Geometrical Construction of Supertwistor Theory

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We propose a geometrical construction of supertwistor theory based on the SUSY Hopf map. Penrose introduced twistors as more fundamental quantities than space-time itself. Twistor variables $Z^a = (Z^1, Z^2, Z^3, Z^4) = (Z_+ . Z_-)$ and space-time variables $x^\mu = (t, x, y, z)$ are related by the incidence relation

$$Z_+ = i\eta_{\mu\nu}\sigma^\mu x^\nu Z_-. \tag{1}$$

First, we briefly discuss close relations between the Hopf maps and twistors. There exist three (bosonic) Hopfs maps, which are explicitly given by $S^2 \simeq S^3/S^1$, $S^4 \simeq S^4/S^3$ and $S^8 \simeq S^{15}/S^7$. With a normalized two-component spinor $\phi = (\phi_1, \phi_2)^t$, the first Hopf map is expressed as $\phi \to \hat{x}^i = \phi^\dagger \sigma^i \phi$ $(i = 1, 2, 3)$, and $\hat{x}^i$ satisfy $\hat{x}^i \hat{x}^i = (\phi^\dagger \phi)^2 = 1$. Similarly, with a normalized four-component spinor $\Phi = (\Phi^1, \Phi^2, \Phi^3, \Phi^4) = (\Phi_+, \Phi_-)$, the second Hopf map is denoted as $\Phi \to \hat{x}^a = \Phi^\dagger \Gamma^a \Phi$ $(a = 1, 2, 3, 4, 5)$, where Γ^a are $SO(5)$ gamma matrices, and $\hat{x}^a$ satisfy the normalization condition $\hat{x}^a \hat{x}^a = (\Phi^\dagger \Gamma^a \Phi)^2 = 1$. Φ_+ and Φ_- are related as

$$\Phi_+ = (x^4 + i\sigma^i x^i)\Phi_-, \tag{2}$$

with $x^\mu = \hat{x}^\mu/(1 + \hat{x}^5)$ $(\mu = 1, 2, 3, 4)$. Indeed, (2) is Euclidean version of (1).

The SUSY Hopf map is denoted as $S^{2|2} \simeq S^{3|2}/S^1$, and is explicitly $\psi \to x^i = 2\psi^\ddagger l^i \psi$ and $\theta^\alpha = 2\psi^\ddagger l^\alpha \psi$. ($l^i$ and l^α are $OSp(1|2)$ fundamental representation matrices). With the normalized three-component spinor ψ (two components are Grassmann even, while one component is Grassmann odd), $\hat{x}^i$ and $\hat{\theta}^\alpha$ automatically satisfy the relation $\hat{x}^i \hat{x}^i + \epsilon_{\alpha\beta}\hat{\theta}^\alpha \hat{\theta}^\beta = (\psi^\ddagger \psi)^2 = 1$. Inspired by the analogies between the original incidence relations, we introduce a super-incidence relation

$$\mathcal{Z}_+ = 2i(\eta_{\mu\nu} x^\mu l^\mu + \epsilon_{\alpha\beta}\theta^\alpha l^\beta)\mathcal{Z}_-. \tag{3}$$

Our supertwistor variables are $\mathcal{Z}^A = (\mathcal{Z}_+, \mathcal{Z}_-) = (Z^1, Z^2, Z^3, Z^4, \xi^1, \xi^2)$, four of which are Grassmann even, while the remaining two are Grassmann odd. From the super-hermiticity of l^i and l^α, $\mathcal{Z}^A$ satisfy the super-null condition $\mathcal{Z}_A^* \mathcal{Z}^A = Z_a^* Z^a + \xi_i^* \xi^i = 0$. The (half of the) norm of twistor Z^a physically represents the spin degrees of freedom, $s = 1/2 Z_a^* Z^a$, but from (1), the norm of twistor is identically zero. To describe the non-zero spin, conventionally, the space-time needs to be complexified. Meanwhile, in the present approach, with N species of fermionic components, the spin is given by $s = 1/2 Z_a^* Z^a = -1/2 \sum_{l=1}^{2N} \xi_l^* \xi^l$, and the Minkowski space-time does not need to be complexified. Further, the number of the fermionic components of the supertwistor is always even integer $2N$. (This condition is necessary to realize the half-integer of integer spins, but it has been imposed by hands in the conventional supertwistor theory.) These are advantages of the present geometrical construction.