Instantons in Deformed Super Yang-Mills Theories

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1. Introduction

non-perturbative properties of supersymmetric gauge theory from string theory

- Effective field theory on D-brane $\rightarrow U(N)$ super Yang-Mills
- We turn on the constant closed string backgrounds.
  (NS-NS B-field, R-R fields)
  - Nekrasov formula for $\mathcal{N}=2$ super Yang-Mills
  - Dijkgraaf-Vafa theory for $\mathcal{N}=1$ supersymmetric gauge theory

  Closed string backgrounds play an important role in these cases.

★ Here we will consider $\mathcal{N}=2$ super Yang-Mills theories in the background of R-R 3-form (with fixed $(2\pi\alpha')^{1/2}\mathcal{F}$).

[Billo-Frau-Fucito-Lerda, 2006]

– Consider D3-D(-1) system in type IIB on $\mathbf{R}^2 \times \mathbf{R}^4/\mathbf{Z}_2$ with constant self-dual R-R 3-form. $(\mathbf{R}^2 \times \mathbf{R}^4/\mathbf{Z}_2 \Rightarrow \mathcal{N}=2)$

– Calculate the deformed effective action of D(-1)-branes ($=$ instanton from the view point of D3 side).

– And compute the instanton partition function using the localization technique.

$\Rightarrow$ R-R 3-form $=$ Ω-background

$$\begin{cases} \text{Derivation from D3 side (the deformed spacetime action)} \\ \text{Extension to the case of } \mathcal{N}=4 \text{ (and } \mathcal{N}=2^* \text{ etc.)} \end{cases}$$

2. Deformation in $\mathcal{N}=2$ Super Yang-Mills

Procedure:

– Consider (fractional) D3-brane in type IIB on $\mathbf{R}^2 \times \mathbf{R}^4/\mathbf{Z}_2$.
We forget about $\mathbf{Z}_2$-orbifolding for a moment ($\Rightarrow \mathcal{N}=4$).

– Calculate the disk amplitude with/without the insertion of
the vertex operator for R-R 3-form $\mathcal{F}^{\mu\nu a}$.

$$V_{\mathcal{F}}(z,\bar{z}) = (2\pi\alpha')\,\mathcal{F}^{\mu\nu a}(\sigma_{\mu\nu})^{\alpha\beta}(\Sigma_a)^{AB} \times \left[S_\alpha(z)S_A(z)e^{-\frac{1}{2}\phi(z)}S_\beta(\bar{z})S_B(\bar{z})e^{-\frac{1}{2}\phi(\bar{z})}\right].$$

$\mu,\nu = 1,\ldots,4$: 4D indices, $\quad A,B = 1,\ldots,4$: $SU(4)_R$ indices,
$a,b = 5,\ldots,10$: transverse 6D indices ($SO(6)_R$).

• $\mathcal{N}=4$ deformation term ($C^{\mu\nu a} \propto (2\pi\alpha')^{1/2}\mathcal{F}^{\mu\nu a}$ kept finite)

$$\mathcal{L}_C =$$
$$\mathrm{Tr}\left[g C^{\mu\nu a}\left(i\varphi_a F_{\mu\nu} - \frac{1}{2}(\bar{\Sigma}_a)_{AB}\Lambda^A \sigma_{\mu\nu}\Lambda^B \right) + \frac{g^2}{2} C^{\mu\nu a} C_{\mu\nu}{}^{b} \varphi_a \varphi_b \right].$$

• orbifolding

$\mathcal{N}=4$: type IIB on $\mathbf{R}^4 \times \mathbf{R}^6 \Rightarrow$
$\mathcal{N}=2$: type IIB on $\mathbf{R}^4 \times \mathbf{R}^2 \times \mathbf{R}^4/\mathbf{Z}_2$

projection of the fields

$$\Lambda^A_\alpha = 0 \text{ for } A = 3,4, \quad \varphi_a,\ C^{\mu\nu a} = 0 \text{ for } a = 7,\ldots,10.$$

Under this reduction, $\mathcal{L}_{\mathcal{N}=4}$ becomes $\mathcal{L}_{\mathcal{N}=2}$. $\quad (A \to I = 1, 2)$

$$\mathcal{L}_{\mathcal{N}=2} = \mathrm{Tr}\left[-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - i\Lambda^{I\alpha}(\sigma^{\mu})_{\alpha\dot{\beta}}D_{\mu}\bar{\Lambda}_{I}^{\dot{\beta}} - D_{\mu}\varphi D^{\mu}\bar{\varphi}\right.$$
$$\left. -\frac{i}{\sqrt{2}}g\Lambda^{I}[\bar{\varphi}, \Lambda_{I}] + \frac{i}{\sqrt{2}}g\bar{\Lambda}_{I}[\varphi, \bar{\Lambda}^{I}] - \frac{1}{2}g^{2}[\varphi, \bar{\varphi}]^{2}\right].$$

$\mathcal{N} = 2$ deformation term $\quad$ ([Billo-Frau-Fucito-Lerda] for $\bar{C} = 0$)

$\mathcal{L}_{C} =$

$$\mathrm{Tr}\left[ig(C^{\mu\nu}\bar{\varphi} + \bar{C}^{\mu\nu}\varphi)F_{\mu\nu} + \frac{i}{\sqrt{2}}g\bar{C}^{\mu\nu}\Lambda^{I}\sigma_{\mu\nu}\Lambda_{I} + \frac{g^{2}}{2}(C^{\mu\nu}\bar{\varphi} + \bar{C}^{\mu\nu}\varphi)^{2}\right].$$

Here $C^{\mu\nu}$ and $\bar{C}^{\mu\nu}$ are defined by

$$C^{\mu\nu} = -\frac{i}{\sqrt{2}}(C^{\mu\nu5} + iC^{\mu\nu6}), \quad \bar{C}^{\mu\nu} = \frac{i}{\sqrt{2}}(C^{\mu\nu5} - iC^{\mu\nu6}).$$

3. Instanton Solution of Deformed $\mathcal{N}=2$ Super Yang-Mills

- gauge field part

$$
\begin{aligned}
\mathcal{L}_{\mathrm{E}} &= \mathrm{Tr}\left[\tfrac{1}{2}(F^{-}_{\mu\nu})^2\right] + \cdots \\
&= \mathrm{Tr}\left[\tfrac{1}{2}\Big(F^{+}_{\mu\nu} - ig(C^{\mu\nu}\bar{\varphi} + \bar{C}^{\mu\nu}\varphi)\Big)^2\right] + \cdots ,
\end{aligned}
$$

where $F^{\pm}_{\mu\nu} = \frac{1}{2}(F_{\mu\nu} \pm \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma})$. This means that the (anti-)self-dual equations are

$$
\begin{aligned}
F^{-}_{\mu\nu} &= 0 \quad \text{for self-dual case,} \\
F^{+}_{\mu\nu} - ig(C^{\mu\nu}\bar{\varphi} + \bar{C}^{\mu\nu}\varphi) &= 0 \quad \text{for anti-self-dual case.}
\end{aligned}
$$

• weak coupling expansion for self-dual case

$$
\begin{aligned}
A_\mu &= g^{-1}A_\mu^{(0)} + g^{1}A_\mu^{(1)} + \cdots , & F_{\mu\nu} &= g^{-1}F_{\mu\nu}^{(0)} + g^{1}F_{\mu\nu}^{(1)} + \cdots , \\
\Lambda^I &= g^{-\frac{1}{2}}\Lambda^{(0)I} + g^{\frac{3}{2}}\Lambda^{(1)I} + \cdots , & \bar\Lambda_I &= g^{\frac{1}{2}}\bar\Lambda_I^{(0)} + g^{\frac{5}{2}}\bar\Lambda_I^{(1)} + \cdots , \\
\varphi &= g^{0}\varphi^{(0)} + g^{2}\varphi^{(1)} + \cdots , & \bar\varphi &= g^{0}\bar\varphi^{(0)} + g^{2}\bar\varphi^{(1)} + \cdots .
\end{aligned}
$$

From this expansion, we obtain the self-dual instanton equation from the equation of motion.

• self-dual equation for leading-order fields ($\bar\Lambda_I^{(0)}$ is subleading.)

$$
\begin{aligned}
&F_{\mu\nu}^{(0)-} = 0, \quad (\bar\sigma^\mu)^{\dot\alpha\beta} D_\mu \Lambda_\beta^{(0)I} = 0, \\
&D^2\varphi^{(0)} + i\sqrt{2}\Lambda^{(0)I}\Lambda_I^{(0)} + iC^{\mu\nu}F_{\mu\nu}^{(0)} = 0, \\
&D^2\bar\varphi^{(0)} + i\bar C^{\mu\nu}F_{\mu\nu}^{(0)} = 0.
\end{aligned}
$$

• brief review on ADHM construction

1. $\Delta_{\dot{\alpha}} = a_{\dot{\alpha}} + b^{\alpha}\sigma_{\mu\alpha\dot{\alpha}}x^{\mu}$: $(N+2k)\times 2k$ matrix

ADHM constraint

$$\bar{\Delta}^{\dot{\alpha}}\Delta_{\dot{\beta}} = f^{-1}\delta^{\dot{\alpha}}{}_{\dot{\beta}}\ , \quad f\ :\ x^{\mu}\text{-dependent } k\times k \text{ matrix.}$$

canonical form of $\Delta_{\dot{\alpha}}$

$$\Delta_{(u+i\alpha),j\dot{\alpha}} = \begin{pmatrix} w_{uj\dot{\alpha}} \\ (a'_{ij}+\delta_{ij}x)_{\alpha\dot{\alpha}} \end{pmatrix} \longrightarrow \begin{cases} (\vec{\tau})^{\dot{\alpha}}{}_{\dot{\beta}}(\bar{w}^{\dot{\beta}}w_{\dot{\alpha}} + \bar{a}'^{\dot{\beta}\alpha}a'_{\alpha\dot{\alpha}}) = 0, \\ a'_{\mu} = \bar{a}'_{\mu}. \end{cases}$$

2. U : $(N+2k)\times N$ matrix, zero-mode of $\bar{\Delta}$, i.e. $\bar{\Delta}U = 0$.

self-dual gauge field

$$A^{(0)}_{\mu} = -i\bar{U}\partial_{\mu}U \ \Rightarrow\ F^{(0)}_{\mu\nu} = -4i\bar{U}b^{\alpha}(\sigma_{\mu\nu})_{\alpha}{}^{\beta}f\bar{b}_{\beta}U\ :\ \text{self-dual.}$$

- fermionic part

ansatz ($\mathcal{M}^I$: constant $(N+2k)\times k$ matrix)

$$\Lambda_\alpha^{(0)I} = \bar{U}(\mathcal{M}^I f \bar{b}_\alpha - b_\alpha f \bar{\mathcal{M}}^I)U,$$

$$\Rightarrow \ \bar{\sigma}^{\mu\dot{\alpha}\alpha} D_\mu \Lambda_\alpha^{(0)I} = \bar{U} b^\alpha f \left(\bar{\Delta}^{\dot{\alpha}} \mathcal{M}^I + \bar{\mathcal{M}}^I \Delta^{\dot{\alpha}}\right) f \bar{b}_\alpha U = 0.$$

fermionic ADHM constraint

$$\bar{\Delta}^{\dot{\alpha}} \mathcal{M}^I + \bar{\mathcal{M}}^I \Delta^{\dot{\alpha}} = 0.$$

canonical form

$$\mathcal{M}^I_{(u+i\alpha),j} = \begin{pmatrix} \mu^I_{uj} \\ (\mathcal{M}'^I_\alpha)_{ij} \end{pmatrix} \longrightarrow \begin{cases} \bar{\mu}^I w_{\dot{\alpha}} + \bar{w}_{\dot{\alpha}} \mu^I + [\mathcal{M}'^{\alpha I}, a'_{\alpha\dot{\alpha}}] = 0, \\ \mathcal{M}'^I_\alpha = \bar{\mathcal{M}}'^I_\alpha. \end{cases}$$

• solution of leading-order self-dual equation from ADHM

$$A_\mu^{(0)} = -i\bar{U}\partial_\mu U, \quad \Lambda_\alpha^{(0)I} = \bar{U}(\mathcal{M}^I f\bar{b}_\alpha - b_\alpha f\bar{\mathcal{M}}^I)U,$$

$$\varphi^{(0)} = i\frac{\sqrt{2}}{4}\epsilon_{IJ}\bar{U}\mathcal{M}^I f\bar{\mathcal{M}}^J U + \bar{U}\left(\begin{smallmatrix}\phi & 0\\ 0 & \chi\mathbf{1}_2+\mathbf{1}_k C\end{smallmatrix}\right)U,$$

$$\bar{\varphi}^{(0)} = \bar{U}\left(\begin{smallmatrix}\bar{\phi} & 0\\ 0 & \bar{\chi}\mathbf{1}_2+\mathbf{1}_k\bar{C}\end{smallmatrix}\right)U,$$

where $\phi = \lim_{|x|\to\infty}\varphi^{(0)}$, $\bar{\phi} = \lim_{|x|\to\infty}\bar{\varphi}^{(0)}$, and

$$\chi = \boldsymbol{L}^{-1}\left(-i\tfrac{\sqrt{2}}{4}\epsilon_{IJ}\bar{\mathcal{M}}^I\mathcal{M}^J + \bar{w}^{\dot{\alpha}}\phi w_{\dot{\alpha}} + C^{\mu\nu}[a'_\mu, a'_\nu]\right),$$

$$\bar{\chi} = \boldsymbol{L}^{-1}\left(\bar{w}^{\dot{\alpha}}\bar{\phi}w_{\dot{\alpha}} + \bar{C}^{\mu\nu}[a'_\mu, a'_\nu]\right),$$

$$\boldsymbol{L} = \frac{1}{2}\{\bar{w}^{\dot{\alpha}}w_{\dot{\alpha}}, *\} + \left[a'_\mu, [a'^\mu, *]\right].$$

comment

1. In the case of $\phi=\bar{\phi}=\bar{C}^{\mu\nu}=0$, the solution

$$
\begin{aligned}
A_\mu &= A_\mu^{(0)}, & & \\
\Lambda_\alpha^I &= \Lambda_\alpha^{(0)I}, & \bar{\Lambda}_{I\dot{\alpha}} &= 0, \\
\varphi &= \varphi^{(0)}\Big|_{\phi=0}\,, & \bar{\varphi} &= \bar{\varphi}^{(0)}\Big|_{\bar{\phi}=\bar{C}^{\mu\nu}=0} = 0\,,
\end{aligned}
$$

is an exact solution of the equation of motion.

2. In the case of $\bar{C}^{\mu\nu}=0$, the leading-order solution is reduced to the solution constructed by [Billo et. al.].

4. Instanton Effective Action

• definition

In the instanton background, the action can be evaluated as

$$S_{\mathrm{E}} = \frac{8\pi^2}{g^2}k + g^0 S_{\mathrm{eff}}^{(0)} + \mathcal{O}(g^2), \quad k: \text{ instanton number.}$$

This $S_{\mathrm{eff}}^{(0)}$ is called the instanton effective action.

$$S_{\mathrm{eff}}^{(0)} = \int d^4x \, \mathrm{Tr}\Big[\big(D_\mu \bar{\varphi}^{(0)}\big) D^\mu \varphi^{(0)} + \frac{i}{\sqrt{2}} \Lambda^{(0)\alpha I} \big[\bar{\varphi}^{(0)}, \Lambda^{(0)}_{\alpha I}\big] \\ - i\big(C^{\mu\nu}\bar{\varphi}^{(0)} + \bar{C}^{\mu\nu}\varphi^{(0)}\big) F^{(0)}_{\mu\nu} - \frac{i}{\sqrt{2}} \bar{C}^{\mu\nu} \Lambda^{(0)I} \sigma_{\mu\nu} \Lambda^{(0)}_I\Big].$$

- instanton effective action in terms of ADHM moduli

$$S^{(0)}_{\mathrm{eff}} = 4\pi^2 \mathrm{tr}_k \Big[-i\frac{\sqrt{2}}{4}\epsilon_{IJ}\bar{\mu}^I \bar{\phi}\mu^J + \frac{1}{2}\bar{w}^{\dot{\alpha}}(\bar{\phi}\phi + \phi\bar{\phi})w_{\dot{\alpha}} - \bar{\chi}\boldsymbol{L}\chi$$
$$- i\frac{\sqrt{2}}{8}\bar{C}^{\mu\nu}\epsilon_{IJ}\mathcal{M}'^I\sigma_{\mu\nu}\mathcal{M}'^J + \frac{1}{4}C^{\mu\nu}\bar{C}_{\mu\nu}\bar{w}^{\dot{\alpha}}w_{\dot{\alpha}}\Big].$$

calculation from D(-1) side in D3-D(-1) system [Billo et. al.]

$$S^{(0)}_{\mathrm{str}} = 4\pi^2 \mathrm{tr}_k \Big[-i\frac{\sqrt{2}}{4}\epsilon_{IJ}\bar{\mu}^I \bar{\phi}\mu^J + \frac{1}{2}\bar{w}^{\dot{\alpha}}(\bar{\phi}\phi + \phi\bar{\phi})w_{\dot{\alpha}} - \bar{\chi}\boldsymbol{L}\chi$$
$$- i\frac{\sqrt{2}}{8}\bar{C}^{\mu\nu}\epsilon_{IJ}\mathcal{M}'^I\sigma_{\mu\nu}\mathcal{M}'^J - \frac{1}{4}C^{\mu\nu}\bar{C}_{\mu\nu}a'_\rho a'^\rho\Big].$$

There is a discrepancy in $C\bar{C}$-term.

Then we should have additional contribution to the original Lagrangian as

$$\mathcal{L}' = \mathrm{Tr}\Big[g^2 C_{\mu\rho} x^\rho \bar{C}_{\nu\sigma} x^\sigma F^{\mu\lambda} F^{\nu}{}_{\lambda}\Big]$$
$$\to -\pi^2 C^{\mu\nu} \bar{C}_{\mu\nu} \mathrm{tr}_k \big[a'_\rho a'^\rho + \bar{w}^{\dot{\alpha}} w_{\dot{\alpha}}\big]$$

Since this term depends on x^μ explicitly, we cannot calculate it from the disk amplitude of the open string.

comment

1. $\mathcal{L}'$ does not contribute to the self-dual equation for leading-order fields. Then the instanton solution is unchanged by $\mathcal{L}'$.

2. $\mathcal{L} + \mathcal{L}'$ basically corresponds to the (truncated version of) super Yang-Mills Lagrangian on the Ω-background.

5. Summary and Outlook

Summary

1. We have constructed the (constrained) instanton solution of $\mathcal{N} = 2$ super Yang-Mills theory deformed by R-R 3-form.

2. We have also calculated the instanton effective action. But in order to compare with the result of the calculation from D(-1) side in D3-D(-1) system, we should have additional contribution.

Outlook

1. $\mathcal{N} = 4$ (and $\mathcal{N} = 2^*$ etc.) case
   - The instanton solution can be obtained in a similar way.
   - The integration over x^μ can be performed and we obtain the similar result.

2. instanton calculus (integration over ADHM moduli)
   - (deformed) BRST invariance, localization formula

3. interpretation of the additional term
   - second-order interaction between D3-branes and R-R 3-form (cf. argument from κ-symmetry of DBI $+$ CS)

4. deformation by other R-R fields (other 3-form, 5-form, 1-form)