Wilson loops in Five Dimensional Cohomological Field Theories

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Gauge theories

Srings, gravities

5D N=1 SYM

Nekrasov's
partition function

[Nekrasov '02]

Topological string amplitude

[Iqbal-KapshiniPoor 04,
Eguchi-Kanno '03]

Polytopes of certain toric varieties

[Maeda, Nakatsu, YN, Tamakoshi '05]

generalization

Adding matters

Polytopes of certain toric varieties

[YN '06]

Other gauge groups

Adding observables

Observables in 5D SYM

[Baulieu, Nekrasov '97]

5D Chern-Simons
+ etc

Wilosn loops

↔ Framing

[Maeda, Nakatsu, Takasaki , Tamakoshi '04]

↔ ?

They are not calculated in the path integral formalism.

5D cohomological field theory

[Nekrasov '96]

5D N=1 SYM on R^4 x S^1 (all fields are periodic)

$$SO(5) \supset SO(4) = SU(2)_L \times SU(2)_R$$

We respect
only this symm.

twist

$$SU(2)_L \times SU(2)_R \times SU(2)_I \mapsto SU(2)_L \times SU(2)_{\mathrm{diagonal}}$$

Fields $$A_\mu(t)dx^\mu,\ (A_t + i\varphi)(t),\ (A_t - i\varphi)(t),$$

$$\bar{\psi}(t),\ \psi_\mu(t)dx^\mu,\ \frac{1}{2}\zeta_{\mu\nu}(t)dx^\mu \wedge dx^\nu.$$

SUSY transformation

$$
\begin{aligned}
QA(t) &= \psi(t), \quad Q\psi(t) = d_{A(t)}(A_t + i\varphi)(t) - \partial_t A(t), \\
Q(A_t + i\varphi)(t) &= 0.
\end{aligned}
$$

Superpartners

$$
\underline{\begin{pmatrix} (A_t - i\varphi)(t) \\ \bar{\psi}(t) \end{pmatrix} \quad \begin{pmatrix} \zeta^+(t) \\ H^+(t) \end{pmatrix}}
$$

Projection multiplets

Omega background

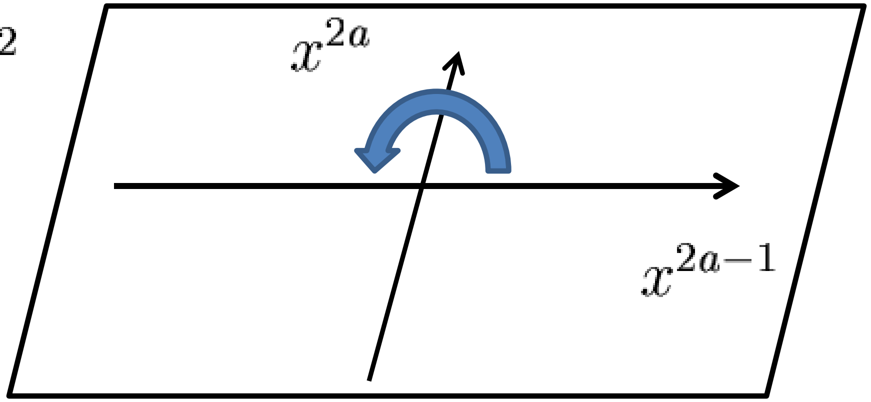
$$ds^2 = ds^2(\epsilon_1, \epsilon_2)$$

[Losev, Marshakov, Nekrasov 02]

Klling vector

$$V^\mu \partial_\mu = \epsilon_1 T_1 + \epsilon_2 T_2$$

$$T_a \curvearrowright \mathbb{R}^2$$



For general ϵ_1 ϵ_2 , the theory does not becomes cohomological. (At least my calculation)

SUSY trans. in Omega background

$$\begin{aligned} Q_\epsilon A(t) &= \psi(t), \qquad Q_\epsilon \psi(t) = \left[d_{A(t)}, \mathcal{H}_\epsilon(t) \right], \\ Q_\epsilon \mathcal{H}_\epsilon(t) &= 0. \end{aligned}$$

$$\mathcal{H}_\epsilon(t) := \frac{d}{dt} + (A_t + i\varphi)(t) + \iota_V(d_{A(t)}) + \frac{1}{2}\Omega_{\mu\nu}(\epsilon) J^{\mu\nu}$$

$$\begin{pmatrix} (A_t - i\varphi)(t) \\ \bar{\psi}(t) \end{pmatrix} \quad \begin{pmatrix} \zeta^+(t) \\ H^+(t) \end{pmatrix}$$

5D gauge theory and SUSY quantum mechanics

[Nekrasov (Omega=0) '97]

$$\begin{pmatrix} (A_t - i\varphi)(t) \\ \bar{\psi}(t) \end{pmatrix} \quad \begin{pmatrix} \zeta^+(t) \\ H^+(t) \end{pmatrix}$$

Integrate out

Integrate out

$$d^*_{A(t)}\psi(t) = 0$$ (Coulomb gauge)

$$d^*_{A(t)} d_{A(t)}(A_t + i\varphi)(t) + 2\psi(t^2) + d^*_{A(t)}\left(\iota_V F_{A(t)} + \frac{dA(t)}{dt}\right) = 0.$$

$$F^{(+)}_{A(t)} = 0, \ \left(d_{A(t)}\psi(t)\right)^{(+)} = 0,$$ (ASD condition)

Remaining fields $A(t),\ \psi(t),\ (A_t + i\varphi)(t)$ with the 4 conditions.

5D SYM can be reduced to SUSY quantum mechanics whose target space is the instanton moduli spaces $\mathcal{M}$.

$$S^{\mathcal{N}=1}_{5D,\, SYM} \mapsto S^{\mathcal{N}=1/2}_{SQM}$$

$$S^{\mathcal{N}=1/2}_{SQM} = \int dt\, \frac{1}{2} G_{IJ} \dot{m}^I \dot{m}^J + \underline{\frac{1}{2} G_{IJ} V^I_{\mathcal{M}} \dot{m}^J} + \frac{1}{2} \chi^I \left(G_{IJ} \partial_t + \dot{m}^L G_{IK} \Gamma^K_{LJ} \right) \chi^J$$

$$\begin{aligned} Q_\epsilon m^I &= \chi^I, \\ Q_\epsilon \chi^I &= -\dot{m}^I - \underline{V^I_{\mathcal{M}}}. \end{aligned}$$

$V_{\mathcal{M}} \simeq \iota_V F_{A(t)}|_{\mathcal{M}}$: Killing vector on inst. moduli $\mathcal{M}$

The partition function is:

$$
\begin{aligned}
Z_{5D,\,SYM} &= \sum_{k=0}^{\infty} Q^k Z_{SQM\,on\,\mathcal{M}_k} \\
&= \sum_{k=0}^{\infty} Q^k \int_{\mathcal{M}_k} \hat{A}_V(\mathcal{M}_k) \\
&= \sum_{k=0}^{\infty} Q^k \mathrm{Ind}_{equiv}(D_{Dirac\,on\,\mathcal{M}_k}).
\end{aligned}
$$

where $Q = (R\Lambda)^2$

We derived that the partition function is given by the index by using path integral formalism.

Wilson loops

$$\mathcal{O}^{(0)}(x) := \mathrm{Tr}P\left(e^{\int dt(A_t+i\varphi)(t)(x)}\right)$$

In general, this is not a BPS observable.

$$Q_\epsilon \mathcal{O}^{(0)}(x) \neq 0 \qquad \text{if } x \neq 0.$$

$$\mathcal{F}(t) := F_{A(t)} + \psi(t) + (A_t + i\varphi)(t)$$

$$\mathcal{O}(x) := \mathrm{Tr}P\left(e^{\int dt\mathcal{F}(t)(x)}\right)$$

$$\mathcal{O} = \mathcal{O}^{(0)} + \mathcal{O}^{(1)} + \cdots + \mathcal{O}^{(4)}$$

$$Q_\epsilon \int_{\mathbb{R}^4} \mathcal{O}^{(4)} = 0.$$

BPS observable

Universal bundle

$$\mathcal{F}(t) := F_{A(t)} + \psi(t) + (A_t + i\varphi)(t)$$

forms the loop space analog of the equivariant curvature of the universal bundle $\mathcal{E}$.

Universal bundle

$$\begin{array}{ccc} E & \overrightarrow{/\mathcal{G}} & \mathcal{E} \\ \downarrow & & \downarrow \\ \mathcal{A} \times R^4 & \overrightarrow{/\mathcal{G}} & \mathcal{A}/\mathcal{G} \times R^4 \end{array}$$

$\mathcal{A}$: space of gauge connections

loop space analog of the equivariant Bianchi identity

$$(d_{A(t)} - \iota_V + Q_\epsilon)(F_{A(t)} + \psi(t) + (A_t + i\varphi)(t)) = \underline{\frac{dA(t)}{dt}}.$$

五次元特有

Equivariant descendant equation

$$(d - \iota_V + Q_\epsilon)\mathcal{O} = 0$$

$$\left[\begin{array}{rcl} Q_\epsilon \mathcal{O}^{(0)} - \iota_V \mathcal{O}^{(1)} &=& 0, \\ d\mathcal{O}^{(0)} + Q_\epsilon \mathcal{O}^{(1)} - \iota_V \mathcal{O}^{(2)} &=& 0, \\ \cdots && \\ d\mathcal{O}^{(3)} + Q_\epsilon \mathcal{O}^{(4)} &=& 0. \end{array}\right]$$

$$\Rightarrow \quad Q_\epsilon \int_{\mathbb{R}^4} \mathcal{O}^{(4)} = 0.$$

We can use the localization formula in correlation function:

$$\langle \int_{\mathbb{R}^4} \mathcal{O}^{(4)} \rangle = \langle \frac{4\pi^2}{\epsilon_1 \epsilon_2} \mathrm{Tr} P e^{\int dt (A_t + i\varphi)(t)(x=0)} \rangle$$

Localized to the origin.

- This should be related to the equivariant curvature of $\mathcal{E} \to \mathcal{M}$

- When the gauge group is (N.C.) U(1), the ordering is irrelevant. In that case, we can understand the effect of the Wilson loops in terms of the SUSY quantum mechanics.

When the gauge group is U(1):

$$\langle \int_{\mathbb{R}^4} \mathcal{O} \rangle = \int_{L\mathcal{M}} [\mathcal{D}X] e^{-S_{SQM}^{\mathcal{N}=1/2}} \frac{4\pi^2}{\varepsilon_1 \varepsilon_2} \mathrm{Tr}\left(e^{\int dt (A_t + i\varphi)(t)(x=0)} \right)$$

$$= \int_{L\mathcal{M}} [\mathcal{D}X] \frac{4\pi^2}{\varepsilon_1 \varepsilon_2} e^{-S_{SQM}^{\mathcal{N}=1/2} \underbrace{- \int_{S^1} -i\dot{m}^I \epsilon_I + \frac{i}{2} \chi^I F_{IJ} \chi^J}} \mathrm{Tr} e^{\mu(V_{\mathcal{M}})}$$

Interaction terms between charged particle and the gauge field ϵ_I , which is the connection of $\mathcal{E}$

$$= \frac{4\pi^2}{\varepsilon_1 \varepsilon_2} \int_{L\mathcal{M}} [\mathcal{D}X] e^{-S_{SQM}^{\mathcal{N}=1/2}{}_{charged}} \mathrm{Tr} e^{\mu(V_{\mathcal{M}})}$$

$$= \frac{4\pi^2}{\varepsilon_1 \varepsilon_2} \int_{\mathcal{M}} \hat{A}_V(\mathcal{M}) Ch_V(\mathcal{E}) = \frac{4\pi^2}{\varepsilon_1 \varepsilon_2} \mathrm{Ind}_{equiv}(D_{Dirac\, twisted})$$

$\mu(V_{\mathcal{M}})$: moment

By using Nakajima's method

$$\langle \int_{\mathbb{R}^4} \mathcal{O}^{(4)} \rangle = \sum_{\lambda} Q^{|\lambda|} q^{\kappa(\lambda)} s_{\lambda}(q^{\rho})^2 \frac{1}{\hbar^2} \hat{\mathcal{O}}(\lambda)$$

where $\epsilon_1 = -\epsilon_2 = i\hbar,\ q = e^{-R\hbar}$

$$\hat{\mathcal{O}}(\lambda) = 1 + (1 - q^{-1}) \sum_{i=1}^{l(\lambda)} (q^{\lambda_i - i + 1} - q^{-i+1})$$

k times wrapping $$\mathcal{O}_k(x) := \mathrm{Tr} P e^{k \int dt \mathcal{F}(t)(x)}$$

The generating function is

$$\langle \exp \left(\sum_{k=1}^{\infty} t_k \int_{\mathbb{R}^4} \mathcal{O}_k^{(4)} \right) \rangle = \sum_{\lambda} Q^{|\lambda|} q^{\kappa(\lambda)} s_\lambda(q^\rho)^2 \exp \left(\sum_{k=1}^{\infty} \frac{t_k}{\hbar^2} \hat{\mathcal{O}}_k(\lambda) \right)$$

where $$\hat{\mathcal{O}}_k(\lambda) = 1 + (1 - q^{-k}) \sum_{i=1}^{l(\lambda)} (q^{k(\lambda_i - i + 1)} - q^{k(-i+1)})$$

This function correspond to the tau function of the 1-Toda hierarchy.

[Nakatsu, Takasaki '07]

Semi-classical limit

By letting $\hbar \to 0$, we obtained an extended Seiberg-Witten curve.

$$\mathcal{C}_\beta \;:\; y + y^{-1} = \frac{1}{R\Lambda}(e^{-Rz} - \beta)\,, \qquad z \in \mathbb{C}\,,$$

where $$\beta = \beta(t_1, t_2, \cdots)$$

Conclusion

•We obtained that the one point functions of the Wilson loops are given by the index of the twisted Dirac operator on the moduli space of instantons.

• We obatined a deformed extended Seiberg-Witten curve.

Discussion

•Non-abelian?   -> It may be calculable.

•5D Chern-Simons? -> We can calculate it by using a similar method.

•Topological Strings? -> I’m not sure.

•4D limit? -> In the 4D theory, the observables correspond to the irrelevant operators. It is hard to think that the observables deform the low energy theory.