On the moduli space of semilocal strings and lumps

Naoto YOKOI

Institute of Physics, Univ. of Tokyo (Komaba)

1. Introduction to (Non-Abelian) Semilocal Vortex
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1 Introduction to (Non-Abelian) Semilocal Vortex

Abrikosov-Nielsen-Olesen (ANO) Vortex in Abelian Higgs Model

\[ S = \int d^4x \left( -\frac{1}{4} F_{\mu\nu}^2 + \frac{1}{2} |\mathcal{D}_\mu \Phi|^2 - \frac{\lambda}{8} (|\Phi|^2 - v^2)^2 \right). \]

Finite Tension Soln. of Eq. of Motion: For BPS Case,

\[(\mathcal{D}_x + i\mathcal{D}_y) \Phi = 0, \quad B + \frac{1}{2} (|\Phi|^2 - v^2) = 0.\]

ANO Vortex as Squeezed Magnetic Flux in Type II Superconductor

- Flux Energy is Proportional to Length
  \[\implies \text{“Probe Monopoles” are Confined.}\]

- Stability from Non Simply-Connected Vacuum Manifold
  \[\Longleftarrow \pi_1(S^1) = \pi_1(U(1)) = \mathbb{Z}.\]

- Characterized by Only Positions on the plane.
Extension to Multi-Flavor Case: \( \Phi \rightarrow \Phi_I (I = 1, \cdots, N_f) \)

What Happens for the Simplest \( N_f = 2 \) Case?

1. Vacuum Manifold Changes to \( S^3 \).  
   \[ \Phi_I^\dagger \Phi_I = v^2 \]
   \[ \pi_1(S^3) = 1 \text{ (Trivial)} \Rightarrow \text{Stable Solution?} \]

From Analysis of Perturbation (by Hindmarsh)

\[ \text{For } \lambda/e^2 \leq 1, \text{ Stable Solutions Do Exist and Classified by } \pi_1(U(1)). \]

2. For \( \lambda/e^2 = 1 \), Vortex Solutions Have Another Kind of Parameters.

\[ \text{These “Size” Moduli Determine Transverse Size of Vortex!} \]

3. Large \( r \) Behavior is Quite Different from ANO \( \Rightarrow \) “Lump” in Sigma Models.

These Vortex Solutions are Called Semilocal Vortex (or String) (Vachaspati-Achucarro).

\[ \text{Why This Semilocal Vortex is Interesting?} \]
Another Extension of ANO Vortex from $U(1)$ to $U(N)$ Gauge Theory
In Such Extensions, There Exists Another Type of Vortex Solutions.

◊ Non-Abelian (NA) Vortex  (Hanany-Tong, Auzzi-Bolognesi-Evslin-Konishi-Yung)

● NA-Vortex Has Also Another Kind of Parameters
  $\rightarrow$ “Orientation” Moduli from $SU(N)$ Color-Flavor Diag. Symmetry.
We Have Studied Non-Abelian Ver. of Electric-Magnetic Duality and Confinement from

◊ Orientation Moduli of NA-Vortex.  (Cf. Eto et. al., hep-th/0611313)

Quantum Mechanically, Our Analysis on NA-Duality Requires Certain Number of Flavors.
$\rightarrow$ NA-Vortex Becomes Semilocal and “Size” and “Orientation” Moduli Both Appear!

Knowledge of Moduli Space for NA-Semilocal Vortex Gives New Hints for NA-Duality.

We Discuss the Moduli Space of NA-Semilocal Vortex Using Moduli Matrix Formalism.
2 Analysis of Moduli Space by Moduli Matrix

- Short Review of Local Non-Abelian Vortex

Consider the $U(N_c)$ Gauge Theory with Higgs Scalars

$$\mathcal{L} = \text{Tr} \left[ -\frac{1}{2g^2} F_{\mu\nu} F^{\mu\nu} - \frac{2}{g^2} D_\mu \Phi^\dagger D^\mu \Phi - D_\mu H D^\mu H^\dagger \right.$$
$$\left. - \lambda (\xi 1_{N_c} - H H^\dagger)^2 \right] + \text{Tr} \left[ (H^\dagger \Phi - m H^\dagger)(\Phi H - m H) \right],$$

where $\Phi$: Adjoint Higgs, $H$: $N_f$ Fund. Higgs in ($N_c \times N_f$) Matrix Form.

For Local $N_f = N_c$ Case, the Vacuum Becomes

$$\langle \Phi \rangle = m 1_{N_c}, \quad \langle H \rangle = \sqrt{\xi} 1_{N_c}.$$

- The Vacuum Preserves Color-Flavor Diagonal Sym. $SU(N_c)_{C+F}$.

- Eq. of Motion for BPS Vortex ($\lambda = g^2/4$):

$$\left( D_x + iD_y \right) H = 0, \quad F_{xy} + \frac{g^2}{2} (\xi 1_N - H H^\dagger) = 0.$$

- “Non-Abelian” Zero Modes from the Breaking of $SU(N_c)_{C+F}$ by Vortex.
Moduli Matrix Formalism for Non-Abelian Vortex (Eto-Isozumi-Nitta-Ohashi-Sakai)

Solutions for the Eq. of Motion \( z = x + i y \):

\[
H = S^{-1}(z, \bar{z}) \ H_0(z), \quad A_x + i A_y = -2i \ S^{-1} \ \partial_{\bar{z}} S(z, \bar{z}).
\]

• \( S(z, \bar{z}) \) Satisfies a Nonlinear “Master Equation”:

\[
\partial_{\bar{z}} (\Omega^{-1} \partial_{\bar{z}} \Omega) = \frac{g^2}{4} (\xi \ 1_N - \Omega^{-1} \ H_0 \ H_0^\dagger). \quad (\Omega \equiv SS^\dagger)
\]

• \( H_0(z) \) is Moduli Matrix Encoding All Moduli Parameters up to the \( V \)-Transformation :

\[
H_0(z) \rightarrow V(z) H_0(z), \quad S(z, \bar{z}) \rightarrow V(z) S(z, \bar{z}) \quad (V(z) \) is Hol. Matrix).
\]

• Vortex Number (Flux) \( k \) is Encoded in \( \det H_0(z) \sim z^k \) at \( z \rightarrow \infty \).

Another Construction of Moduli Space by Kähler Quotient

\[
\begin{align*}
\{ H_0(z) | & \ \
\det H_0 \sim z^k \} \quad \iff \quad \{ Z, \Psi | \ (k \times k) \) and \ (N_c \times k) \ Const. Matrix \} \\
\{ V(z) | & \ V \in GL(N_c; C) \} \quad \iff \quad \{ U | U \in GL(k; C) \}
\end{align*}
\]

where \( Z \sim U \ Z \ U^{-1} \) and \( \Psi \sim \Psi U^{-1} \).

Simplest Example : 1-Vortex in \( U(N) \) Theory \( \rightarrow \ \mathcal{M} = \mathbb{C} P^{N-1} \).
• Moduli Space for Semilocal Non-Abelian Vortex with $N_f > N_c$

- Non-Trivial Degenerate Higgs Vacua Appear:
  \[ \mathcal{V}_{\text{Higgs}} \simeq \frac{SU(N_f)}{SU(N_c) \times SU(N_f - N_c) \times U(1)} \]
  \[ \implies SU(N_c)_{C+F} \times SU(N_f - N_c) \text{ Global Symmetry is Preserved.} \]

- Moduli Matrix Becomes Rectangular:
  
  \[ H_0(z) = (D(z), Q(z)), \]
  
  where $D(z) : N_c \times N_c$ Matrix and $Q(z) : N_c \times (N_f - N_c)$ Matrix.

  \[ \implies \text{Additional "Size" Moduli Appear from } Q(z). \]

- Vortex No. $k \iff \det H_0 H_0^\dagger \sim |z|^{2k} \quad (|z| \sim \infty).$

However, Kähler Quotient Construction Can be Applied to Semilocal Case:

\[
\left\{ \begin{array}{l}
H_0^{(k)}(z) \\
V(z)
\end{array} \right\} \iff \left\{ \begin{array}{l}
Z, \Psi, \tilde{\Psi} \mid (k \times k), (N_c \times k), (k \times (N_c - N_f)) \text{ Matrix} \\
U \mid U \in GL(k; C)
\end{array} \right\},
\]

where

\[ \{ Z, \Psi, \tilde{\Psi} \sim \{ UZU^{-1}, \Psi U^{-1}, U\tilde{\Psi} \} \]
Structure of the Quotient

From Construction with $H_0(z)$, $GL(k; C)$ Action Should be FREE on $(Z, \Psi)$:

$$\{UZU^{-1}, \Psi U^{-1}\} = \{Z, \Psi\} \implies U = 1.$$  

With this Condition, the Quotient $\{Z, \Psi, \tilde{\Psi}\}/GL(k; C)$ is Equivalent to

$$\{ (Z, \Psi, \tilde{\Psi}) | [\tilde{Z}^\dagger, Z] + \Psi^\dagger \Psi - \tilde{\Psi} \tilde{\Psi}^\dagger - r = 0 \}/U(k) \quad (r > 0).$$

$\implies$ D-flat Conditions for Some 2-Dimensional Gauge Theory $\sim$ D-Brane Set-Up

The Exchange Such That:

1. $N_c \rightarrow N_f - N_c \equiv \tilde{N}_c$
2. $GL(k; C)$ Free on $(Z, \Psi) \rightarrow GL(k; C)$ Free on $(Z, \tilde{\Psi})$
3. $r > 0 \rightarrow r < 0$.

Gives A Different Moduli Space of Vortex in $U(N_f - N_c)$ Gauge Theory.

These Moduli Spaces Corresspond to Two Different Reg. of A Parent Space!

Note: Vacuum of the Theory is Invariant under $N_c \rightarrow \tilde{N}_c$. 
Simplest Example of Moduli Space:

1-Vortex in $U(2)$ Gauge Theory with $N_f = 3$ ($GL(1; C) = U(1)^C = C^*$)

$$(Z, \Psi, \tilde{\Psi}) \sim (Z, \lambda^{-1}\Psi, \lambda\tilde{\Psi}), \quad \lambda \in C^*,$$

where $Z$, $\tilde{\Psi}$: Constant and $\Psi$: 2-Vector.

Except for Position Moduli $Z$, Internal Moduli Space Appears to be

$W \subset \mathbb{CP}^2[1, 1, -1] : (y_1, y_2, y_3) \sim (\lambda y_1, \lambda y_2, \lambda^{-1} y_3) \quad (\neq (0, 0, 0))$.

This Space is NON-Hausdorff Space!

Because Two Distinct Points $(a, b, 0)$ and $(0, 0, 1)$ Has NO Disjoint Neighborhoods:

$$(\epsilon a, \epsilon b, 1) \sim (a, b, \epsilon), \quad \text{where } \epsilon \text{ is Arbitrarily Small.}$$

In Order to Make the Space Hausdorff, We Should Eliminate Either Point:

- Two “Regularizations” $\Rightarrow$ Two Different Manifolds

This Corresponds to the Choice Between $U(2)$ Theory and “Dual” $U(1)$ Theory
Two “Regularized” Spaces as Moduli Spaces of “Dual” Theories

1. \( W \mathbb{C}P^2[1, 1, -1] \equiv W \mathbb{C}P^2[1, 1, -1] - (0, 0, 1) \)
   
   Moduli Space of \( U(2) \) Theory \( \implies \mathcal{M}_{2, 3} = \widetilde{C}^2 : \) Blow Up of \( C^2 \)

2. \( W \mathbb{C}P^2[1, 1, -1] \equiv W \mathbb{C}P^2[1, 1, -1] - \mathbb{C}P^1 \)
   
   Moduli Space of \( U(1) \) Theory \( \implies \mathcal{M}_{1, 3} = C^2 \)

\( GL(k, \mathbb{C}) \) Free Condition \( \iff \) Eliminating “Irregular” Subspace.

Self-Dual Case : 1-Vortex in \( U(2) \) Theory with \( N_f = 4 \):

Parent Space is \( W \mathbb{C}P^3[1, 1, -1, -1] \implies \) Two Reg. Give Same Space.

\( \implies \) Resolved Conifold : \( \mathcal{O}(-1) \oplus \mathcal{O}(-1) \rightarrow \mathbb{C}P^1 \)

Generalization to \( U(N_c) \) with \( N_f \) : Parent Space is \( W \mathbb{C}P^{N_f-1}[1^{N_c}, -1^{N_f-N_c}] \).

1. \( \mathcal{M}_{N_c, N_f} = W \mathbb{C}P^{N_f-1}[1^{N_c}, -1^{\widetilde{N}_c}] : \mathcal{O}(-1) \oplus \widetilde{N}_c \rightarrow \mathbb{C}P^{N_c-1} \)

2. \( \mathcal{M}_{\widetilde{N}_c, N_f} = W \mathbb{C}P^{N_f-1}[1^{N_c}, -1^{\widetilde{N}_c}] : \mathcal{O}(-1) \oplus N_c \rightarrow \mathbb{C}P^{\widetilde{N}_c-1} \).
Lump Solution in Strong Coupling Limit

LEET of Strong Coupling Limit $\implies$ Non-Linear Sigma Model on $\mathcal{V}_{\text{Higgs}}$.

This Sigma Model Has Codim. 2 Lump Solitons from $\pi_2(\mathcal{V}_{\text{Higgs}}) = \mathbb{Z}$.

$\implies$ In the Strong Coupling Limit, Our Vortex Becomes this Lump Soliton.

Moduli Space of Smooth $k$-Lump Soliton is Also Determined by Moduli Matrix:

$$\mathcal{M}_{N_c,N_f}^{\text{lump}} \equiv \left\{ (Z, \Psi, \bar{\Psi}) | GL(k, \mathbb{C}) \text{ free on } (Z, \Psi) \text{ and } (Z, \bar{\Psi}) \right\} / GL(k, \mathbb{C})$$

$$= \mathcal{M}_{N_c,N_f} \cap \mathcal{M}_{\tilde{N}_c,N_f}.$$  

Finally, We Have the Following Diamond Diagram:

$$\mathcal{M}_{N_c,N_f} \left( \mathbb{CP}^2_{(1,1,-1)} \right) \quad \text{Seiberg-like Duality} \quad \mathcal{M}_{\tilde{N}_c,N_f} \left( \mathbb{C}^2 \right)$$

$$\mapsto \quad g^2 \to \infty \quad \mathcal{M}_{N_c,N_f} \left( \mathbb{C}^2 \right) \quad \tilde{g}^2 \to \infty$$
3 Worldsheet Effective Dynamics of Moduli

Worldsheet Effective Theory on Vortex

Possible to Obtain Eff. Theory by Promoting the Moduli to Slowly-Moving Fields

\[
\downarrow
\]

2-Dim. Non-Linear Sigma Model on Our Moduli Space

In SUSY Context, Moduli Matrix Can Provide the Kahler Potential:

\[
K = \text{Tr} \int d^2 z \left( \xi \log \Omega + \Omega^{-1} H_0 H_0^\dagger + \mathcal{O}(1/g^2) \right).
\]

Note: This Gives Standard \(\mathbb{C}P^N\) Metric for Local NA-Vortex.

Crucial Difference from Local Vortex is

Existence of Non-Normalizable Moduli (such as “Size” Moduli).

Actually, Large \(r\) Behavior of Kahler Pot. Becomes (\(L\) : IR Cut-Off)

\[
K \simeq 2\pi \xi \log L \text{ Tr } |\Psi \tilde{\Psi}|^2 + \text{const.} + \mathcal{O}(L^{-1}),
\]
Explicit Form of Kähler Potential for $U(2)$ Theory with $N_f = 3$:

$$K_{{N_c=2, N_f=3}} = \xi \pi |c|^2 (1 + |b|^2) \log \frac{L^2}{|c|^2 (1 + |b|^2)} + \mathcal{O}(L^0).$$

Replacement ($\tilde{c} = c$, $\tilde{b} = c b$) Gives $K_{{N_c=1, N_f=3}}$ of $U(1)$ Dual Theory.

Number of Normalizable Moduli ($\sim$ Dynamical Fields) Depends on

◊ Rank of Matrix $\Psi \tilde{\Psi}$.

$\implies$ Number of Normalizable Moduli Does Change on Some Submanifold!

Simple Example: 1-Vortex in $U(N_c)$ Theory $\implies$ Rank$(\Psi \tilde{\Psi}) = \ell \leq 1$.

1. For $\ell = 1$ ($c \neq 0$), Only Position Moduli is Normalizable ($\sim \mathbb{C}$).

2. For $\ell = 0$ ($c = 0$), Space of Normalizable Moduli Becomes $\mathbb{C} \times \mathbb{C}P^{N_c-1}$.

◊ Non-Trivial Moduli Enhancement Occurs!

Note: More Interesting Phenomena Occur in general $k$-Vortex Case.
4 Summary and Discussion

□ Summary

• We Have Discussed Aspects of the Moduli Space of Semilocal Non-Abelian Vortex in $U(N_c)$ Gauge Theory with $N_f > N_c$ by Using the Moduli Matrix Formalism.

• We Have Found A Geometrical Correspondence of the Moduli Spaces of Vortex in the $U(N_c)$ Theory and $U(N_f - N_c)$ Theory.

• We Have Also Studied Effective Theory of Moduli on the Worldsheet of Vortex.

□ Discussion

1. Understanding of Bulk Gauge Theory Dynamics Using Vortex Dynamics

   ➞ Seiberg Duality from Semilocal NA-Vortex Moduli


3. Dynamics of NON-BPS Semilocal NA-Vortex