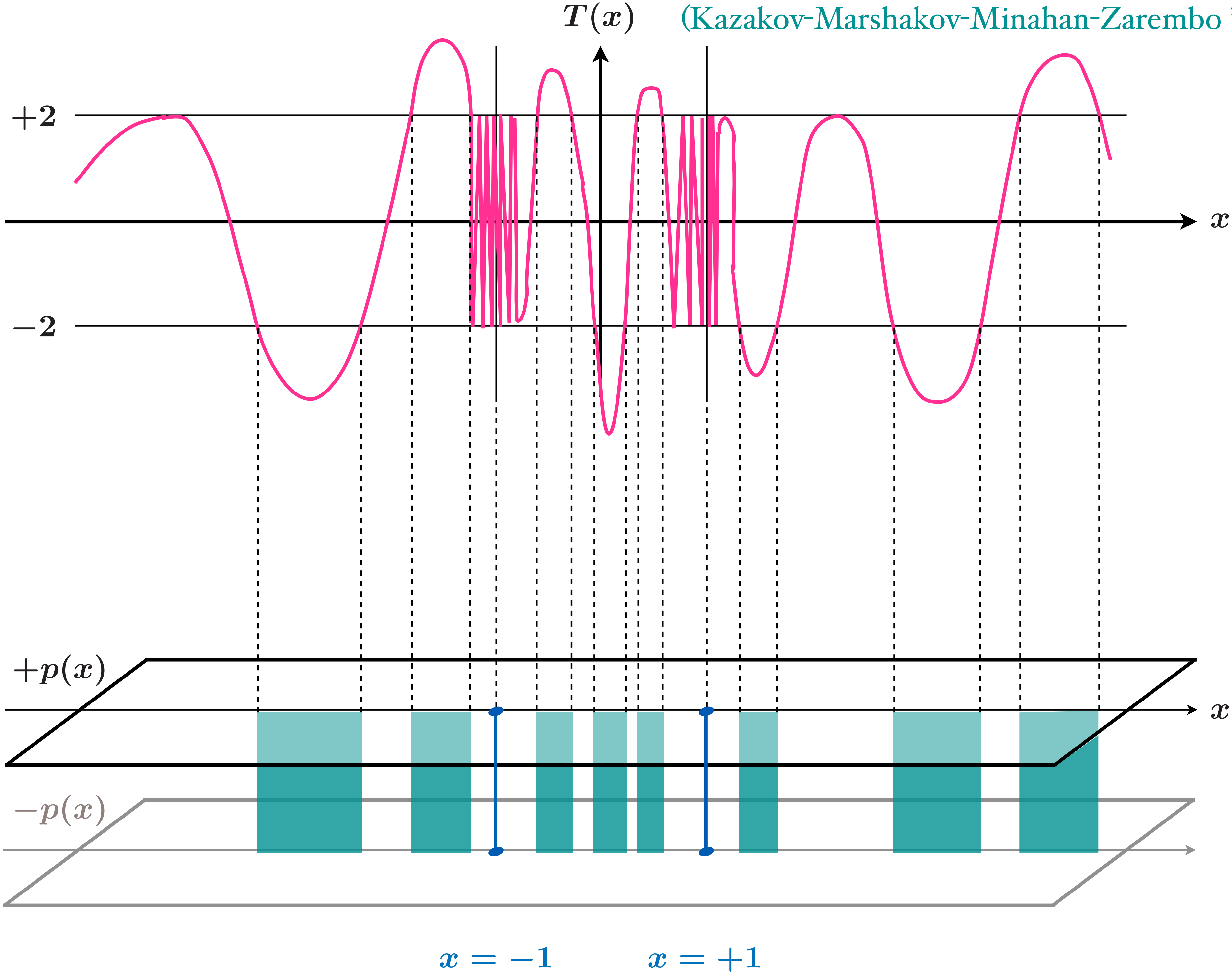
$$\Omega(x) \sim \begin{pmatrix} e^{ip(x)} & 0 \\ 0 & e^{-ip(x)} \end{pmatrix}$$

$p(x)$: quasi-momentum

$$T(x) := \mathrm{Tr}\,\Omega(x) = 2\cos p(x)$$

(transfer matrix eigenvalue)

(Kazakov-Marshakov-Minahan-Zarembo '04)



- Virasoro Constraints

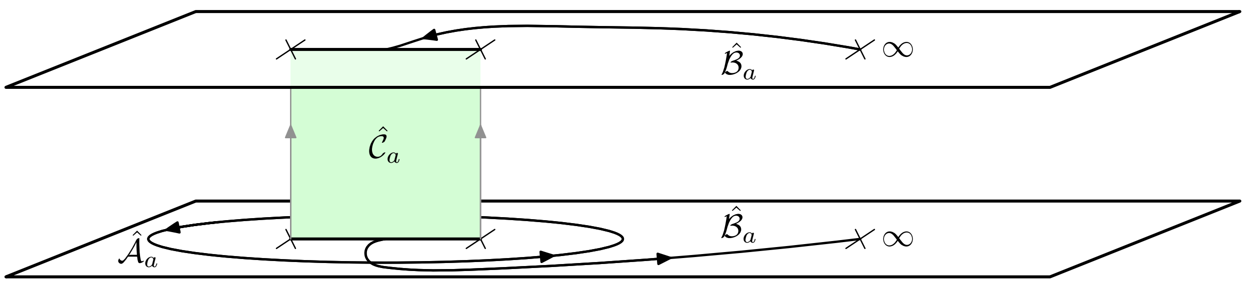
$$\frac{1}{2}\mathrm{Tr}j_{\pm}^2 = -\kappa^2 \quad \Longrightarrow \quad p(x) \sim -\frac{\pi\kappa}{x \mp 1} \qquad (x \to \pm 1)$$

- Single-valuedness

$$\oint dp = 2\pi\mathbb{Z}$$

$$\oint_{\hat{\mathcal{A}}_a} dp = 0, \qquad \int_{\hat{\mathcal{B}}_a} dp = 2\pi\hat{n}_a$$

$\hat{n}_a$: mode number



Explicit form of general finite gap solution

(Dorey-Vicedo '06)

$$X_1 + iX_2 = C_1 \frac{\theta(2\pi \int_{\infty^+}^{0^+} \vec{\omega} - \oint_{\vec{b}} dQ - \vec{D})}{\theta(\oint_{\vec{b}} dQ + \vec{D})} \exp\left(-i \int_{\infty^+}^{0^+} dQ\right)$$

$$X_3 + iX_4 = C_2 \frac{\theta(2\pi \int_{\infty^-}^{0^+} \vec{\omega} - \oint_{\vec{b}} dQ - \vec{D})}{\theta(\oint_{\vec{b}} dQ + \vec{D})} \exp\left(-i \int_{\infty^-}^{0^+} dQ\right)$$

$$\theta(\vec{z}) = \sum_{\vec{m}\in\mathbb{Z}^g} \exp\left(i\vec{m}\cdot\vec{z} + \pi i(\Pi\vec{m})\cdot\vec{m}\right)$$: Riemann theta function

$dQ = \sigma dp + \tau dq$ $\quad p$: quasi-momentum $\quad q$: quasi-energy

ω_j : normalized holomorphic differentials $\left(\oint_{\mathcal{A}_i} \omega_j = \delta_{ij}\right)$

$b_j = \mathcal{B}_j - \mathcal{B}_{g+1}$: closed B-cycles

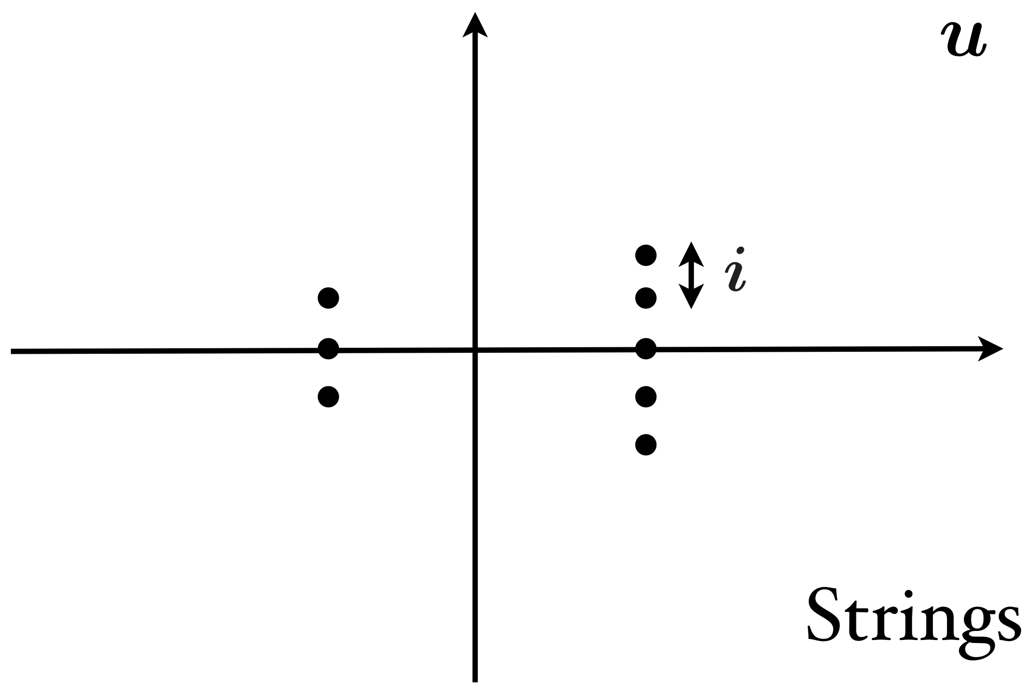
$\vec{D}, C_1, C_2$: constants

Finite gap solution on Yang-Mills side

- Traditional thermodynamic limit

$$\left(\frac{u_p + \frac{i}{2}}{u_p - \frac{i}{2}}\right)^L = \prod_{\substack{q=1 \\ q \neq p}}^{J} \frac{u_p - u_q + i}{u_p - u_q - i}$$

$$L \to \infty, \quad u_k \sim O(1)$$



- Novel thermodynamic limit

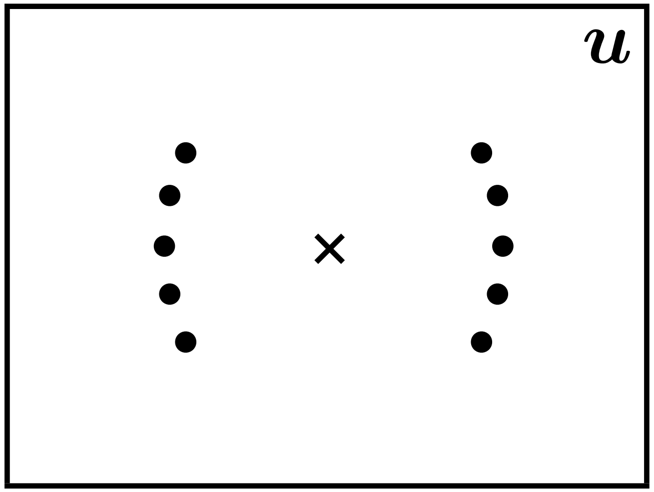
$$\left(\frac{u_p + \frac{i}{2}}{u_p - \frac{i}{2}}\right)^L = \prod_{\substack{q=1 \\ q \neq p}}^{J} \frac{u_p - u_q + i}{u_p - u_q - i}$$

$$L, J \to \infty, \qquad u_k \to L u_k$$

Log of both sides

$$\frac{1}{u_p} + 2\pi n_p = \frac{2}{L} \sum_{q \neq p}^{J} \frac{1}{u_p - u_q}$$

$$n_p \in \mathbb{Z}$$: mode number

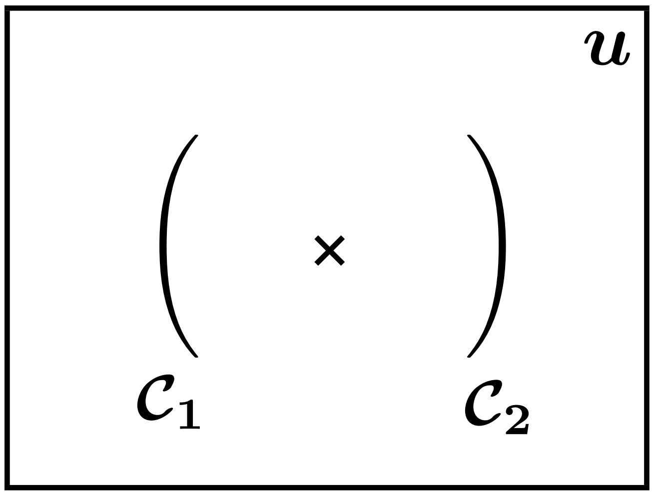


Resolvent

$$G(u) = \frac{1}{L}\sum_{q=1}^{J}\frac{1}{u-u_q}$$

$$\Downarrow$$

$$\Downarrow$$



$$G(u) = \int_{\mathcal{C}} \frac{dv\,\rho(v)}{u-v}$$

BAE

$$\frac{1}{u} + 2\pi n_a = 2\not{G}(u)$$

for $u \in \mathcal{C}_a$

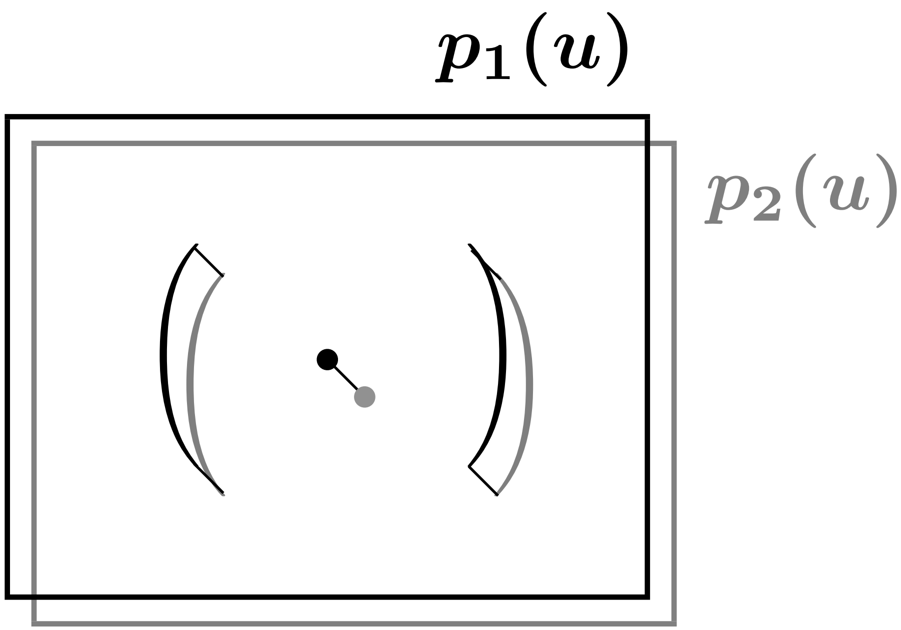
Quasi-momenta

$$p_1(u) = -p_2(u) = G(u) - \frac{1}{2u}$$

BAE

$$\frac{1}{u} + 2\pi n_a = 2\not{G}(u)$$

$$\Leftrightarrow \quad \boxed{p_1(u+i0) = p_2(u-i0) + 2\pi n_a \quad (u \in \mathcal{C}_a)}$$



hyper-elliptic curve

Comparison with Yang-Mills side

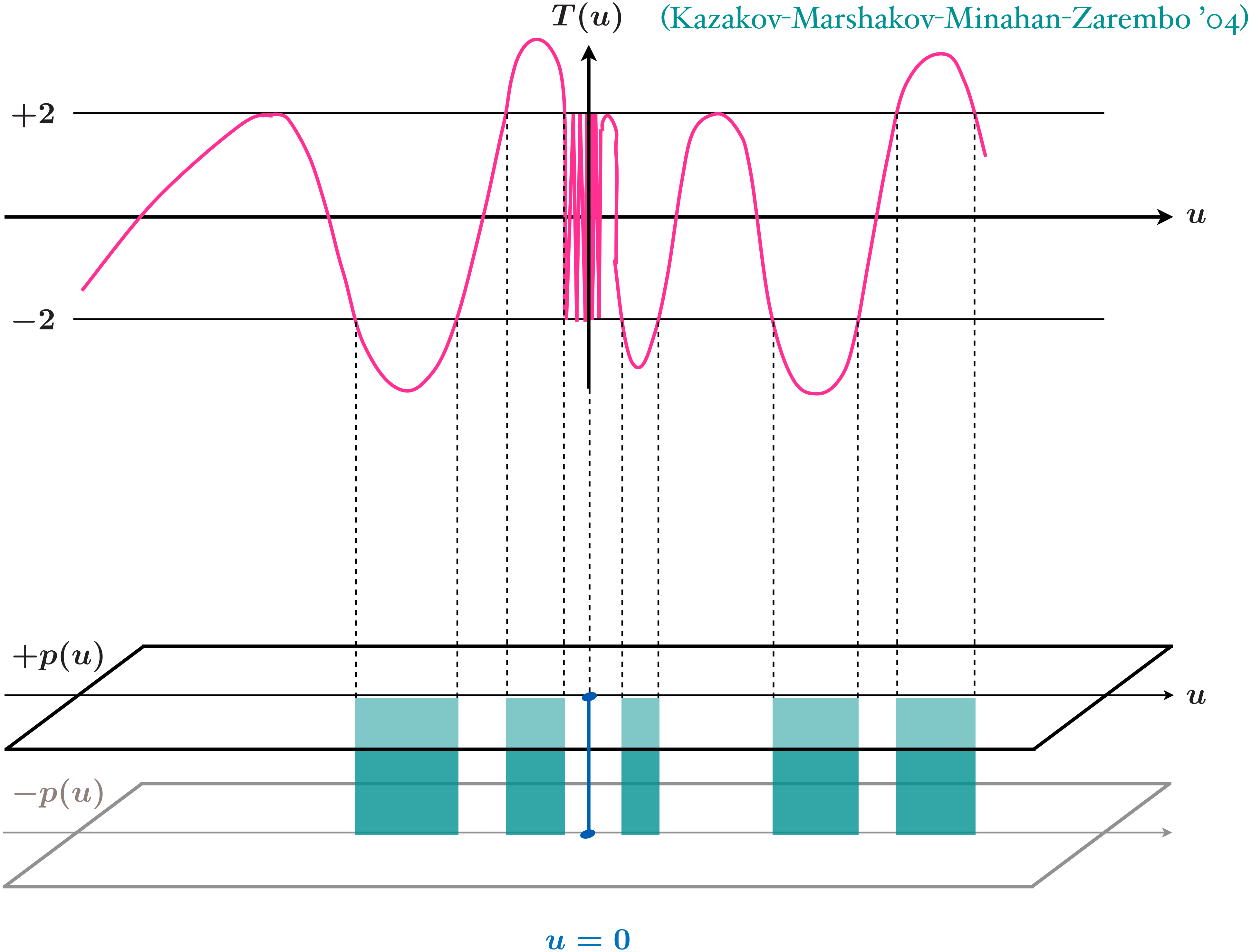
Frolov-Tseytlin limit: $$\frac{L}{\sqrt{\lambda}} \to \infty$$

with rescaling

$$u = \frac{\sqrt{\lambda}}{4\pi} x$$

Interior cuts $(-1 < x < 1)$
Poles $(x = \pm 1)$

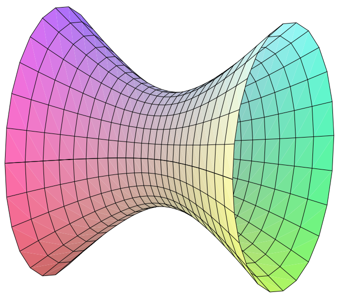
$\Rightarrow$ degenerate into the origin $u = 0$



T(u)
(Kazakov-Marshakov-Minahan-Zarembo '04)
+2
u
−2
+p(u)
u
−p(u)
u = 0

Classical Superstring on $AdS_5 \times S^5$

Classical IIB Superstrings on



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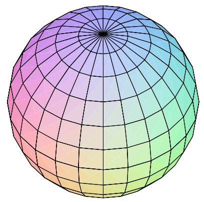


AdS_5 S^5

=

Coset sigma-model on

$$\frac{\mathbf{PSU(2,2|4)}}{\mathbf{Sp(1,1) \times Sp(2)}}$$

$$\begin{matrix} X^i(\sigma,\tau) \\ \psi_\alpha(\sigma,\tau) \end{matrix} \quad \longrightarrow \quad g(\sigma,\tau) \in \mathbf{PSU(2,2|4)}$$

$$J = -g^{-1}dg$$

decomposition w.r.t. $\mathbb{Z}_4$-grading

$$J = H + Q_1 + P + Q_2$$

Sigma-Model Action

(Metsaev-Tseytlin '98)
(Roiban-Siegel '02)

$$S_\sigma = \frac{\sqrt{\lambda}}{2\pi} \int \left(\tfrac{1}{2} \mathrm{str} P \wedge *P - \tfrac{1}{2} \mathrm{str} Q_1 \wedge Q_2 + \Lambda \wedge \mathrm{str} P \right)$$

Lax Connection

(Bena-Polchinski-Roiban '03)

$$\begin{aligned} A(z) \; = \; & H + \left(\tfrac{1}{2} z^2 + \tfrac{1}{2} z^{-2}\right) P \\ & + \left(-\tfrac{1}{2} z^2 + \tfrac{1}{2} z^{-2}\right) (*P - \Lambda) + z^{-1} Q_1 + z\, Q_2 \end{aligned}$$

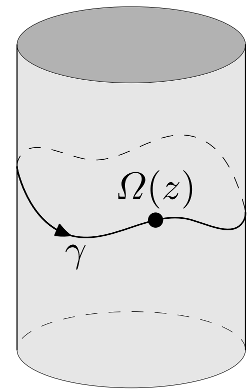
$\left\{ \begin{array}{l} \text{Bianchi Identity} \qquad dJ - J \wedge J = 0 \\ \text{Equation of Motion} \end{array} \right.$

$\Leftrightarrow$ Flatness Condition

$$dA(z) - A(z) \wedge A(z) = 0$$

Monodromy Matrix

$$\Omega(z) = \frac{\mathbf{P}\exp\int_0^{2\pi} d\sigma\, A(z)}{\mathbf{P}\exp\int_0^{2\pi} d\sigma\, A(1)}$$



Physical quantity: Conjugacy class of $\Omega(z)$

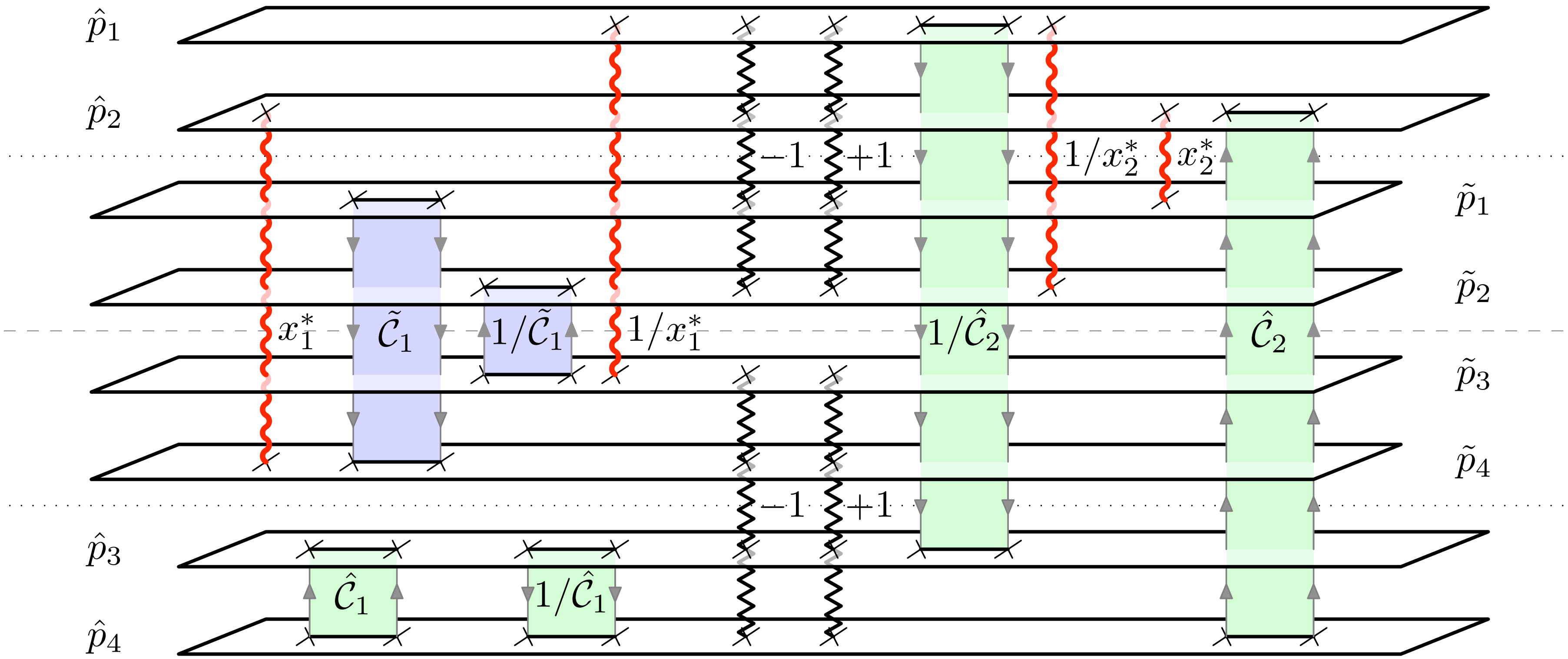
($\Rightarrow$ Generating functions of conserved charges)

Eigenvalues of the Monodromy Matrix

$$\begin{aligned}\Omega^{\mathrm{diag}}(z) &= u(z)\Omega(z)u(z)^{-1} \\ &= \mathrm{diag}(e^{i\tilde{p}_1}, e^{i\tilde{p}_2}, e^{i\tilde{p}_3}, e^{i\tilde{p}_4} | e^{i\hat{p}_1}, e^{i\hat{p}_2}, e^{i\hat{p}_3}, e^{i\hat{p}_4})\end{aligned}$$

$\tilde{p}_i(z), \hat{p}_i(z)$: quasi-momenta

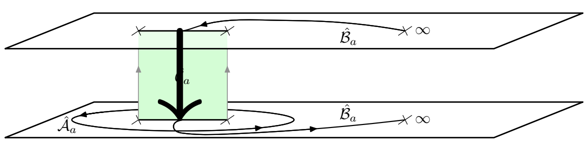
Spectral Curve for the Sigma-Model



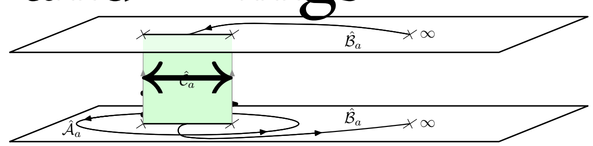
(Beisert-Kazakov-K.S.-Zarembo '05)

Distribution of cuts with

mode numbers



and fillings

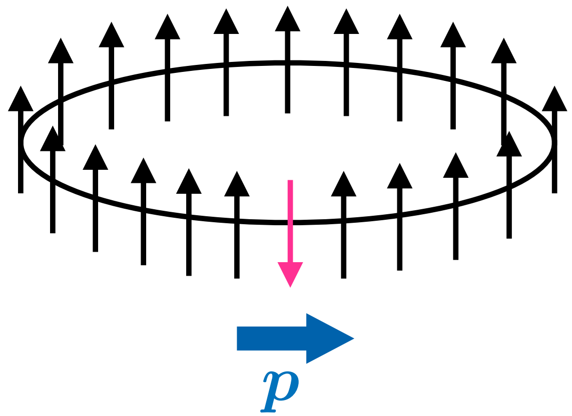


determines a classical solution

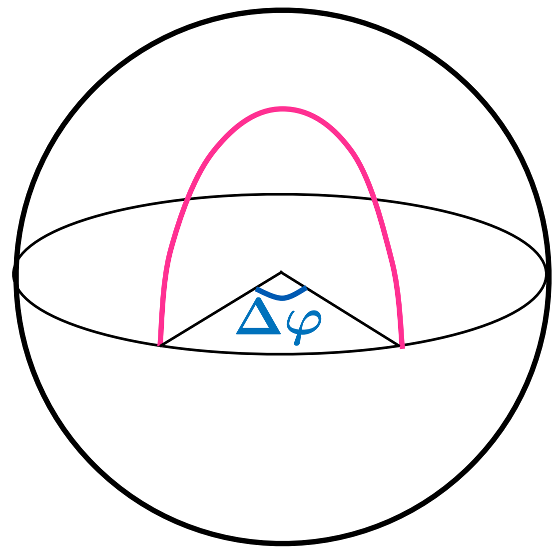
Giant Magnons

(Hofman-Maldacena '06)

(Okamura & Suzuki's talk)



$\Longleftrightarrow$

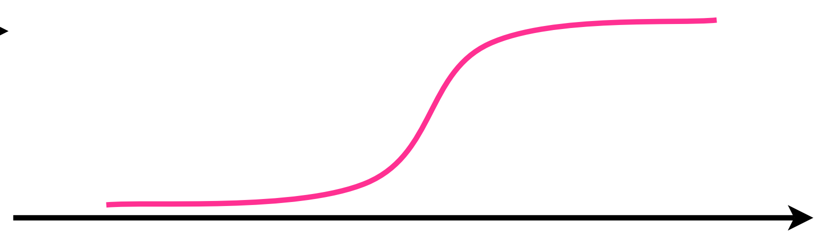


O(3)-sigma model

HM limit:

$$J \to \infty$$
$$E - J = \text{fixed}$$
$$\lambda = \text{fixed}$$
$$p = \text{fixed}$$

$\Longleftrightarrow$

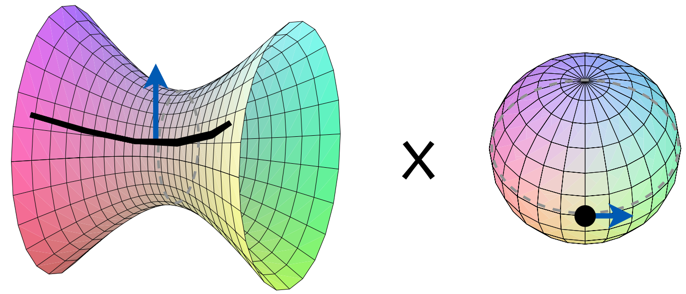


Sine-Gordon model

$$\left(\begin{array}{r} \text{BMN limit:} \\ \lambda \to \infty \\ n = pJ = \text{fixed} \end{array} \right)$$

Non compact sector and log S scaling for $S \gg J$

$$\mathcal{O} = \mathrm{Tr}(D^{s_1} Z D^{s_2} Z) \quad \Longleftrightarrow$$



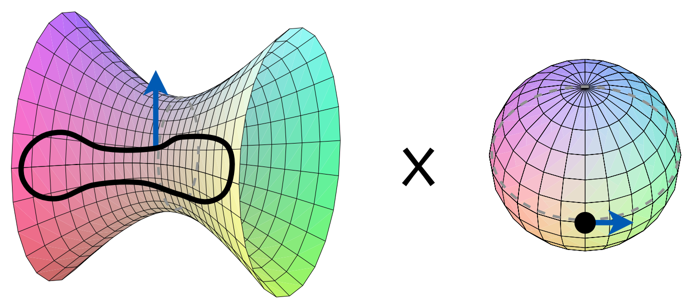
twist-two operator

folded string

(Gubser-Klebanov-Polyakov ’02)

- log S scaling is universal

$$\mathcal{O} = \mathrm{Tr}(D^{s_1} Z D^{s_2} Z \cdots D^{s_J} Z) \quad \Longleftrightarrow$$



$$\Delta - S \sim c\lambda \log S$$

(one-loop)

$$\Delta - S \sim c\sqrt{\lambda} \log S$$

(classical level)

(Callan-Gross ’73) (Korchemsky ’95) etc.

(K.S.-Satoh ’06)

Towards quantization of strings on $AdS_5 \times S^5$

- Green-Schwarz, pure spinor, ...

  several difficulties in conventional approach

Green-Schwarz string as an integrable particle model?

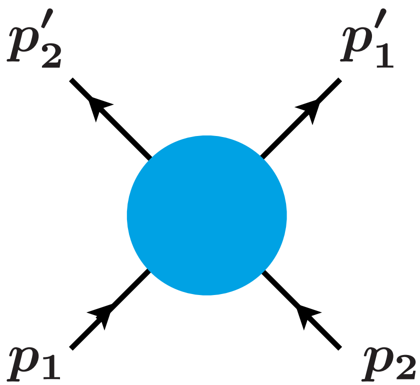
- Conformal gauge vs Uniform gauge

worldsheet Lorentz symmetry ($\Rightarrow$ crossing symmetry)	
relativistic	non-relativistic
global symmetry (no good subsector)	
unbroken	broken
Virasoro constraint	
???	easy

Relativistic particle model

O(4)-sigma model (SU(2) principal chiral field model)

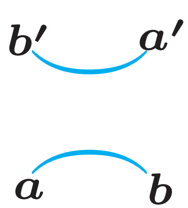
- Zamolodchikovs' S-matrix (Zamolodchikov-Zamolodchikov '77)



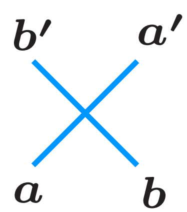
$$= \quad \delta(p_1 - p'_1)\delta(p_2 - p'_2)\hat{S}_{a\ b}^{a'b'}(\theta_1 - \theta_2)$$

$$\vec{p} = (m\cosh\pi\theta, m\sinh\pi\theta)$$

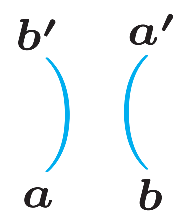
$$\hat{S}_{a\ b}^{a'b'}(\theta) = \sigma_1(\theta)$$



$$+ \sigma_2(\theta)$$



$$+ \sigma_3(\theta)$$



- Unitarity

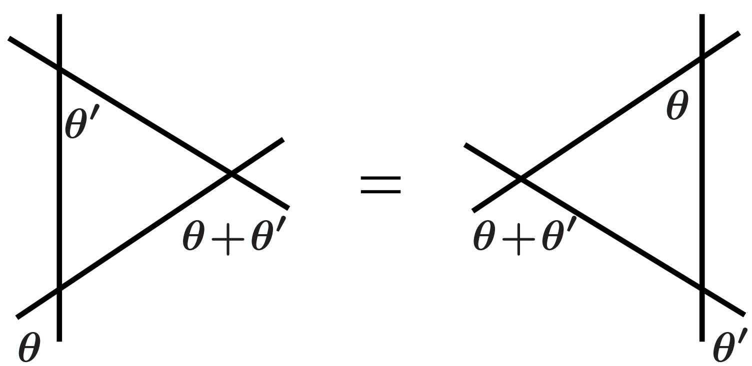
$$\hat{S}_{b_1 b_2}^{c_1 c_2}(\theta)\hat{S}_{a_1 a_2}^{b_1 b_2}(-\theta) = \hat{I}_{a_1 a_2}^{c_1 c_2}$$

- Crossing Symmetry

$$\hat{S}_{a\ b}^{a' b'}\big(i\pi - \theta\big) = \hat{S}_{a\ b'}^{a' b}(\theta)$$

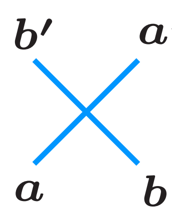
- Associativity

$$\hat{S}_{c_1 c_2}^{b_1 b_2}(\theta)\hat{S}_{a_1 c_3}^{c_1 b_3}(\theta+\theta')\hat{S}_{a_2 a_3}^{c_2 c_3}(\theta') = \hat{S}_{a_1 a_2}^{c_1 c_2}(\theta)\hat{S}_{c_1 a_3}^{b_1 c_3}(\theta+\theta')\hat{S}_{c_2 c_3}^{b_2 b_3}(\theta')$$

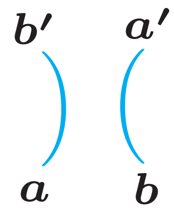


$\hat{S}(\theta)$ is constrained up to an overall factor (CDD ambiguity)

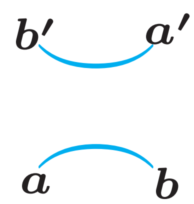
$$\hat{S}_{a\;b}^{a'b'}(\theta) = S_0(\theta)^2 \Bigg[$$



$$- \frac{i}{\theta}$$



$$- \frac{i}{i-\theta}$$



$$\Bigg]$$

Minimal Solution

$$S_0(\theta) = i\,\frac{\Gamma(-\frac{\theta}{2i})\Gamma(\frac{1}{2}+\frac{\theta}{2i})}{\Gamma(\frac{\theta}{2i})\Gamma(\frac{1}{2}-\frac{\theta}{2i})}$$

Yang equations

$$e^{-i\mu p(\theta_\alpha)} = \prod_{\alpha \neq \beta} \hat{S}(\theta_\alpha - \theta_\beta)$$

: Matrix equation

⇨ Can be diagonarized by nested Bethe ansatz

$$e^{-i\mu \sinh \pi\theta_\alpha} = \prod_{\beta \neq \alpha} S_0^2(\theta_\alpha - \theta_\beta) \prod_j \frac{\theta_\alpha - u_j + i/2}{\theta_\alpha - u_j - i/2} \prod_k \frac{\theta_\alpha - v_k + i/2}{\theta_\alpha - v_k - i/2}$$

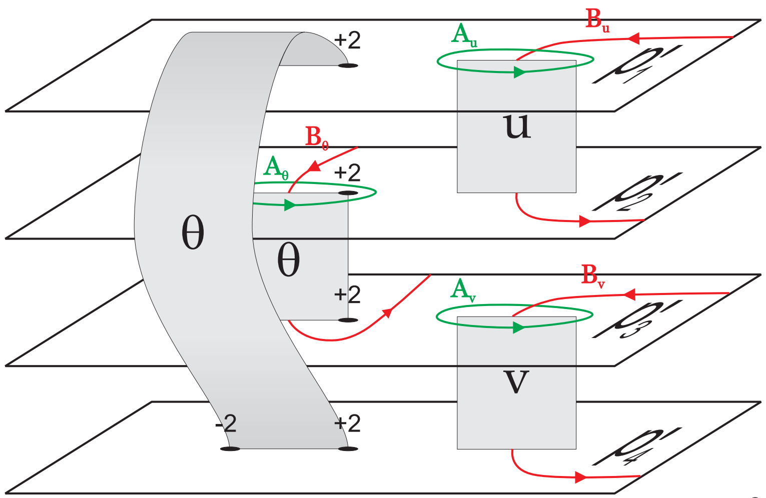
$$1 = \prod_\beta \frac{u_j - \theta_\beta - i/2}{u_j - \theta_\beta + i/2} \prod_{i \neq j} \frac{u_j - u_i + i}{u_j - u_i - i}$$

$$1 = \prod_\beta \frac{v_k - \theta_\beta - i/2}{v_k - \theta_\beta + i/2} \prod_{l \neq k} \frac{v_k - v_l + i}{v_k - v_l - i}$$

The above BAEs reproduce

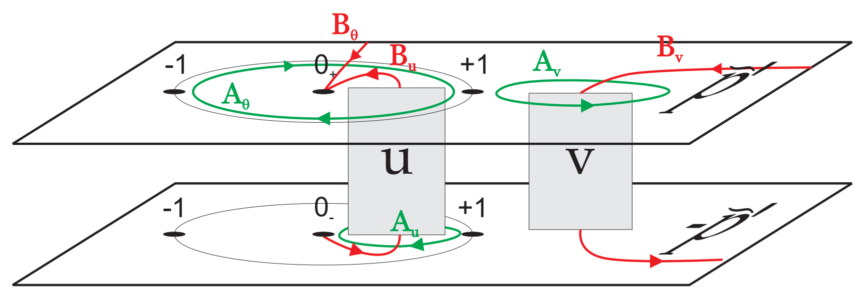
- Classical solution (Gromov-Kazakov-K.S.-Vieira ’06)
    - macroscopic number of particles (1-cut for θ)
    - macroscopic number of magnons
    - double scaling limit
    - eliminate θ

Single theta-cut solution



⟺

Classical solution (KMMZ)



$$z = x + \frac{1}{x}$$

The above BAEs reproduce

(Arutyunov-Frolov-Staudacher '04)

(Gromov-Kazakov '06)

- AFS 'string Bethe ansatz equations'
  - macroscopic number of particles (1-cut for θ)
  - ~~macroscopic number of~~ magnons
  - double scaling limit
  - eliminate θ

(Beisert-Dippel-Staudacher '04)

BDS all-loop Bethe Ansatz equations

+ dressing factor

introduced to repair the 3-loop discrepancy

Non-relativistic particle model

- AFS's dressing factor

$\Rightarrow \Rightarrow \Rightarrow$ overall factor of the string S-matrix

$$\mathcal{S}_{\mathfrak{psu}(2,2|4)} = S_0 \left[\mathcal{S}_{\mathfrak{su}(2|2)} \otimes \mathcal{S}_{\mathfrak{su}(2|2)}\right]$$

constrained by symmetry

small λ — large λ

$$S_0 = 1 + \mathcal{O}(\lambda^3) \overset{?}{=} S_0 = \exp\left(2i\sqrt{\lambda}\sum_{n=0}^{\infty}\theta_n(p_1,p_2)\left(\frac{1}{\sqrt{\lambda}}\right)^n\right)$$

(SYM perturbation)

(classical string) → $\theta_0(p_1,p_2)$ (Arutyunov-Frolov-Staudacher '04)

(worldsheet 1-loop) → $\theta_1(p_1,p_2)$ (Hernández-López '06)

(Janik '06) (crossing symmetry) → $\theta_n(p_1,p_2)$ (Beisert-Hernández-López '06)

Summary

- The spectral problem of the dilatation operator is fully solved at one-loop

- Magnon picture may solve the problem even at all loops for infinitely long operators

- General solutions of classical strings on the AdS background are available

- Integrable formulation of quantum strings is in progress

Prospects

- Mismatch between the both sides
- Finite length corrections, wrapping effects

Related subjects

- Integrability in $\mathcal{N} < 4$ theories
- Open spin chain
- Plane-wave matrix model, LLM background