Two-dimensional $\mathcal{N} = (2, 2)$ super Yang-Mills theory on computer

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It will be very exciting if non-perturbative questions in SUSY gauge theories can be studied numerically at one’s will!

- spontaneous SUSY breaking
- string/gauge correspondence
- test of various “solutions” (e.g., Seiberg-Witten)

\[
\{Q, Q^\dagger\} \sim P
\]

\textbf{SUSY vs lattice !}

\textbf{SUSY restores only in the continuum limit !}
Present status:

- For 4d $\mathcal{N} = 1$ SYM (gaugino condensation, degenerate vacua, Veneziano-Yankielowicz effective action, etc.), numerically promising formulation exists

- Even in this “simplest realistic” model, no conclusive evidence of SUSY has been observed

- Investigation of low-dimensional SUSY gauge theories (simpler UV structure) would thus be useful to test various ideas

- Kaplan et. al., Sugino, Catterall, Sapporo group...

- SUSY QM (16 SUSY charges!) $\leftrightarrow$ Takeuchi-kun
In this work, we carry out a (very preliminary) Monte Carlo study of Sugino’s lattice formulation of $2d \mathcal{N} = (2, 2)$ SYM (4 SUSY charges)

F. Sugino, JHEP 03 (2004) 067 [hep-lat/0401017]
Two-dimensional square lattice (size $L$)

$$\Lambda = \{ x \in a\mathbb{Z}^2 \mid 0 \leq x_\mu < L \}$$

The lattice action

$$S = Q a^2 \sum_{x \in \Lambda} \left( O_1(x) + O_2(x) + O_3(x) + \frac{1}{a^4 g^2} \text{tr} \{ \chi(x) H(x) \} \right),$$

where

$$O_1(x) = \frac{1}{a^4 g^2} \text{tr} \left\{ \frac{1}{4} \eta(x) [\phi(x), \bar{\phi}(x)] \right\}$$

$$O_2(x) = \frac{1}{a^4 g^2} \text{tr} \left\{ -i \chi(x) \hat{\Phi}_{\text{TL}}(x) \right\}$$

$$O_3(x) = \frac{1}{a^4 g^2} \text{tr} \left\{ i \sum_{\mu=0}^1 \psi_\mu(x) \left( \bar{\phi}(x) - U(x, \mu) \bar{\phi}(x + a\hat{\mu}) U(x, \mu)^{-1} \right) \right\}$$
A lattice counterpart of the BRST-like transformation \( Q \)

\[
Q U(x, \mu) = i\psi_\mu(x)U(x, \mu)
\]

\[
Q \psi_\mu(x) = i\psi_\mu(x)\psi_\mu(x) - i \left( \phi(x) - U(x, \mu)\phi(x + a\hat{\mu})U(x, \mu)^{-1} \right)
\]

\[
Q \phi(x) = 0
\]

\[
Q \chi(x) = H(x) \quad Q H(x) = [\phi(x), \chi(x)]
\]

\[
Q \bar{\phi}(x) = \eta(x) \quad Q \eta(x) = [\phi(x), \bar{\phi}(x)]
\]

\[
Q^2 = 0 \text{ on gauge invariant quantities}
\]

From this nilpotency, the lattice action is manifestly invariant under one of four super-transformations, \( Q \).
More explicitly

\[ S = a^2 \sum_{x \in \Lambda} \left( \sum_{i=1}^{3} \mathcal{L}_{B_i}(x) + \sum_{i=1}^{6} \mathcal{L}_{F_i}(x) + \frac{1}{a^4 g^2} \text{tr} \left\{ H(x) - \frac{1}{2} i \hat{\Phi}_{TL}(x) \right\} \right)^2 \]

where

\[ \mathcal{L}_{B_1}(x) = \frac{1}{a^4 g^2} \text{tr} \left\{ \frac{1}{4} [\phi(x), \phi(x)]^2 \right\} \]

\[ \mathcal{L}_{B_2}(x) = \frac{1}{a^4 g^2} \text{tr} \left\{ \frac{1}{4} \hat{\Phi}_{TL}(x)^2 \right\} \]

\[ \mathcal{L}_{B_3}(x) = \frac{1}{a^4 g^2} \text{tr} \left\{ \sum_{\mu=0}^{1} \left( \phi(x) - U(x, \mu) \phi(x + a\hat{\mu}) U(x, \mu)^{-1} \right) \right\} \]

\[ \times \left( \bar{\phi}(x) - U(x, \mu) \bar{\phi}(x + a\hat{\mu}) U(x, \mu)^{-1} \right) \]
and

\[ L_{F1}(x) = \frac{1}{a^4 g^2} \text{tr} \left\{ -\frac{1}{4} \eta(x)[\phi(x), \eta(x)] \right\} \]

\[ L_{F2}(x) = \frac{1}{a^4 g^2} \text{tr} \left\{ -\chi(x)[\phi(x), \chi(x)] \right\} \]

\[ L_{F3}(x) = \frac{1}{a^4 g^2} \text{tr} \left\{ -\psi_0(x)\psi_0(x) \left( \bar{\phi}(x) + U(x, 0)\bar{\phi}(x + a\hat{0})U(x, 0)^{-1} \right) \right\} \]

\[ L_{F4}(x) = \frac{1}{a^4 g^2} \text{tr} \left\{ -\psi_1(x)\psi_1(x) \left( \bar{\phi}(x) + U(x, 1)\bar{\phi}(x + a\hat{1})U(x, 1)^{-1} \right) \right\} \]

\[ L_{F5}(x) = \frac{1}{a^4 g^2} \text{tr} \left\{ i\chi(x)Q\Phi(x) \right\} \]

\[ L_{F6}(x) = \frac{1}{a^4 g^2} \text{tr} \left\{ -i \sum_{\mu=0}^{1} \psi_{\mu}(x) \left( \eta(x) - U(x, \mu)\eta(x + a\hat{\mu})U(x, \mu)^{-1} \right) \right\} \]
Advantage of this formulation

• $Q$-invariance (a part of the supersymmetry) is manifest even with finite lattice spacings and volume (probably, so far the unique formulation?)

• global $U(1)_R$ symmetry (this is a chiral symmetry!)

\[
\begin{align*}
U(x,\mu) & \rightarrow U(x,\mu) & \psi_\mu(x) & \rightarrow e^{i\alpha}\psi_\mu(x) \\
\phi(x) & \rightarrow e^{2i\alpha}\phi(x) \\
\chi(x) & \rightarrow e^{-i\alpha}\chi(x) \\
\overline{\phi}(x) & \rightarrow e^{-2i\alpha}\overline{\phi}(x) \\
H(x) & \rightarrow H(x) \\
\eta(x) & \rightarrow e^{-i\alpha}\eta(x)
\end{align*}
\]

is also manifest
Possible disadvantage of the formulation

- The pfaffian $\text{Pf}\{iD\}$ resulting from the integration of fermionic variables is generally a complex number (lattice artifact)

- would imply the sign (or phase) problem in Monte Carlo simulation

Continuum limit:

\[ a \to 0, \text{ while } g \text{ and } L \text{ are kept fixed} \]

It can be argued that the full SUSY of the 1PI effective action for elementary fields is restored in this limit.

- Power counting

- Scalar mass terms are the only source of SUSY breaking

  \[ \iff \text{super-renormalizability} \]

- Exact \( Q \)-invariance forbids the mass terms
Monte Carlo study ($SU(2)$ only)
For SUSY, quantum effect of fermions is vital!

Quenched approximation ($S_B$ bosonic action)

$$\langle \mathcal{O} \rangle = \frac{\int d\mu_B \mathcal{O} e^{-S_B}}{\int d\mu_B e^{-S_B}}$$

is meaningless, though it provides a useful standard.

Here we adopt the re-weighting method

$$\langle \langle \mathcal{O} \rangle \rangle = \frac{\int d\mu \mathcal{O} e^{-S}}{\int d\mu e^{-S}} = \frac{\langle \mathcal{O} \text{Pf}\{iD\} \rangle}{\langle \text{Pf}\{iD\} \rangle}$$

(potential overlap problem)
We developed a hybrid Monte Carlo algorithm code for the action $S_B$ by using a C++ library, FermiQCD/MDP

For each configuration, we compute the inverse (i.e., fermion propagator) and the determinant of the lattice Dirac operator $iD$ by using the LU decomposition

Expressing the determinant of the Dirac operator as

$$\det\{iD\} = re^{i\theta}, \quad -\pi < \theta \leq \pi$$

(the complex phase is lattice artifact) we define

$$\text{Pf}\{iD\} = \sqrt{r} e^{i\theta/2}, \quad \therefore (\text{Pf}\{iD\})^2 = \det\{iD\}$$

However, with this prescription, the sign may be wrong
To estimate the systematic error introduced with this, we compute also the phase-quenched average

$$\langle\langle O \rangle\rangle_{\text{phase-quenched}} = \frac{\langle O | Pf\{iD\} \rangle}{\langle | Pf\{iD\} \rangle}$$
Parameters in our Monte Carlo study \((\beta = 2N_c/(a^2g^2))\)

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<td>9.0</td>
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<td>0.666666</td>
<td>0.8</td>
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This sequence corresponds to the fixed physical lattice size \(L_g = 4.0\)

For each value of \(\beta\), we stored 1000–10000 independent configurations extracted from 10^6 trajectories of the molecular dynamics

Statistical error is estimated by the jackknife analysis

(The constant \(\epsilon\) for the admissibility is fixed to be \(\epsilon = 2.6\))
One-point SUSY Ward-Takahashi identities
Since the action is $Q$-exact, we have $\langle \langle S \rangle \rangle = 0$, or
\[
\sum_{i=1}^{3} \langle \langle L_{Bi}(x) \rangle \rangle + \sum_{i=1}^{6} \langle \langle L_{Fi}(x) \rangle \rangle + \frac{1}{a^4 g^2} \left\langle \left\langle \text{tr} \left\{ H(x) - \frac{1}{2} i \hat{\Phi}_{TL}(x) \right\}^2 \right\rangle \right\rangle = 0
\]
but
\[
\sum_{i=1}^{6} \langle \langle L_{Fi}(x) \rangle \rangle = -2(N_c^2 - 1) \frac{1}{a^2}
\]
and
\[
\frac{1}{a^4 g^2} \left\langle \left\langle \text{tr} \left\{ H(x) - \frac{1}{2} i \hat{\Phi}_{TL}(x) \right\}^2 \right\rangle \right\rangle = \frac{1}{2}(N_c^2 - 1) \frac{1}{a^2}
\]
Thus
\[
\sum_{i=1}^{3} \langle \langle L_{Bi}(x) \rangle \rangle - \frac{3}{2} (N_c^2 - 1) \frac{1}{a^2} = 0
\]
Expectation values of $\sum_{i=1}^{3} \mathcal{L}_{B_i}(x) - \frac{3}{2}(N_c^2 - 1) \frac{1}{a^2}$
• The real part is consistent with the expected identity within $1.5\sigma$ ($\Rightarrow$ strongly supports the correctness of our code/algorithm)

• The imaginary part is consistent with zero

• No notable difference of the phase-quenched average ($\Rightarrow$ systematic error due to wrong-sign determination is negligible)

• Clear distinction from the quenched average ($\Rightarrow$ effect of dynamical fermions is properly included)

• Effect of quenching starts at 2-loop $\sim g^2 \ln(a/L)$
Another exact relation

\[ \langle QO_1(x) \rangle = \langle L_{B1}(x) \rangle + \langle L_{F1}(x) \rangle = 0 \]
Expectation values of $\mathcal{L}_{B_1}(x) + \mathcal{L}_{F_1}(x)$
• The relation is confirmed within $2\sigma$ (note the difference in scale of vertical axis compared to the previous figure)

• The quenched average is certainly inconsistent with the SUSY relation

• No clear separation between the re-weighted average and the quenched one ($\leftarrow$ The effect of quenching starts at 3-loop $\sim a^2g^4 \ln(a/L)$)
Another relation

\[ \langle \langle Q\mathcal{O}_2(x) \rangle \rangle = \frac{1}{a^4 g^2} \left\langle \left. \text{tr} \left\{ -iH(x)\Phi_{\text{TL}}(x) \right\} \right\rangle + \langle \langle \mathcal{L}_{F5}(x) \rangle \rangle = 0 \]

but

\[ H(x) = \frac{1}{2} i\Phi_{\text{TL}}(x) \]

and thus

\[ 2 \langle \langle \mathcal{L}_{B2}(x) \rangle \rangle + \langle \langle \mathcal{L}_{F5}(x) \rangle \rangle = 0 \]
Expectation values of $2\mathcal{L}_{B2}(x) + \mathcal{L}_{F5}(x)$
The situation is again similar with the last piece of the relation

\[ \langle \langle \mathcal{Q}_3(x) \rangle \rangle = \langle \langle \mathcal{L}_{B3}(x) \rangle \rangle + \langle \langle \mathcal{L}_{F3}(x) \rangle \rangle + \langle \langle \mathcal{L}_{F4}(x) \rangle \rangle + \langle \langle \mathcal{L}_{F6}(x) \rangle \rangle = 0 \]
Expectation values of $\mathcal{L}_{B3}(x) + \mathcal{L}_{F3}(x) + \mathcal{L}_{F4}(x) + \mathcal{L}_{F6}(x)$
So far, we have observed WT identities implied by the exact $Q$-symmetry of the lattice action

The continuum theory is invariant also under other fermionic transformations, $Q_{01}$, $Q_0$ and $Q_1$

\[ Q_{01}A_\mu = -\epsilon_{\mu\nu}\psi_\mu \]
\[ Q_{01}\phi = 0 \]
\[ Q_{01}\eta = 2H \]
\[ Q_{01}\overline{\phi} = -2\chi \]
\[ Q_{01}\psi_\mu = i\epsilon_{\mu\nu}D_\nu\phi \]
\[ Q_{01}H = \frac{1}{2}[\phi,\eta] \]
\[ Q_{01}\chi = -\frac{1}{2}[\phi,\overline{\phi}] \]
*Another fermionic symmetry* $Q_1$ *is obtained by further ex-
change* $\psi_0 \leftrightarrow \psi_1$

Invariance under these transformations is expected to be restored only in the continuum limit
In the supersymmetric continuum theory

\[
\langle\langle Q_0 \frac{1}{g^2} \text{tr} \left\{ -\frac{1}{2} \chi[\phi, \bar{\phi}] \right\} \rangle\rangle_{\text{continuum}} = \frac{1}{g^2} \langle\langle \text{tr} \left\{ \frac{1}{4} [\phi, \bar{\phi}]^2 \right\} \rangle\rangle_{\text{continuum}} + \frac{1}{g^2} \langle\langle \text{tr} \left\{ -\chi[\phi, \chi] \right\} \rangle\rangle_{\text{continuum}} = 0
\]

Corresponding to this relation, one might expect

\[
\langle\langle \mathcal{L}_{B1}(x) \rangle\rangle + \langle\langle \mathcal{L}_{F2}(x) \rangle\rangle \rightarrow 0?
\]

holds in the continuum limit \( a \rightarrow 0 \)
Expectation values of $\mathcal{L}_{B1}(x) + \mathcal{L}_{F2}(x)$
• It appears that the average approaches a non-zero number around 0.15 (not zero)

• This does not contradict with SUSY restoration. The argument of SUSY restoration is not applied to correlation functions containing composite operators

• Composite operators $\mathcal{L}_{B_1}(x)$ and $\mathcal{L}_{F_2}(x)$ induce logarithmic UV divergence at 2-loop level. If SUSY of the 1PI effective action is restored, this 2-loop level divergence should be the only source of UV divergence

• Moreover, that remaining 2-loop level divergence is cancelled out in the sum $\langle \mathcal{L}_{B_1}(x) \rangle + \langle \mathcal{L}_{F_2}(x) \rangle$
• This argument indicates that, if SUSY in the 1PI effective action restores, \( \langle \mathcal{L}_{B_1}(x) \rangle + \langle \mathcal{L}_{F_2}(x) \rangle \) approaches a constant (but not necessarily zero) as \( ag \to 0 \).

• The behavior is consistent with this picture based on a restoration of SUSY.

• Within almost 1\( \sigma \) the re-weighted average and the quenched average are degenerate and this also appears consistent with a perturbative picture (\( \leftrightarrow \) The effect of quenching starts at 3-loop \( \sim a^2 g^4 \ln(a/L) \)).

• So, the figure is consistent with the scenario of SUSY restoration, but, it may be dangerous to conclude the restoration of SUSY from the above result alone.
Another example:

\[
\langle \langle Q_0 \frac{1}{g^2} \text{tr} \left\{ -\frac{1}{2} \psi_0 [\phi, \bar{\phi}] \right\} \rangle \rangle_{\text{continuum}} = \frac{1}{g^2} \langle \langle \text{tr} \left\{ \frac{1}{4} [\phi, \bar{\phi}]^2 \right\} \rangle \rangle_{\text{continuum}} + \frac{1}{g^2} \langle \langle \text{tr} \left\{ -\psi_0 [\psi_0, \bar{\phi}] \right\} \rangle \rangle_{\text{continuum}} = 0
\]

and one might expect

\[
\langle \langle \mathcal{L}_{B_1}(x) \rangle \rangle + \langle \langle \mathcal{L}_{F_3}(x) \rangle \rangle \rightarrow 0?
\]

in the continuum limit \( a \rightarrow 0 \).
Expectation values of $\mathcal{L}_{B1}(x) + \mathcal{L}_{F3}(x)$
Yet another:

\[ \langle \mathcal{L}_{B_1}(x) \rangle + \langle \mathcal{L}_{F_4}(x) \rangle \rightarrow 0? \]
Expectation values of $\mathcal{L}_{B1}(x) + \mathcal{L}_{F4}(x)$
Gauge invariant scalar bi-linear operators

Classical “moduli space”

\[ [\phi, \bar{\phi}] = 0 \]

This degeneracy is not lifted to all order of loop expansion (the so-called flat directions)

Gauge-invariant scalar bi-linear operators

\[ a^{-2} \text{tr}\{\phi(x)\bar{\phi}(x)\} \]
\[ a^{-2} \text{tr}\{\phi(x)\phi(x)\} \]
$a^{-2} \text{tr}\{\phi(x)\bar{\phi}(x)\}$ is invariant under the global $U(1)_R$ transformation

$$\phi(x) \rightarrow e^{2i\alpha} \phi(x) \quad \bar{\phi}(x) \rightarrow e^{-2i\alpha} \bar{\phi}(x)$$

The continuum limit of this quantity itself is meaningless, because it is a bare quantity and suffers from UV divergence. Power counting shows that the over-all UV divergence comes from the simplest 1-loop diagram and

$$\sim \ln(a/L)g^2$$

If SUSY of the 1PI effective action is restored in the continuum limit, this 1-loop divergence is the only source of UV divergence
So we define the renormalized operator (the normal product)

\[ \mathcal{N}[a^{-2} \text{tr}\{\phi(x)\overline{\phi}(x)\}] \equiv a^{-2} \text{tr}\{\phi(x)\overline{\phi}(x)\} - (N_c^2 - 1)c(a/L)g^2 \]

This subtraction must remove all the UV divergence of the composite operator

\[
c(a/L = 1/N) = \frac{1}{2N^2} \sum_{n_0=0}^{N-1} \sum_{n_1=0}^{N-1} \frac{1}{\sum_{\mu=0}^{1} \left(1 - \cos \frac{2\pi}{N} n_\mu\right)}
\]

40
Expectation values of $\mathcal{N}[a^{-2} \text{tr}\{\phi(x)\bar{\phi}(x)\}]$
• Clear separation between the re-weighted average and the quenched one (quantum effect of dynamical fermions)

• Fermions actually uplifts the expectation value!

• The expectation value appears to approach some finite number (in a unit of $g^2$) in the continuum limit after the renormalization

• Without the renormalization, there is a tendency that the expectation values grow as $ag \to 0$

• If SUSY is restored in the continuum limit, the expectation value is expected to become independent of $ag$ as $a \to 0$. The behavior in figure is more or less consistent with this expectation (though we need much data to conclude this)
Expectation values of $a^{-2} \text{tr}\{\phi(x)\phi(x)\}$
Conclusion

• Preliminary numerical study of Sugino’s lattice formulation of 2d $\mathcal{N} = (2,2)$ SYM

• WT identities associated with the $Q$-symmetry were confirmed in fair accuracy ($\Rightarrow$ re-weighting method is basically working)

• On the other hand, all results are consistent with the basic scenario of SUSY restoration (encouraging), though we could not conclude the restoration of full SUSY in a definite level
Prospects

• Much larger lattice with (RIKEN) PC cluster

• Two-point functions (conservation of SUSY current, mass spectrum)

• Wilson loops (screening by adjoint fermions?)

• $2d \mathcal{N} = (4, 4) \text{ SYM}$ (and $2d \mathcal{N} = (8, 8) \text{ SYM}$)
RIKEN Symposium

Quantum Field Theory and Symmetry

12/22 (Sat.) and 12/23 (Sun.)

You Are Welcome!

To be announced in sg-I (hopefully) soon
Appendix

Comparison with

Catterall, JHEP 04 (2007) 015 [hep-lat/0612008]
Expectation values of $\mathcal{L}_{B1}(x) + \mathcal{L}_{F1}(x)$
Expectation values of $2\mathcal{L}_{B2}(x) + \mathcal{L}_{F5}(x)$
Expectation values of $\mathcal{L}_{B3}(x) + \mathcal{L}_{F3}(x) + \mathcal{L}_{F4}(x) + \mathcal{L}_{F6}(x)$
Expectation values of $\mathcal{L}_{B1}(x) + \mathcal{L}_{F2}(x)$
Expectation values of $\mathcal{L}_{B_1}(x) + \mathcal{L}_{F_3}(x)$