Magnetic monopole loops supported by a meron pair as the quark confiner

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§ Introduction

⊙ Wilson's criterion for quark confinement:

Wilson loop operator: $W_C[\mathscr{A}]$
$=$ trace of the holonomy operator for Yang-Mills connection

$$W_C[\mathscr{A}] := \mathrm{tr}\left[\mathscr{P}\exp\left\{ig\oint_C dx^\mu \mathscr{A}_\mu(x)\right\}\right]/\mathrm{tr}(\mathbf{1})$$

$$S_{\mathrm{YM}}[\mathscr{A}] = \int d^D x \frac{1}{2}\mathrm{tr}[\mathscr{F}_{\mu\nu}(x)\mathscr{F}_{\mu\nu}(x)]$$

$$W(C) := \langle W_C[\mathscr{A}]\rangle_{\mathrm{YM}} = Z_{\mathrm{YM}}^{-1}\int d\mu[\mathscr{A}]e^{-S_{\mathrm{YM}}[\mathscr{A}]}W_C[\mathscr{A}]$$

Area law of the Wilson loop average $W(C) := \langle W_C[\mathscr{A}]\rangle_{\mathrm{YM}} \sim \exp(-\sigma|\mathrm{Area}(C)|)$
$\Longrightarrow$ linear potential of static inter-quark potential $V(r)$; $V(r) \sim \sigma r$ for large r

⊙ Topological field configuration as dominant configurations in the functional integral:

Abelian magnetic monopoles, Non-Abelian magnetic monopole, center vortices, Yang-Mills instantons, merons, elliptic solution, Hopfion, calorons, ...

In the dual superconductor picture, it is expected that (Abelian or non-Abelian) magnetic monopoles exist and are condensed to cause the dual Meissner effect.

D=2: Yang-Mills theory is exactly calculable, $V(r) = \sigma r$, $\sigma = c_2(N)\frac{g^2}{2} = \frac{N^2-1}{2N}\frac{g^2}{2}$.

Coulomb potential = linear potential in D=2!

⊙ Dual superconductor picture was valid in the following models where confinement was shown in the analytical way.

D=3:

• compact QED$_3$ in Georgi-Glashow model [Polyakov, 1977]

$\longrightarrow$ magnetic monopole plasma, sine-Gordon theory described by the dual variable

D=4:

• (Lattice) compact QED$_4$ (in the strong coupling region) [Polyakov, 1975]

$\longrightarrow$ magnetic monopole plasma ;

U(1) link variable $\rightarrow$ monopole current variable [Banks, Myerson and Kogut, 1977]

• N=2 SUSY YM$_4$ [Seiberg and Witten, 1994] ...

⊙ How about (ordinary non-SUSY) YM$_3$, YM$_4$ and QCD$_4$?

Can we introduce magnetic monopoles in these theories?

$\odot$ Abelian projection and the resulting magnetic monopole [G. 't Hooft, 1981]:

Even in the pure Yang-Mills theory (without adjoint Higgs scalar fields), Abelian magnetic monopoles can be introduced as the gauge fixing defect of partial gauge fixing: $G = SU(N) \to H = U(1)^{N-1}$ [Abelian projection]

$G = SU(N)$ non-Abelian Yang-Mills fields
$\to H = U(1)^{N-1}$ Abelian gauge fields + Abelian magnetic monopoles + electrically charged matter fields

Let $\phi(x)$ be a Lie-algebra $\mathscr{G}$-valued functional of the Yang-Mills field $\mathscr{A}_\mu(x)$. Suppose that it transforms in the adjoint representation under the gauge transformation:

$$\phi(x) \to \phi'(x) := U(x)\phi(x)U^\dagger(x) \in \mathscr{G} = su(N), \quad U(x) \in G, \quad x \in \mathbb{R}^D.$$

e.g., $\phi(x) = \mathscr{F}_{12}(x)$, $\mathscr{F}_{\mu\nu}\mathscr{F}_{\mu\nu}$, $\mathscr{F}_{\mu\nu}(x)D^2\mathscr{F}_{\mu\nu}(x)$

For $G = SU(2)$, the location of magnetic monopole is determined by simultaneous zeros of $\phi(x)$:

$$\phi^A(x) = 0 \quad (A = 1, 2, 3).$$

$\Longrightarrow$ The magnetic monopole is a topological object of co-dimension 3.

D=3: 0-dimensional point defect $\to$ magnetic monopole

D=4: 1-dimensional line defect $\to$ magnetic monopole loop (closed loop)

• Numerical simulation on the lattice in the Maximal Abelian gauge (MAG):

For the SU(2) Cartan decomposition: $\mathscr{A}_\mu = A_\mu^a \frac{\sigma^a}{2} + A_\mu^3 \frac{\sigma^3}{2}$ $(a = 1, 2)$,

Abelian-projected Wilson loop $\left\langle \exp\left\{ ig \oint_C dx^\mu A_\mu^3(x) \right\} \right\rangle_{\rm YM}^{\rm MAG} \sim e^{-\sigma_{Abel}|S|}$!?

· Abelian dominance $\Leftrightarrow$ $\sigma_{Abel} \sim \sigma_{NA}$ (92±4)% [Suzuki & Yotsuyanagi,PRD42,4257,1990]
The magnetic monopole of the Dirac type appears in the diagonal part A_μ^3 of $\mathscr{A}_\mu(x)$.

$$A_\mu^3 = \text{Monopole part} + \text{Photon part},$$

· Monopole dominance $\Leftrightarrow$ $\sigma_{monopole} \sim \sigma_{Abel}$ (95)%
[Stack, Neiman & Wensley, hep-lat/9404014][Shiba & Suzuki, hep-lat/9404015]

MAG is given by minimizing the functional $F_{\rm MAG}$ w.r.t. the gauge transf. Ω.

$$F_{\rm MAG}[\mathscr{A}] := \frac{1}{2}(A_\mu^a, A_\mu^a) = \int d^D x \frac{1}{2} A_\mu^a(x) A_\mu^a(x) \quad (a = 1, 2)$$

$$\delta_\omega F_{\rm MAG} = (\delta_\omega A_\mu^a, A_\mu^a) = ((D_\mu[A]\omega)^a, A_\mu^a) = -(\omega^a, D_\mu^{ab}[A^3]A_\mu^b)$$

The continuum form is $D_\mu^{ab}[A^3]A_\mu^b := [\partial_\mu \delta^{ab} - g\epsilon^{ab3} A_\mu^3(x)]A_\mu^b(x) = 0$ $(a, b = 1, 2)$.
In general, MAG fixes G/H, leaving H unbroken.

• Numerical simulations for Abelian monopole current

[Chernodub &

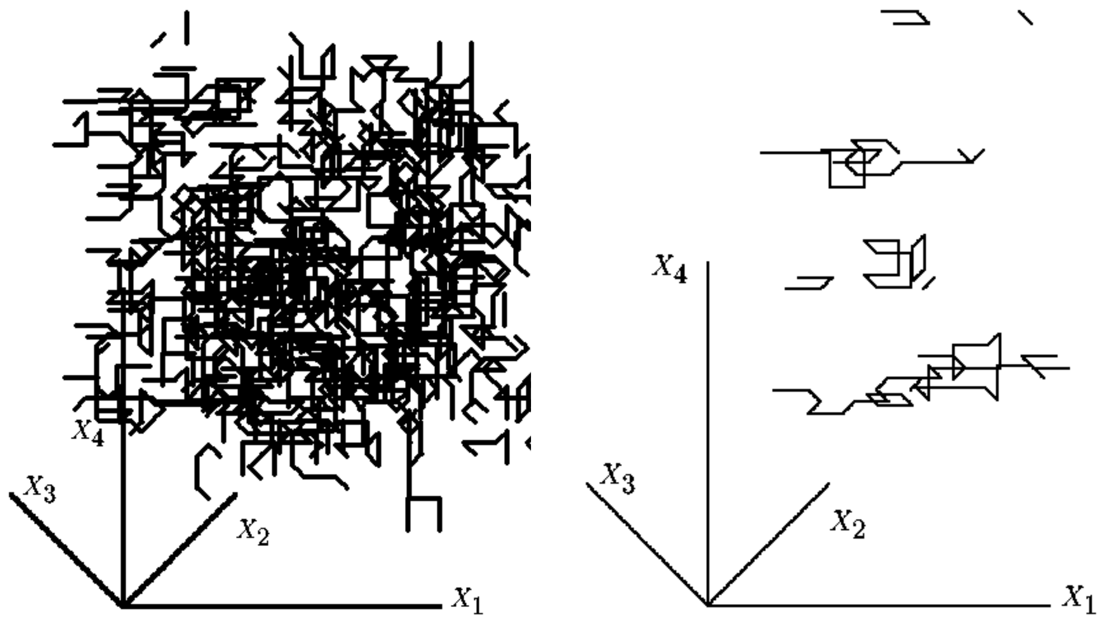


Figure 1: The abelian monopole currents for the confinement (a) ($\beta = 2.4$, 10^4 lattice) and the deconfinement (b) phases ($\beta = 2.8$, $12^3 \cdot 4$ lattice).

It is important to notice that the nature of the defects depends on the order of the zeros. For first-order zeros, one obtains magnetic monopoles. The defects from zeros of second order are Hopfion which is characterized by a topological invariant called Hopf index for the Hopf map $S^3 \to SU(2)/U(1) \simeq S^2$ with non-trivial Homotopy $\pi_3(S^2) = \mathbb{Z}$.

It is rather delicate whether magnetic monopole loops on the lattice in the MAG can survive in the continuum limit.

To clairfy these issues, we need analytical solutions of magnetic monopole loop in in $D = 4$ pure Yang-Mills theory.

The purpose of this talk is to give an analytical solution representing circular magnetic monopole loops joining a pair of merons in the four-dimensional Euclidean SU(2) Yang-Mills theory.

This is achieved by solving the differential equation for the adjoint color (magnetic monopole) field in the two–meron background field within the recently developed reformulation of the Yang-Mills theory.

Our analytical solution corresponds to the numerical solution found by Montero and Negele on a lattice.

This result strongly suggests that a meron pair is the most relevant quark confiner in the original Yang-Mills theory, as Callan, Dashen and Gross suggested long ago.

original Yang-Mills: merons

$\Longleftrightarrow$ **dual Yang-Mills: magnetic monopole loops**

§ What are merons?

	instanton	meron
discovered by	BPST 1975	DFF 1976
$D_\nu \mathscr{F}_{\mu\nu} = 0$	YES	YES
self-duality $*\mathscr{F} = \mathscr{F}$	YES	NO
Topological charge Q_P	$(0), \pm 1, \pm 2, \cdots$	$(0), \pm 1/2, \pm 1, \cdots$
charge density D_P	$\frac{6\rho^4}{\pi^2} \frac{1}{(x^2+\rho^2)^4}$	$\frac{1}{2}\delta^4(x-a) + \frac{1}{2}\delta^4(x-b)$
solution $\mathscr{A}_\mu^A(x)$	$g^{-1}\eta_{\mu\nu}^A \frac{2(x-a)_\nu}{(x-a)^2+\rho^2}$	$g^{-1}\left[\eta_{\mu\nu}^A \frac{(x-a)_\nu}{(x-a)^2} + \eta_{\mu\nu}^A \frac{(x-b)_\nu}{(x-b)^2}\right]$
Euclidean	finite action $S_{\mathrm{YM}} = (8\pi^2/g^2)\lvert Q_P\rvert$	(logarithmic) divergent action
tunneling	between $Q_P = 0$ and $Q_P = \pm 1$ vacua in the $\mathscr{A}_0 = 0$ gauge	$Q_P = 0$ and $Q_P = \pm 1/2$ vacua in the Coulomb gauge
multi-charge solutions	Witten, 't Hooft, Jackiw-Nohl-Rebbi, ADHM	??? not known
Minkowski	trivial	everywhere regular finite, non-vanishing action

An instanton dissociates into two merons?

§ Relevant works (excluding numerical simulations)

papers	original configuration	dual counterpart	method
CG95	one instanton	a straight magnetic line	MAG (analytical)
BOT96	one instanton	no magnetic loop	MAG (numerical)
BHVW00	one instanton	no magnetic loop	LAG (analytical)
RT00	one meron	a straight magnetic line	LAG (analytical)
BOT96	instaton-antiinstanton	a magnetic loop	MAG (numerical)
	instaton-instaton	a magnetic loop	MAG (numerical)
RT00	instaton-antiinstanton	two magnetic loops	LAG (numerical)
Ours KFSS08	one instanton	no magnetic loop	New (analytical)
0806.3913	one meron	a straight magnetic line	New (analytical)
[hep-th]	two merons	circular magnetic loops	New (analytical)

CG95=Chernodub & Gubarev, [hep-th/9506026], JETP Lett. **62**, 100 (1995).
BOT96=Brower, Orginos & Tan, [hep-th/9610101], Phys.Rev.D **55**, 6313–6326 (1997).
BHVW00=Bruckmann, Heinzl, Vekua & Wipf, [hep-th/0007119], Nucl.Phys.B **593**, 545–561 (2001). Bruckmann, [hep-th/0011249], JHEP 08, 030 (2001).
RT00=Reinhardt & Tok, [hep-th/0011068], Phys.Lett. B**505**, 131–140 (2001). hep-th/0009205.
BH03=Bruckmann & Hansen, [hep-th/0305012], Ann.Phys. **308**, 201–210 (2003).

§ Reformulating Yang-Mills theory in terms of new variables

SU(2) Yang-Mills theory written in terms of $\mathbb{A}_\mu^A(x)$ $(A=1,2,3)$	$\Longleftrightarrow$ NLCV	A reformulated Yang-Mills theory written in terms of new variables: $\boldsymbol{n}^A(x), c_\mu(x), \mathbb{X}_\mu^A(x)$ $(A=1,2,3)$

We introduce a "color unit field" $\boldsymbol{n}(x)$ of unit length with three components

$$\boldsymbol{n}(x) = n_A(x)T_A, T_A = \sigma_A/2 \quad \Longleftrightarrow \mathbf{n}(x) = (n_1(x), n_2(x), n_3(x))$$

$$\text{i.e.,tr}[\boldsymbol{n}(x)\boldsymbol{n}(x)] = 1/2 \quad \text{or} \quad \mathbf{n}(x)\cdot\mathbf{n}(x) = n_A(x)n_A(x) = 1$$

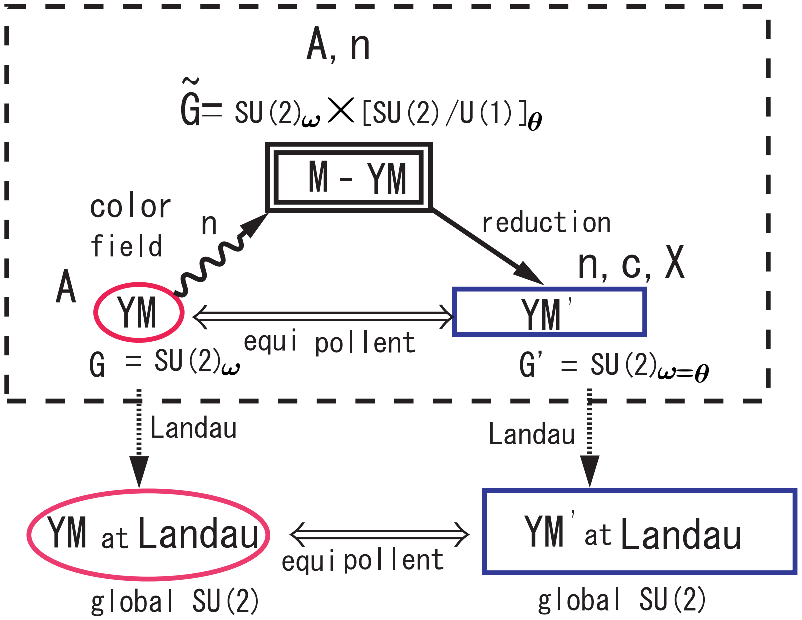
Expected role of the color field:

- The color field $\boldsymbol{n}(x)$ plays the role of recovering color symmetry which will be lost in the conventional approach, e.g., the MA gauge.
- The color field $\boldsymbol{n}(x)$ carries topological defects responsible for non-perturbative phenomena, e.g., quark confinement.

New variables $\boldsymbol{n}^A(x), c_\mu(x), \mathbb{X}_\mu^A(x)$ should be given as functionals of the original $\mathbb{A}_\mu^A(x)$.

A new viewpoint of the Yang–Mills theory

$$\delta_\theta \mathbf{n}(x) = g\mathbf{n}(x) \times \theta(x) = g\mathbf{n}(x) \times \theta_\perp(x)$$



$$\delta_\omega \mathbb{A}_\mu(x) = D_\mu[\mathbb{A}]\omega(x)$$

By introducing a color field, the original Yang-Mills (YM) theory is enlarged to the master Yang–Mills (M-YM) theory with the enlarged gauge symmetry $\tilde{G}$. By imposing the reduction condition, it is reduced to the equipollent Yang-Mills theory (YM') with the gauge symmetry G'. The overall gauge fixing condition can be imposed without breaking color symmetry, e.g. Landau gauge.
[K.-I.K., Murakami & Shinohara, hep-th/0504107; Prog.Theor.Phys. **115**, 201 (2006).]
[K.-I.K., Murakami & Shinohara, hep-th/0504198; Eur.Phys.C**42**, 475 (2005)](BRST)

§ Bridge between instanton and magnetic monopole

(i) For a given SU(2) Yang-Mills field $\mathbf{A}_\mu(x) = \mathbf{A}_\mu^A(x)\frac{\sigma_A}{2}$, the color field $\mathbf{n}(x)$ is obtained by solving the reduction differential equation (RDE):

$$\mathbf{n}(x) \times D_\mu[\mathbf{A}]D_\mu[\mathbf{A}]\mathbf{n}(x) = \mathbf{0},$$

where the color field has the unit length

$$\mathbf{n}(x) \cdot \mathbf{n}(x) = 1.$$

(ii) Once the color field $\mathbf{n}(x)$ is known, the gauge-invariant "magnetic-monopole current" k is constructed by

$$k := \delta * f = *df,$$

where f is the *gauge-invariant* two-form defined from the connection one-form $\mathbf{A}$ by

$$f_{\alpha\beta}(x) = \partial_\alpha[\mathbf{n}(x) \cdot \mathbf{A}_\beta(x)] - \partial_\beta[\mathbf{n}(x) \cdot \mathbf{A}_\alpha(x)] + ig^{-1}\mathbf{n}(x) \cdot [\partial_\alpha \mathbf{n}(x) \times \partial_\beta \mathbf{n}(x)].$$

The current k is conserved in the sense that $\delta k = 0$.
In $D = 4$ dimensions, $k_\mu = \frac{1}{2}\epsilon_{\mu\nu\alpha\beta}\partial_\nu f_{\alpha\beta}$. The magnetic charge $q_m = \int d^3\tilde{\sigma}_\mu k_\mu$ is gauge invariant and satisfies the Dirac quantization condition:

$$q_m = 4\pi g^{-1} n, \quad n \in \mathbb{Z} = \{\cdots, -2, -1, 0, +1, +2, \cdots\}$$

We now give a new form of the RDE (eigenvalue-like eq.):

$$-D_\mu[\mathbf{A}]D_\mu[\mathbf{A}]\mathbf{n}(x) = \lambda(x)\mathbf{n}(x) \quad (\lambda(x) \geq 0).$$

Once $\mathbf{n}(x)$ satisfying the RDE is known, the value of the reduction functional $F_{\rm rc}$ is immediately calculable as an integral of $\lambda(x)$ over the spacetime $\mathbb{R}^D$ as

$$\begin{aligned} F_{\rm rc} &= \int d^Dx \frac{1}{2}(D_\mu[\mathbf{A}]\mathbf{n}(x)) \cdot (D_\mu[\mathbf{A}]\mathbf{n}(x)) \\ &= \int d^Dx \frac{1}{2}\mathbf{n}(x) \cdot (-D_\mu[\mathbf{A}]D_\mu[\mathbf{A}]\mathbf{n}(x)) \\ &= \int d^Dx \frac{1}{2}\mathbf{n}(x) \cdot \lambda(x)\mathbf{n}(x) = \int d^Dx \frac{1}{2}\lambda(x). \end{aligned}$$

For a given Yang-Mills field $\mathbf{A}_\mu(x)$, look for the unit vector field $\mathbf{n}(x)$ such that $-D_\mu[\mathbf{A}]D_\mu[\mathbf{A}]\mathbf{n}(x)$ is proportional to $\mathbf{n}(x)$ with the smallest value of the reduction functional $F_{\rm rc}$ which is an integral of the scalar function $\lambda(x)$ over the spacetime $\mathbb{R}^D$.

Our method should be compared with that of the conventional Laplacian Abelian gauge (LAG). Bruckmann et al. [hep-th/0007119], Nucl.Phys.B **593**, 545–561 (2001).

§ Meron solution [De Alfaro, Fubini and Furlan, 1976, 1977]

One meron solution at the origin $x = 0$ (non pure gauge everywhere)

$$\mathbf{A}_\mu^{\mathrm{M}}(x) = g^{-1}\eta_{\mu\nu}^{A}\frac{x_\nu}{x^2}\frac{\sigma_A}{2} = \frac{1}{2}ig^{-1}U(x)\partial_\mu U^{-1}(x), \quad U(x) = \frac{\bar{e}_\alpha x_\alpha}{\sqrt{x^2}} \in SU(2)$$

$$D_P(x) := \frac{1}{16\pi^2}\mathrm{tr}(\mathbf{F}_{\mu\nu} * \mathbf{F}_{\mu\nu}) = \frac{1}{2}\delta^4(x), \quad Q_p := \int d^4x D_P(x) = \frac{1}{2}.$$

$$\downarrow \text{Conformal transformation : } x_\mu \to z_\mu = 2a^2\frac{(x+a)_\mu}{(x+a)^2} - a_\mu,$$

meron-antimeron solution (one meron at $x = a$ and one antimeron at $x = -a$)

$$\mathbf{A}_\mu^{\mathrm{M}}(x) \to \partial_\mu z_\nu \mathbf{A}_\nu^{\mathrm{M}}(z) = \ldots$$

$$\downarrow \text{Singular gauge transformation : } U(x+a),$$

meron-meron or dimeron solution (one meron at $x = a$ and another meron at $x = -a$)

$$\mathbf{A}_\mu^{\mathrm{MM}}(x) = -g^{-1}\left[\eta_{\mu\nu}^{A}\frac{(x+a)_\nu}{(x+a)^2} + \eta_{\mu\nu}^{A}\frac{(x-a)_\nu}{(x-a)^2}\right]\frac{\sigma_A}{2}, \quad D_P(x) = \frac{1}{2}\delta^4(x+a) + \frac{1}{2}\delta^4(x-a)$$

§ Smeared meron pair [Callan, Dashen and Gross, 1978]

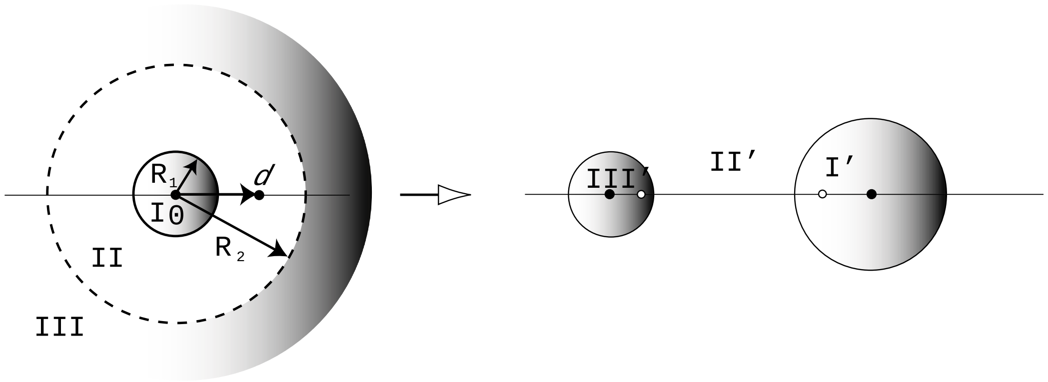


Figure 2: The concentric sphere geometry for a smeared meron (left panel) is transformed to the smeared two meron configuration (right panel) by the conformal transformation including the inversion about the point d.

$$\mathbf{A}_\mu^{\mathrm{sMM}}(x) = \frac{\sigma_A}{2}\eta^A_{\mu\nu}x_\nu \times \begin{cases} \frac{2}{x^2+R_1^2} & \text{I: } \sqrt{x^2} < R_1 \\ \frac{1}{x^2} & \text{II: } R_1 < \sqrt{x^2} < R_2 \\ \frac{2}{x^2+R_2^2} & \text{III: } \sqrt{x^2} > R_2 \end{cases},$$

$$S_{\mathrm{YM}}^{\mathrm{sMM}} = \frac{8\pi^2}{g^2} + \frac{3\pi^2}{g^2}\ln\frac{R_2}{R_1}, \quad Q_P^{\mathrm{I}} = \frac{1}{2}, \quad Q_P^{\mathrm{II}} = 0, \quad Q_P^{\mathrm{III}} = \frac{1}{2},$$

One-instanton limit: $|R_1 - R_2| \downarrow 0$ $(R_2/R_1 \downarrow 1)$.
One-meron limit: $R_2 \uparrow \infty$ or $R_1 \downarrow 0$ $(R_2/R_1 \uparrow \infty)$. $S_{\mathrm{YM}}^{\mathrm{sMM}}$ logarithmic divergence

§ Circular magnetic monopole loops joining the smeared meron pair

The RDE is conformal covariant and gauge covariant, while the reduction functional is conformal invariant and gauge invariant.
The minimum of the reduction functional is achieved by

$$\lambda(x) = \begin{cases} \frac{8x^2}{(x^2+R_1^2)^2} & \text{I: } 0 < \sqrt{x^2} < R_1; (J,L) = (1,0),\ \mathbf{n}_A(x) = Y^A_{(1,0)} = \text{const.} \\ \frac{2(\hat{b}\cdot x)^2}{x^2[x^2-(\hat{b}\cdot x)^2]} & \text{II: } R_1 < \sqrt{x^2} < R_2; (J,L) = (\frac{1}{2},\frac{1}{2}),\ \mathbf{n}_A(x) \simeq Y^A_{(1/2,1/2)} = \text{hedgehog} \\ \frac{8R_2^2}{x^2(x^2+R_2^2)^2} & \text{III: } R_2 < \sqrt{x^2}; (J,L) = (0,1),\ \mathbf{n}_A(x) = Y^A_{(0,1)}(x) = \text{Hopf} \end{cases}$$

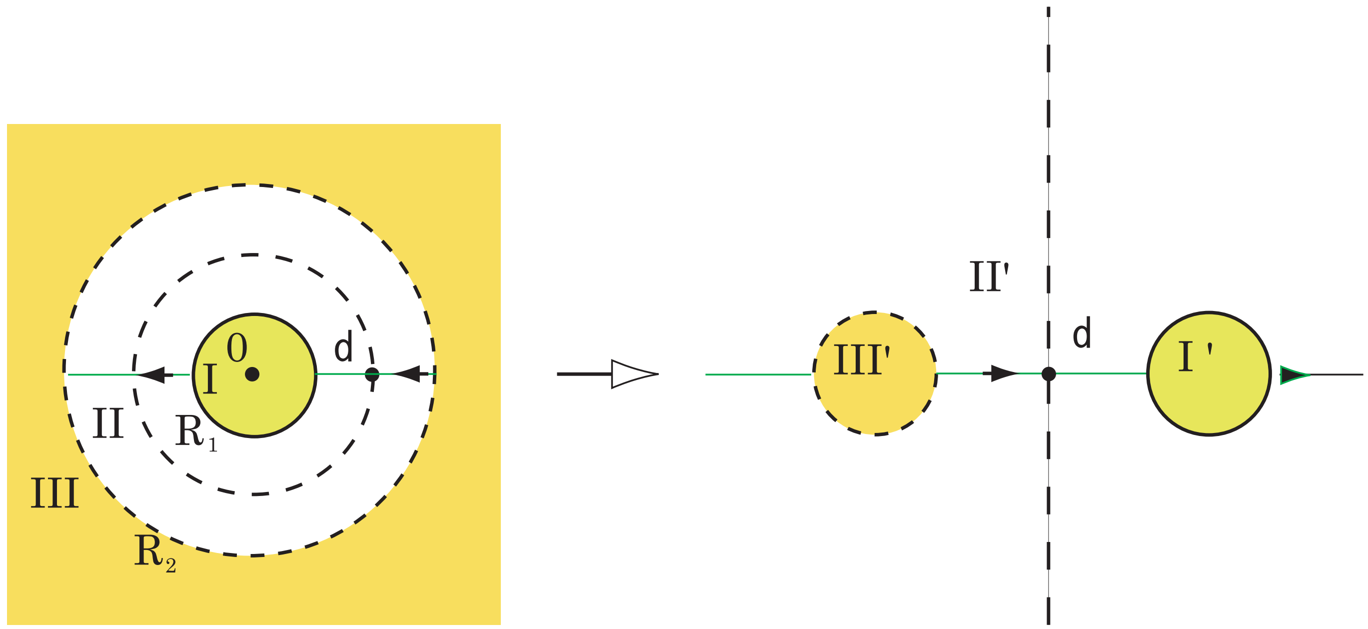
$$F_{\rm rc} = \int_{\mathbb{R}^4} d^4x \lambda(x) < \infty \quad \text{for } R_1, R_2 > 0.$$

After the conformal transformation and the singular gauge transformation,

$$\bar{\mathbf{n}}(x)_{\rm II'} = \frac{2a^2}{(x+a)^2}\hat{b}_\nu \eta^A_{\mu\nu} z_\mu U^{-1}(x+a)\sigma_A U(x+a)/\sqrt{z^2-(\hat{b}\cdot z)^2},$$

The magnetic monopole is dictated by the simultaneous zeros of $\hat{b}_\nu \eta^A_{\mu\nu} z_\mu$ for $A = 1,2,3$:

$$0 = \hat{b}_\nu \eta^A_{\mu\nu}[2a^2(x_\mu + a_\mu) - (x+a)^2 a_\mu] \quad (A = 1,2,3),$$

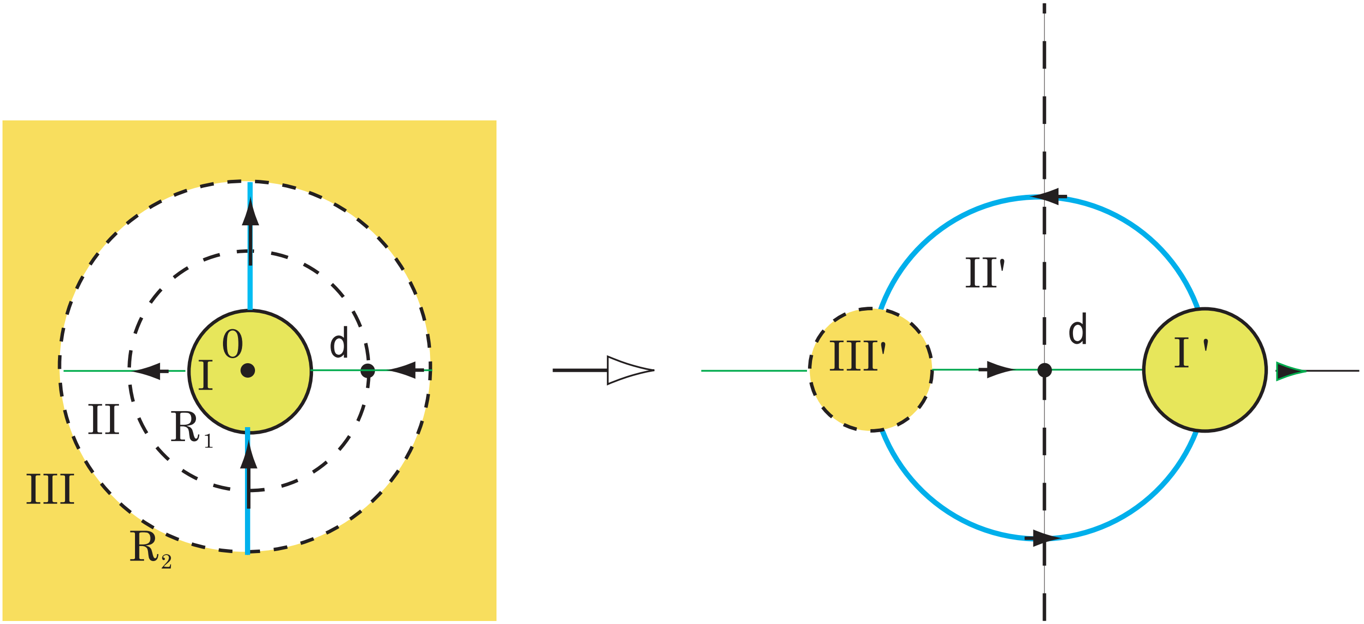


Without loss of generality, we can fix the direction of connecting two merons as $a_\mu := d_\mu/2 = \delta_{\mu 4} T$.

If $\hat{b}_\mu$ is parallel to a_μ, i.e., $\hat{b}_\mu = \delta_{\mu 4}$ (or $\hat{\mathbf{b}} = \mathbf{0}$),

$$x_A = 0 \quad (A = 1, 2, 3) \tag{1}$$

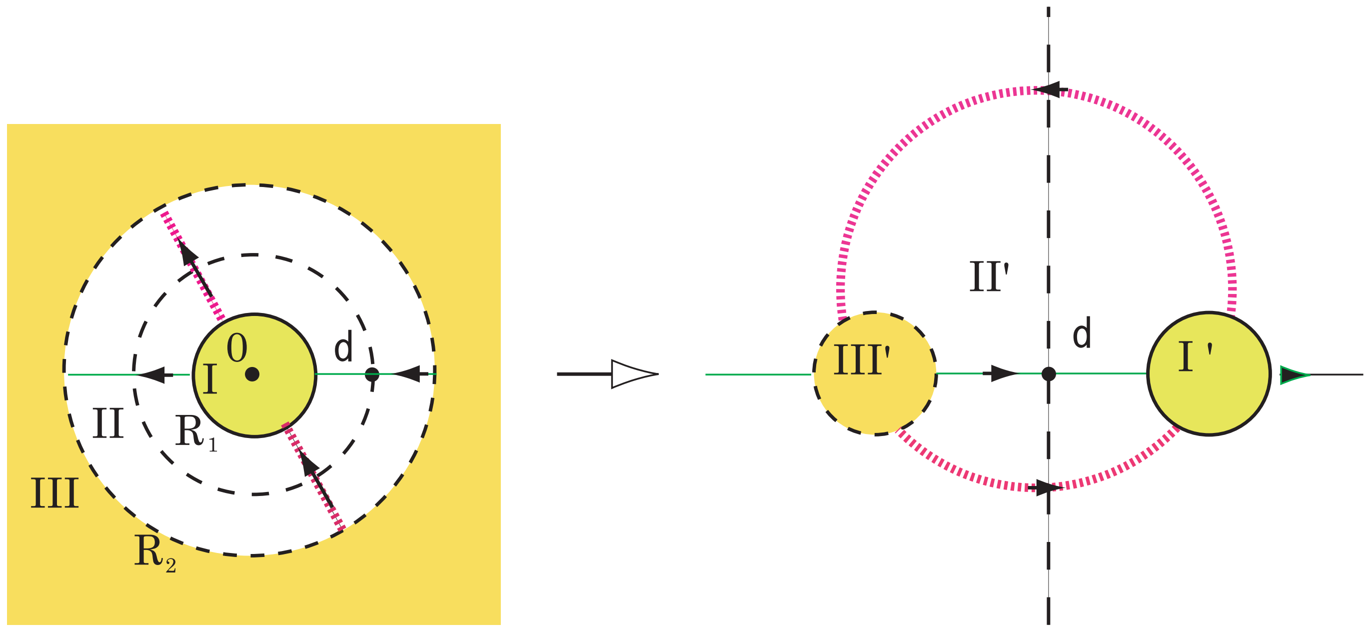
i.e., the magnetic current is a straight line going through two merons at $(\mathbf{0}, \pm T)$.



If $\hat{b}_\mu$ is perpendicular to a_μ (or $\hat{b}_\mu = \delta_{\mu\ell}\hat{b}_\ell$, $\ell = 1, 2, 3$), i.e., $\hat{b}_4 = 0$,

$$x_\ell^2 + x_4^2 = a^2. \tag{2}$$

a circular magnetic monopole loop with its center at the origin 0 in z space and the radius $\sqrt{a^2}$ joining two merons at $(\mathbf{0}, \pm T)$ exists on the plane spanned by a_μ and $\hat{b}_\ell$ $(\ell = 1, 2, 3)$.



Other chices of $\hat{b}_\mu = (\hat{\mathbf{b}}, \hat{b}_4)$

$$\mathbf{x} \times \hat{\mathbf{b}} = \mathbf{0} \quad \& \quad \left(\mathbf{x} + \frac{a\cdot\hat{b}}{|\hat{\mathbf{b}}|}\frac{\hat{\mathbf{b}}}{|\hat{\mathbf{b}}|}\right)^2 + x_4^2 = \left(a^2 + \frac{(a\cdot\hat{b})^2}{|\hat{\mathbf{b}}|^2}\right), \tag{3}$$

where $\hat{\mathbf{b}}$ is the three-dimensional part of unit four $\hat{b}_\mu$ $(\hat{b}_\mu\hat{b}_\mu = \hat{b}_4^2 + |\hat{\mathbf{b}}|^2 = 1)$.
These equations express circular magnetic monopole loops

the center at $\mathbf{x} = -\frac{a\cdot\hat{b}}{|\hat{\mathbf{b}}|}\frac{\hat{\mathbf{b}}}{|\hat{\mathbf{b}}|}$, $x_4 = 0$ with the radius $\sqrt{a^2 + \frac{(a\cdot\hat{b})^2}{|\hat{\mathbf{b}}|^2}}(\geq \sqrt{a^2})$

joining two merons at $\pm a_\mu$ on the plane specified by a_μ and $\hat{\mathbf{b}}$

§ Monopole and vortex content of a meron pair

- A. Montero and J.W. Negele, hep-lat/0202023, Phys.Lett.B533, 322-329 (2002).

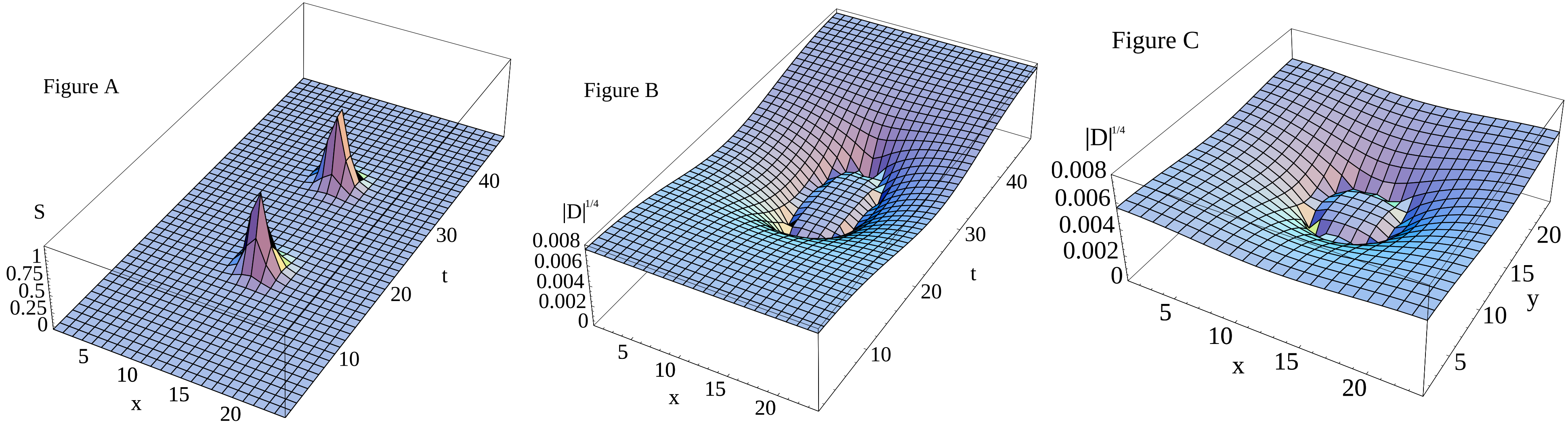


Figure 3: Figure A shows the action density S(t,x,y,z) for the meron pair with $d = 16$ and $c = 1$ (configuration IV) as a function of x and t, with y and z fixed to the values that maximize the action density. Figure B shows the absolute value of the discriminant of the three lowest Laplacian eigenvectors, $D(t, x, y, z)$, to the $1/4$ power as a function of x and t, for the same meron pair configuration and values of the y and z coordinates used in figure A. Figure C shows the absolute value of $D(t, x, y, z)$ to the $1/4$ power as a function of x and y for z fixed to the value that maximizes the action density and t fixed to the midpoint between the two merons.

§ Conclusion and discussion

Summarizing the results,

papers	original configuration	dual counterpart	method
CG95	one instanton	a straight magnetic line	MAG (analytical)
BOT96	one instanton	no magnetic loop	MAG (numerical)
BHVW00	one instanton	no magnetic loop	LAG (analytical)
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BOT96	instaton-antiinstanton	a magnetic loop	MAG (numerical)
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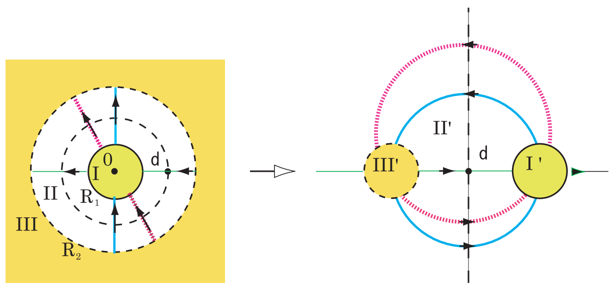
RT00	instaton-antiinstanton	two magnetic loops	LAG (numerical)
Ours	one instanton	no magnetic loop	New (analytical)
	one meron	a straight magnetic line	New (analytical)
KFSS08	two merons	circular magnetic loops	New (analytical)



We have obtained an analytical solution representing circular magnetic monopole loops (supported by a meron pair) in a dual description of the pure Yang-Mills theory.

In the original Yang-Mills theory, a pair of merons can be regarded as the dominant quark confiner. $\rightarrow$ area law of the Wilson loop average

$\odot$ Future subjects to be investigated:

• Extending our results to SU(3):
[K.-I. K., arXiv:0801.1274 [hep-th], Phys. Rev. D **77**, 085029 (2008)]
[K.-I. K., Shinohara and Murakami, arXiv:0803.0176 [hep-th], Prog. Theor. Phys. **120**, 1–50 (2008)]
For the Wilson loop in the fundamental rep.,

$$\boldsymbol{n} \in SU(3)/U(2) \neq SU(3)/[U(1) \times U(1)]$$

• Relationship between other topological objects: For gauge-invariant vortices equivalent to center vortices,
[K.-I. K., arXiv:0802.3829 [hep-th], J. Phys. G: Nucl. Part. Phys. **35**, 085001 (2008)]

• Clarifying the role of elliptic solutions interpolating dimeron and one-instanton:
Cervero, Jacobs & Nohl (1977). one-parameter family of solutions,
k=0: meron, k=1: instanton
dissociation of an instanton into two merons?

• Obtaining the integration measure for collective coordinates:
of circular magnetic monopole loops

• Considering the relationship with the Gribov problem:
non-trivial Coulomb gauge vacua with $Q_P = \pm 1/2$

• D-brane intepretation: D-0 brane $\leftrightarrow$ meron
Drukker, Gross and Itzhaki, [hep-th/0004131], Phys.Rev.D**62**,086007 (2000).

• Evaluating the Wilson loop average ...

Thank you for your attention!

§ Instantons and monopoles

A. Hart and M. Teper, e-Print:hep-lat/9511016, Phys.Lett.B371: 261-269, 1996.

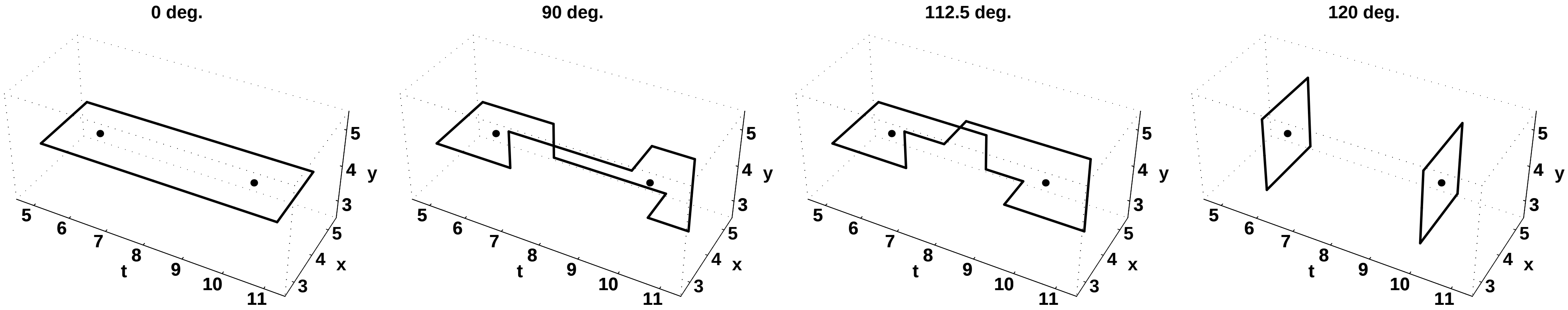


Figure 4: Three dimensional projections of the mutual monopole loop surrounding an instanton–anti-instanton pair (centres marked) of size $\rho = 3$ under increasing rotation angle as detailed in the text. The loops are flat in the fourth direction.

• Reinhardt and Tok, [hep-th/0011068], Phys.Lett.B **505**, 131–140 (2001). hep-th/0009205.

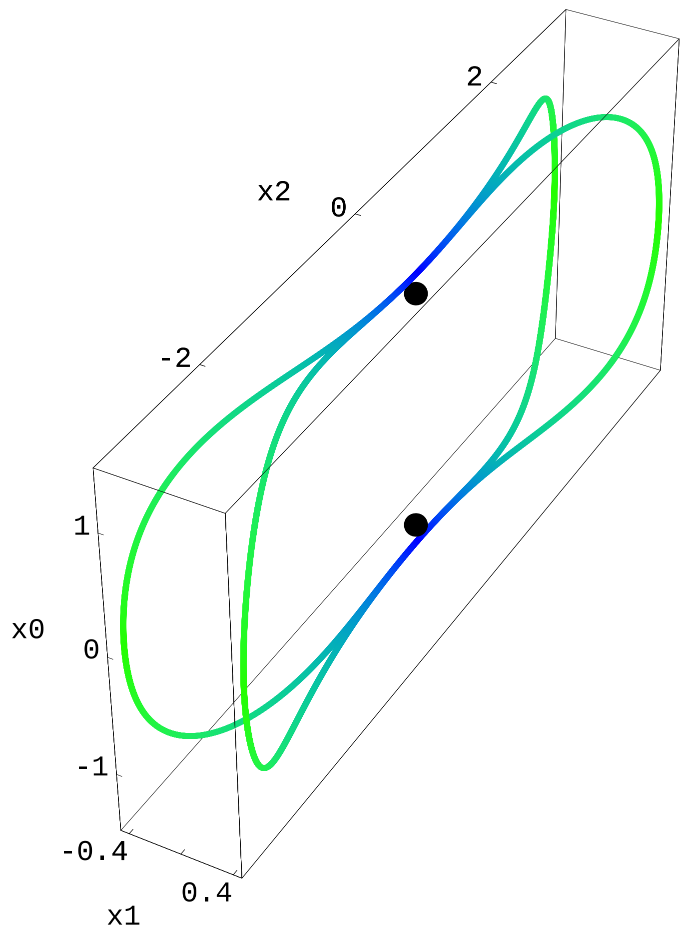


Figure 5: Plot of the two magnetic monopole loops for the gauge potential (**??**) projected onto the $x_1 - x_2 - x_0$-space (dropping the x_3-component). Rotations with angle π around the x_1- , x_2- and x_3-axis interchange the different monopole branches. The thick dots show the positions of the instantons.

§ Simplifying RDE

(1) First, we adopt the CFtHW Ansatz:

$$g\mathbf{A}_\mu(x) = \frac{\sigma_A}{2} g\mathbf{A}_\mu^A(x) = \frac{\sigma_A}{2}\eta_{\mu\nu}^A f_\nu(x), \quad f_\nu(x) := \partial_\nu \ln \Phi(x),$$

where $\eta_{\mu\nu}^A = \eta^{(+)A}_{\mu\nu}$ is the symbol defined by $\eta_{\mu\nu}^A \equiv \eta^{(+)A}_{\mu\nu} := \epsilon_{A\mu\nu 4} + \delta_{A\mu}\delta_{\nu 4} - \delta_{\mu 4}\delta_{A\nu}$.

$$\{[-\partial_\mu\partial_\mu + 2f_\mu f_\mu]\delta_{AB} + 2\epsilon_{ABC}\eta_{\mu\nu}^C f_\nu(x)\partial_\mu\}\mathbf{n}_B(x) = \lambda(x)\mathbf{n}_A(x).$$

The Yang-Mills field in the CFtHW Ansatz satisfies simultaneously

the Lorentz gauge:

$$\partial_\mu \mathbf{A}_\mu^A(x) = 0,$$

and the maximal Abelian gauge (MAG):

$$D_\mu[\mathbf{A}^3]\mathbf{A}_\mu^\pm(x) := (\partial_\mu - ig\mathbf{A}_\mu^3)(\mathbf{A}_\mu^1(x) \pm i\mathbf{A}_\mu^2(x)) = 0.$$

(2) SO(4) symmetry: The angular part is expressed in terms of angular momentum derived from the decomposition: $so(4) \cong su(2) + su(2)$. The generators of SO(4):

$$L_{\mu\nu} = -i(x_\mu \partial_\nu - x_\nu \partial_\mu), \quad \mu, \nu \in \{1, 2, 3, 4\}.$$

yield two independent SU(2) generators ($A \in \{1, 2, 3\}$):

$$M_A := \frac{1}{2}(\mathscr{L}_A - \mathscr{K}_A) = -\frac{i}{2}\bar{\eta}^A_{\mu\nu} x_\mu \partial_\nu, \quad N_A := \frac{1}{2}(\mathscr{L}_A + \mathscr{K}_A) = -\frac{i}{2}\eta^A_{\mu\nu} x_\mu \partial_\nu,$$

using $\mathscr{L}_j := \frac{1}{2}\epsilon_{jk\ell} L_{k\ell}$, $\mathscr{K}_j := L_{j4}$, $j, k, \ell \in \{1, 2, 3\}$. The Casimir operators $\vec{M}^2 := M_A M_A$ and $\vec{N}^2 := N_A N_A$ with eigenvalues half-integers:

$$\vec{M}^2 := M_A M_A \to M(M+1), \quad M \in \{0, 1/2, 1, 3/2, \cdots\},$$

The generators for isospin $S = 1$ are

$$(S_A)_{BC} := i\epsilon_{ABC} = (S_C)_{AB}.$$

It is easy to see that $\vec{S}^2$ is a Casimir operator and $\vec{S}^2$ has the eigenvalue

$$\vec{S}^2 := S_A S_A \to S(S+1) = 2,$$

since $(\vec{S}^2)_{AB} = (S_C)_{AD}(S_C)_{DB} = i\epsilon_{DCA} i\epsilon_{BCD} = 2\delta_{AB}$.

Now we introduce the conserved total angular momentum $\vec{J}$ by

$$\vec{J} = \vec{L} + \vec{S},$$

with the eigenvalue where $\vec{L} = \vec{M}$ or $\vec{L} = \vec{N}$.

$$\vec{J}^2 \to J(J+1), \quad J \in \{L+1, L, |L-1|\},$$

Thus, a complete set of commuting observables is given by the Casimir operators, $\vec{J}^2$, $\vec{L}^2$, $\vec{S}^2$ and their projections, e.g., J_z, L_z, S_z.

By using $\vec{S} \cdot \vec{L} = (\vec{J}^2 - \vec{L}^2 - \vec{S}^2)/2$, the RDE is rewritten in the form: the RDE is rewritten in the form: $f_\nu(x) := \partial_\nu \ln \tilde{\Phi}(x^2) = x_\nu f(x)$

$$\{-\partial_\mu \partial_\mu \delta_{AB} + 2f(x)(\vec{J}^2 - \vec{L}^2 - \vec{S}^2)_{AB} + x_\mu x_\mu f^2(x)(\vec{S}^2)_{AB}\}\mathbf{n}_B(x) = \lambda(x)\mathbf{n}_A(x),$$

(3) The symmetry suggests that $\mathbf{n}(x)$ is separated into the radial and angular part:

$$
\begin{aligned}
\boldsymbol{n}(x) =& \mathbf{n}_A(x)\sigma_A = \psi(R) Y^A_{(J,L)}(\hat{x})\sigma_A, \\
& R := \sqrt{x_\mu x_\mu} \in \mathbb{R}_+, \quad \hat{x}_\mu := x_\mu / R \in S^3
\end{aligned}
$$

where $\vec{Y}_{(J,L)}(\hat{x}) = \{Y^A_{(J,L)}(\hat{x})\}_{A=1,2,3}$ is the vector spherical harmonics on S^3:

$$
\begin{aligned}
\vec{L}^2 Y^A_{(J,L)}(\hat{x}) =& L(L+1) Y^A_{(J,L)}(\hat{x}), \\
\vec{J}^2 Y^A_{(J,L)}(\hat{x}) =& J(J+1) Y^A_{(J,L)}(\hat{x}), \\
\vec{S}^2 Y^A_{(J,L)}(\hat{x})\sigma_A =& S(S+1) Y^A_{(J,L)}(\hat{x})\sigma_A,
\end{aligned}
$$

In this form, the covariant Laplacian reduces to the diagonal form

$$
\begin{aligned}
& [-\partial_\mu \partial_\mu + V(x)] \mathbf{n}_A(x) = \lambda(x) \mathbf{n}_A(x), \\
& V(x) := 2f(x)[J(J+1) - L(L+1) - 2] + 2x^2 f^2(x).
\end{aligned}
$$

Using $-\partial_\mu \partial_\mu = -\partial_R \partial_R - \frac{3}{R}\partial_R + \frac{4\vec{L}^2}{R^2}$, we arrive at

$$
[-\partial_R \partial_R - (3/R)\partial_R + \tilde{V}(x)] \mathbf{n}_A(x) = \lambda(x) \mathbf{n}_A(x), \quad \tilde{V}(x) := 4L(L+1)/x^2 + V(x).
$$

(4) Unit vector condition and angular part

We now take into account the fact that $\mathbf{n}(x)$ has the unit length:

$$1 = \mathbf{n}_A(x)\mathbf{n}_A(x) = \psi(R)\psi(R)Y^A_{(J,L)}(\hat{x})Y^A_{(J,L)}(\hat{x}).$$

If the vector spherical harmonics happens to be normalized at every spacetime point as

$$1 = Y^A_{(J,L)}(\hat{x})Y^A_{(J,L)}(\hat{x}),$$

then we can take without loss of generality

$$\psi(R) \equiv 1.$$

Then, $\mathbf{n}(x)$ is determined only by the vector spherical harmonics:

$$\mathbf{n}_A(x) = Y^A_{(J,L)}(\hat{x}).$$

However, $1 = Y^A_{(J,L)}(\hat{x})Y^A_{(J,L)}(\hat{x})$ is not guaranteed for any set of (J, L) except for some special cases.
Usually, the orthonormality of the vector spherical harmonics is given with respect to the integral over S^3 with a finite volume: $\int_{S^3} d\Omega\, Y^A_{(J,L)}(\hat{x})Y^A_{(J',L')}(\hat{x}) = \delta_{JJ'}\delta_{LL'}$.

§ One instanton case [Simple reproduction of essentials of BOT & BHVW]

One-instanton configuration in the regular gauge with zero size: $f(x) = \frac{2}{x^2}$, and

$$V(x) = \frac{4}{x^2}[J(J+1) - L(L+1)], \quad \tilde{V}(x) = \frac{4}{x^2}J(J+1) \geq 0.$$

$(J, L) = (0, 1)$ gives the lowest value of $\tilde{V}(x)$ at every x. The lowest value of $\lambda(x) \geq 0$ is obtained $\lambda(x) = \tilde{V}(x) = 0$ by setting $\psi(R) \equiv \mathrm{const.}$ if the corresponding vector harmonics is orthonormal.

The vector spherical harmonics $Y_{(0,1)}(\hat{x})$ is 3-fold degenerate $(B = 1, 2, 3)$:

$$\boldsymbol{Y}_{(0,1)}(\hat{x}) = \sum_{B=1}^{3} \hat{a}_B \boldsymbol{Y}_{(0,1),(B)}(\hat{x})$$
$$= \hat{a}_1 \begin{pmatrix} \hat{x}_1^2 - \hat{x}_2^2 - \hat{x}_3^2 + \hat{x}_4^2 \\ 2(\hat{x}_1\hat{x}_2 - \hat{x}_3\hat{x}_4) \\ 2(\hat{x}_1\hat{x}_3 + \hat{x}_2\hat{x}_4) \end{pmatrix} + \hat{a}_2 \begin{pmatrix} 2(\hat{x}_1\hat{x}_2 + \hat{x}_3\hat{x}_4) \\ -\hat{x}_1^2 + \hat{x}_2^2 - \hat{x}_3^2 + \hat{x}_4^2 \\ 2(\hat{x}_2\hat{x}_3 - \hat{x}_1\hat{x}_4) \end{pmatrix} + \hat{a}_3 \begin{pmatrix} 2(\hat{x}_1\hat{x}_3 - \hat{x}_2\hat{x}_4) \\ 2(\hat{x}_2\hat{x}_3 + \hat{x}_1\hat{x}_4) \\ -\hat{x}_1^2 - \hat{x}_2^2 + \hat{x}_3^2 + \hat{x}_4^2 \end{pmatrix},$$

It is easy to check that $Y_{(0,1)}(\hat{x})$ are orthonormal at every point: $\boldsymbol{Y}_{(0,1),(B)}(\hat{x}) \cdot \boldsymbol{Y}_{(1,0),(C)}(\hat{x}) := Y^A_{(0,1),(B)}(\hat{x}) Y^A_{(1,0),(C)}(\hat{x}) = \delta_{BC}$.

The solution is given by the manifestly Lorentz covariant Lie-algebra valued form:

$$\boldsymbol{n}(x) := \mathbf{n}_A(x)\sigma_A = \hat{a}_B Y^A_{(0,1),(B)}(\hat{x})\sigma_A = \hat{a}_B x_\alpha \bar{e}_\alpha \sigma_B x_\beta e_\beta / x^2,$$

$\bar{e}_\mu = (i\sigma_A, \mathbf{1}), \quad e_\mu := (-i\sigma_A, \mathbf{1})$, or in the vector component

$$\mathbf{n}_A(x) = \hat{a}_B Y^A_{(0,1),(B)}(\hat{x}) = \hat{a}_B x_\alpha x_\beta \bar{\eta}^B_{\alpha\gamma} \eta^A_{\gamma\beta} / x^2.$$

It is directly checked that it is indeed the solution of the RDE:

$$-\partial_\mu \partial_\mu \mathbf{n}_A(x) = \frac{8}{x^2} \mathbf{n}_A(x),$$

$$2\epsilon_{ABC} \eta^C_{\mu\nu} f_\nu(x) \partial_\mu \mathbf{n}_B(x) = -8 f(x) \mathbf{n}_A(x) = -\frac{16}{x^2} \mathbf{n}_A(x).$$

Then, for $(J, L) = (0, 1)$, we arrive at $V(x) = -8/x^2 \ \tilde{V}(x) = 0$,

$$\lambda(x) = V(x) + [-\partial_\mu \partial_\mu \mathbf{n}_A(x)]/\mathbf{n}_A(x) \equiv 0 \quad \text{for any A, no sum over A.}$$

Thus this solution is an allowed one, since the solution gives a finite (vanishing) value for the functional F_{rc}=0.

The solution gives a map $\boldsymbol{Y}_{(0,1),(B)}$ from S^3 to S^2, which is known as the standard Hopf map. Therefore, the only zeros of $\phi_A(x)$ in the solution $\mathbf{n}_A(x) = \phi_A(x)/|\phi(x)| = \phi_A(x)/\sqrt{\phi_B(x)\phi_B(x)}$ are the origin and the set of magnetic monopoles consists of the origin only, in other words, the magnetic monopole loop is shrank to a single point. Therefore, we have no monopole loop with a finite and non-zero radius for the Yang-Mills field of one instanton with zero size in the regular gauge.

For one instanton with size ρ, $f(x^2) = \frac{2}{x^2+\rho^2}$,

$$V(x) = \frac{4}{x^2+\rho^2}[J(J+1) - L(L+1)] - \frac{8\rho^2}{(x^2+\rho^2)^2}.$$

The lowest $\lambda(x)$ is realized for distinct set of (J, L) depending on the region of x. This case is obtained by one-instanton limit of two meron case to be discussed later.

One instanton in the singular gauge

$$gA_\mu(x) = \frac{\sigma_A}{2}\bar{\eta}^A_{\mu\nu}x_\nu f(x^2), \quad f(x^2) = \frac{2\rho^2}{x^2(x^2+\rho^2)}.$$

The results in the previous section hold by replacing $\eta^A_{\mu\nu}$ by $\bar{\eta}^A_{\mu\nu}$. In this case, we have

$$V(x) = \frac{4\rho^2}{x^2(x^2+\rho^2)}[J(J+1) - L(L+1) - 2] + \frac{8\rho^4}{x^2(x^2+\rho^2)^2}.$$

We focus on the zero size limit $\rho \to 0$ (or the distant region $x^2 \to \infty$):

$$V(x) \simeq 0, \quad \tilde{V}(x) \simeq 4L(L+1)/x^2.$$

The solution is given at $(J, L) = (1, 0)$, i.e., $\mathbf{n}(x) = \boldsymbol{Y}_{(1,0)}$, which has the lowest value of $\lambda(x)$: $\lambda(x) \equiv 0$. For $(J, L) = (1, 0)$, the state is 3-fold degenerate: $\mathbf{n}(x) = \boldsymbol{Y}_{(1,0)}$ is written as a linear combination of them: $\boldsymbol{Y}_{(1,0)} = (Y^1_{(1,0)}, Y^2_{(1,0)}, Y^3_{(1,0)})^T$

$$\boldsymbol{Y}_{(1,0)} = \sum_{\alpha=1}^{3} \hat{c}_\alpha \boldsymbol{Y}_{(1,0),(\alpha)} = \hat{c}_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \hat{c}_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \hat{c}_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

It constitutes the orthonormal set: $\boldsymbol{Y}_{(1,0),(\alpha)} \cdot \boldsymbol{Y}_{(1,0),(\beta)} := Y^A_{(1,0),(\alpha)} Y^A_{(1,0),(\beta)} = \delta_{\alpha\beta}$.

Therefore, the solution is given by a constant:

$$\mathbf{n}_A(x) = \sum_{\alpha=1}^{3} \hat{c}_\alpha Y^A_{(1,0),(\alpha)} = \hat{c}_A.$$

In this case, $\partial_\mu \mathbf{n}_A(x) = 0$, $\partial_\mu \partial_\mu \mathbf{n}_A(x) = 0$ and

$$\lambda(x) = V(x) = 2x^2 f^2(x) = \frac{8\rho^4}{x^2(x^2+\rho^2)^2}. \tag{1}$$

One-instanton in the singular gauge yields a finite reduction functional:

$$F_{\mathrm{rc}} = \int d^4x \lambda(x) < \infty. \tag{2}$$

§ One meron case [Simple reproduction of Reinhardt & Tok (2001)]

One-meron configuration, $f(x^2) = \frac{1}{x^2}$,

$$V(x) = \frac{2}{x^2}[J(J+1) - L(L+1) - 1], \quad \tilde{V}(x) = \frac{2}{x^2}[J(J+1) + L(L+1) - 1] > 0.$$

For one meron, we find that $(J, L) = (1/2, 1/2)$ gives the lowest $\tilde{V}(x)$. This suggests that the solution might be given by $\boldsymbol{Y}_{(1/2,1/2),(\mu)}(\hat{x}) = \eta^A_{\mu\nu}\hat{x}_\nu \quad (\mu = 1, 2, 3, 4)$

$$\begin{aligned}\boldsymbol{Y}_{(1/2,1/2)}(\hat{x}) &= \sum_{\mu=1}^{4} \hat{b}_\mu \boldsymbol{Y}_{(1/2,1/2),(\mu)}(\hat{x}) \\ &= \hat{b}_1 \begin{pmatrix} -\hat{x}_4 \\ \hat{x}_3 \\ -\hat{x}_2 \end{pmatrix} + \hat{b}_2 \begin{pmatrix} -\hat{x}_3 \\ -\hat{x}_4 \\ \hat{x}_1 \end{pmatrix} + \hat{b}_3 \begin{pmatrix} \hat{x}_2 \\ -\hat{x}_1 \\ -\hat{x}_4 \end{pmatrix} + \hat{b}_4 \begin{pmatrix} \hat{x}_1 \\ \hat{x}_2 \\ \hat{x}_3 \end{pmatrix},\end{aligned}$$

where a unit four-vector $\hat{b}_\mu$ $(\mu = 1, 2, 3, 4)$ denote four coefficients of the linear combination for 4-fold generate $\boldsymbol{Y}_{(1/2,1/2),(\mu)}(\hat{x})$ $(\mu = 1, 2, 3, 4)$.

However, $Y^A_{(1/2,1/2),(\mu)}(\hat{x})$ are non-orthonormal sets at every spacetime point: $\boldsymbol{Y}_{(1/2,1/2),(\mu)}(\hat{x}) \cdot \boldsymbol{Y}_{(1/2,1/2),(\nu)}(\hat{x}) \neq \delta_{\mu\nu}$.

Nevertheless, we find that the unit vector field:

$$\mathbf{n}_A(x) = \hat{b}_\nu \eta^A_{\mu\nu} \hat{x}_\mu / \sqrt{1 - (\hat{b} \cdot \hat{x})^2},$$

constructed from $\boldsymbol{Y}_{(1/2,1/2),(\mu)}(\hat{x}) = \eta^A_{\mu\nu}\hat{x}_\nu \quad (\mu = 1,2,3,4)$, can be a solution of RDE. In fact,

$$-\partial_\mu \partial_\mu \mathbf{n}_A(x) = \frac{2}{x^2 - (\hat{b} \cdot x)^2} \mathbf{n}_A(x),$$

$$2\epsilon_{ABC} \eta^C_{\mu\nu} f_\nu(x) \partial_\mu \mathbf{n}_B(x) = -4f(x)\mathbf{n}_A(x) = -\frac{4}{x^2}\mathbf{n}_A(x).$$

Then, for $(J, L) = (1/2, 1/2)$, we conclude that $V(x) = -2/x^2, \quad \tilde{V}(x) = 1/x^2$,

$$\lambda(x) = \frac{2(\hat{b} \cdot x)^2}{x^2[x^2 - (\hat{b} \cdot x)^2]}.$$

The solution is of the hedgehog type. The magnetic monopole current is obtained as simultaneous zeros of $\hat{b}_\nu \eta^A_{\mu\nu} x_\mu = 0$ for $A = 1, 2, 3$.

Taking the 4th vector, the magnetic monopole current is located at $x_1 = x_2 = x_3 = 0$, i.e., on the x_4 axis. Whereas, if the 3rd vector is taken $\hat{b}_\mu = \delta_{\mu 3}$, the magnetic monopole current flows at $x_1 = x_2 = x_4 = 0$, i.e., on x_3 axis. In general, it turns out that the magnetic monopole current k_μ is located on the straight line parallel to $\hat{b}_\mu$ going through the origin.

The obtained $\lambda(x)$ is invariant under a subgroup $SO(3)$ of the Euclidean rotation $SO(4)$. In other words, once we select $\hat{b}_\mu$, $SO(4)$ symmetry is broken to $SO(3)$ just as in the spontaneously broken symmetry. This result is consistent with a fact that the magnetic monopole current k_μ flows in the direction of $\hat{b}_\mu$ and the symmetry is reduced to the axial symmetry, the rotation group $SO(3)$, about the axis in the direction of a four vector $\hat{b}_\mu$.

It is instructive to point out that the Hopf map $Y_{(0,1)}$ also satisfies the RDE. Therefore, it is necessary to compare the value of the reduction functional of $(J, L) = (1/2, 1/2)$ with that of $(J, L) = (0, 1)$. In the $(J, L) = (0, 1)$ case, we find

$$\lambda_{(0,1)}(x) = \frac{2}{x^2} = \frac{2}{x_1^2 + x_2^2 + x_3^2 + x_4^2}.$$

For instance, we can choose $\hat{b}_\mu = \delta_{\mu 3}$ without loss of generality:

$$\lambda_{(1/2,1/2)}(x) = \frac{2x_3^2}{[x_1^2+x_2^2+x_3^2+x_4^2][x_1^2+x_2^2+x_4^2]}.$$

Note that the integral of $\lambda_{(1/2,1/2)}(x)$ over the whole spacetime $\mathbb{R}^4$ is obviously smaller than that of $\lambda_{(0,1)}(x)$, although $\lambda_{(0,1)}(x) < \lambda_{(1/2,1/2)}(x)$ locally inside a cone with the symmetric axis $\hat{b}_\mu$, i.e., $(\hat{b}\cdot\hat{x})^2 \geq 1/2$.

The reduction functional in $(J,L)=(1/2,1/2)$ case reads

$$F_{\rm rc} = 4\pi^2 \int_0^{L_3} dx_3 x_3,$$

where we have defined $r^2 := x_1^2 + x_2^2 + x_4^2$.

Although $F_{\rm rc}$ remains finite as long as L_3 is finite, it diverges for $L_3 \to \infty$, i.e, when integrated out literally in the whole spacetime $\mathbb{R}^4$. In the next section, we see that this difficulty is resolved for two meron configuration.