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Recent Developments of Topological Vertex

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TOPOLOGICAL STRINGS

- A-model

Twisted topological sigma-model on Riemann surface whose target space is Calabi-Yau

$$\begin{aligned} S^{\mathrm{A}} &= \int_{\Sigma_g} d^2z\sqrt{g}G_{i\bar{j}}g^{\mu\nu}\partial_\mu X^i \partial_\nu X^{\bar{j}} + i\epsilon^{\mu\nu}\partial_\mu X^i \partial_\nu X^{\bar{j}} + \cdots \\ &= \{\mathcal{Q}, V(X,\rho,\chi)\} + \int_{\Sigma_g} X^*(\omega) \end{aligned}$$

$$\Big\downarrow$$

SUSY localization

$$\boldsymbol{\partial_{\bar{z}} X^i = \partial_z X^{\bar{i}} = 0}$$

Thus A-model topological string theory counts holomorphic maps from worldsheets to the Calabi-Yau.

APPLICATIONS OF TOPOLOGICAL STRINGS

. . . . .

Twister amplitudes

Donaldson-Thomas

Seiberg-Witten Theory

Knot Theory

topological strings

Dijkgraaf-Vafa

Gromov-Witten Theory

Black Holes

Mirror Symmetry

Matrix Models

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1. Topological Vertex

- Topological Strings

We want to compute A-model topological string amplitudes for Calabi-Yau X

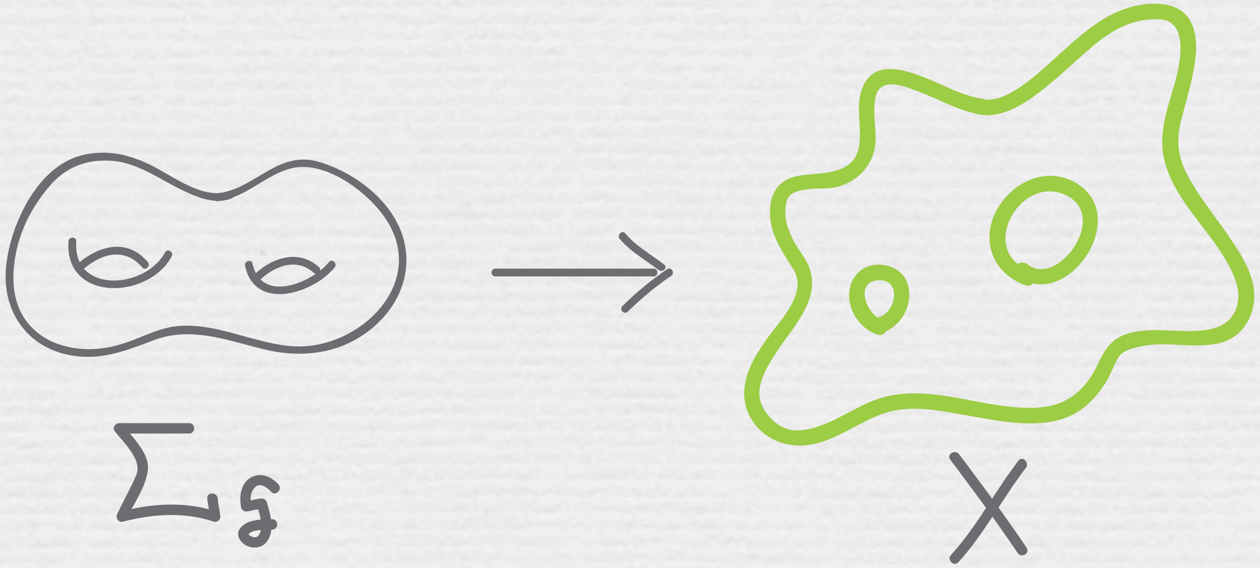
$$F_g = \sum_{\Sigma \in H_2(X,\mathbb{Z})} N_{g,\Sigma} e^{-t_\Sigma}$$

Gromov-Witten invariant

$f : \Sigma_g \to \Sigma \subset X$

$$Z = \exp \sum_{g=0}^{\infty} \hbar^{2g-2} F_g(t)$$

$\hbar$: topological string coupling constant



Thus topological string theory is constructed as a model of strings which propagate on Calabi-Yau 3-fold. In general, it is very hard to get the full partition function via straightforward computation. However, for the certain class of Calabi-Yau's, **we can compute the partition function exactly** using dualities !!

→ **Example : topological vertex**

- Modern “definition” of topological strings [Gopakumar-Vafa, ‘98]

What is the meaning of topogical strings in the context of superstrings ?

→ **BPS state counting**

A-model on X $\xrightarrow[\text{[BCOV]}]{\text{embedding}}$ F-terms of **effective theory of Type IIA superstrings** compactified on X

The general form of the effective action is

graviphoton

$$\sum_{g=0}^{\infty}\int d^4 x\, d^4 \theta\, W^{2g}\, F_g(t_i) = \int d^4 x \left[\tau_{ij}\, F^i_{\mu\nu}\, F^{j\mu\nu} + \sum_{g=1}^{\infty} F_g(t_i)\, {R_+}^2\, {F_+}^{2g-2}\right]$$

effective coupling of massless $U(1)$ gauge fields

$$\tau_{ij} = \partial_i \partial_j F_0(t_i)$$

free energy = prepotential

It was argued that vector multiplets and hyper multiplets are decoupled in the effective action (up to two derivative).

prepotential → Kahler moduli fields → vector mult.

IIA dilaton → hyper mult.

We can evaluate it in strong superstring coupling limit $g_s \longrightarrow \infty$

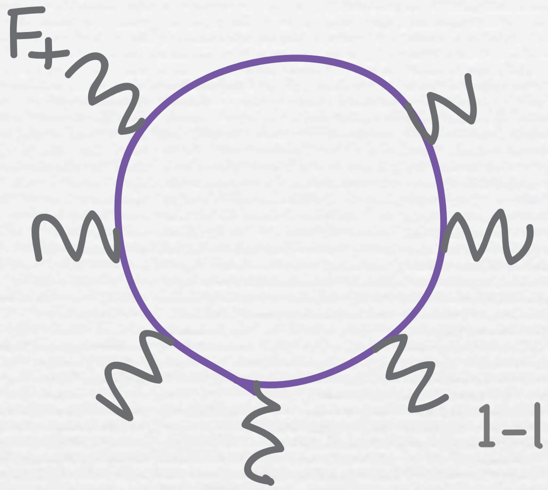
$$m_\Sigma = \frac{t_\Sigma}{g_s} \longrightarrow \text{massless}$$

⇒ BPS states coming from D2-branes wrapping 2-cycles inside X become massless

↓ integrating out these BPS particles with small masses

effective action (F-terms)

Topological strings count degeneracies of wrapped M2-branes (dimensions of M2-brane moduli space with spins (j_L, j_R))



1-loop amplitude of BPS particles coupling to the background field

$$\mathcal{F} = \sum_{\Sigma \in H_2(M,\mathbb{Z})} \sum_{n \in Z} \sum_{j_L, j_R} \boxed{N_\Sigma^{(j_L, j_R)}} \log\det{}_{(j_L,j_R)} \left(\Delta + {m_{(\Sigma,n)}}^2 + 2m_{(\Sigma,n)}\, \sigma_L\, F_+\right)$$

$$= \sum_{\Sigma \in H_2(M,\mathbb{Z})} \sum_{k=1}^{\infty} \sum_{j_L} N_\Sigma^{j_L} \, (-1)^{-2\, j_L} \, e^{-k\, T_\Sigma} \, \frac{\sum_{l=-\,j_L}^{j_L} q^{-2kl}}{k\left(q^{k/2} - q^{-k/2}\right)^2}$$

$$F_+ = -\hbar$$

$$N_\Sigma^{j_L} = \sum_{j_R} (-1)^{-2\, j_R} (2\, j_R + 1)\, N_\Sigma^{(j_L, j_R)}$$

$$\longrightarrow \quad \sum_{j_L} N_\Sigma^{j_L} [j_L] = \sum_{g=0}^{\infty} n_\Sigma^g [2(0) + (1/2)]^{\otimes\, g}$$

change of representation basis

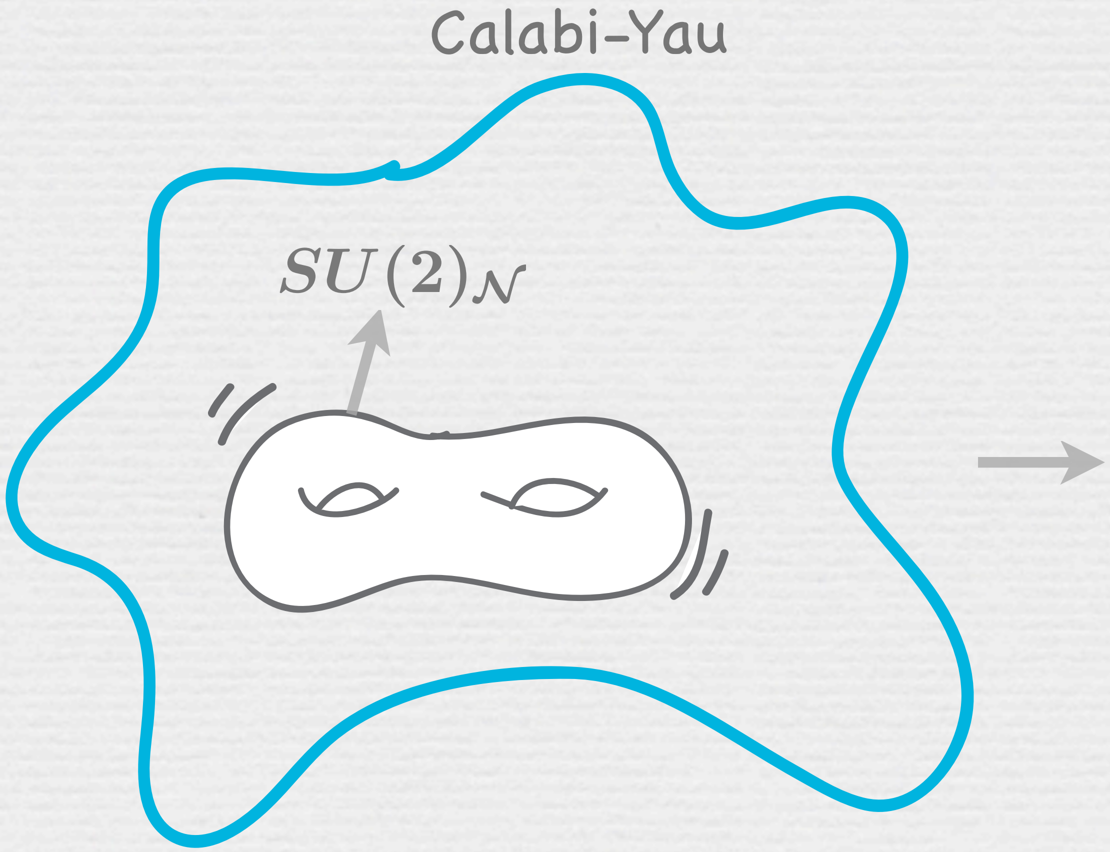
Gopakumar-Vafa invariants

World volume fields on a single wrapped M2-brane are

$$SO(8)_R \to SU(2)_L \times SU(2)_R \times SU(2)_{\mathcal{N}}$$

ψ $\quad \mathbf{8}_s \longrightarrow \quad (0, \frac{1}{2}) \otimes (\frac{1}{2})$ superdeformation

ϕ $\quad \mathbf{8}_v \longrightarrow \quad (0, 0) \otimes (\frac{1}{2})$ deformation



$\mathcal{M}_{g,d}$

the moduli space
of deformations

→ supersymmetric sigma model on $\mathcal{M}_{g,d}$. We can count boundstates by quantizing these zero modes.

→ The degeneracies of bound states of wrapped M2-branes are extracted from the Hilbert space $H^*(\mathcal{M}_{g,d})$

↑

The Hilbert space is graded with $SU(2)_R$ R-charge of fermi fields. We take trace over the charges with the weight $(2j_R^3+1)(-1)^{2j_R^3}$.

Let us consider curves inside the C-Y mfd in class $\Sigma = \sum_i d_i[\Sigma_i]$

Then the Gopakumar-Vafa invariants are given by

$$n_d^g = (-1)^{\dim \mathcal{M}_{g,d}} \chi(\mathcal{M}_{g,d})$$

- Local Calabi-Yau manifolds & toric Calabi-Yau manifolds

Uses of toric Calabi-Yau

- Total space which realize a Calabi-Yau manifold as a hypersurface inside it
  (generalization of quintic for $\mathbb{CP}^4$)

  → Construction of mirror pair (Batyrev)

- Local models of Calabi-Yau manifolds
  (describe the structure in the neighborhood of singurarity)

  → Geometric engineering

  AdS/CFT

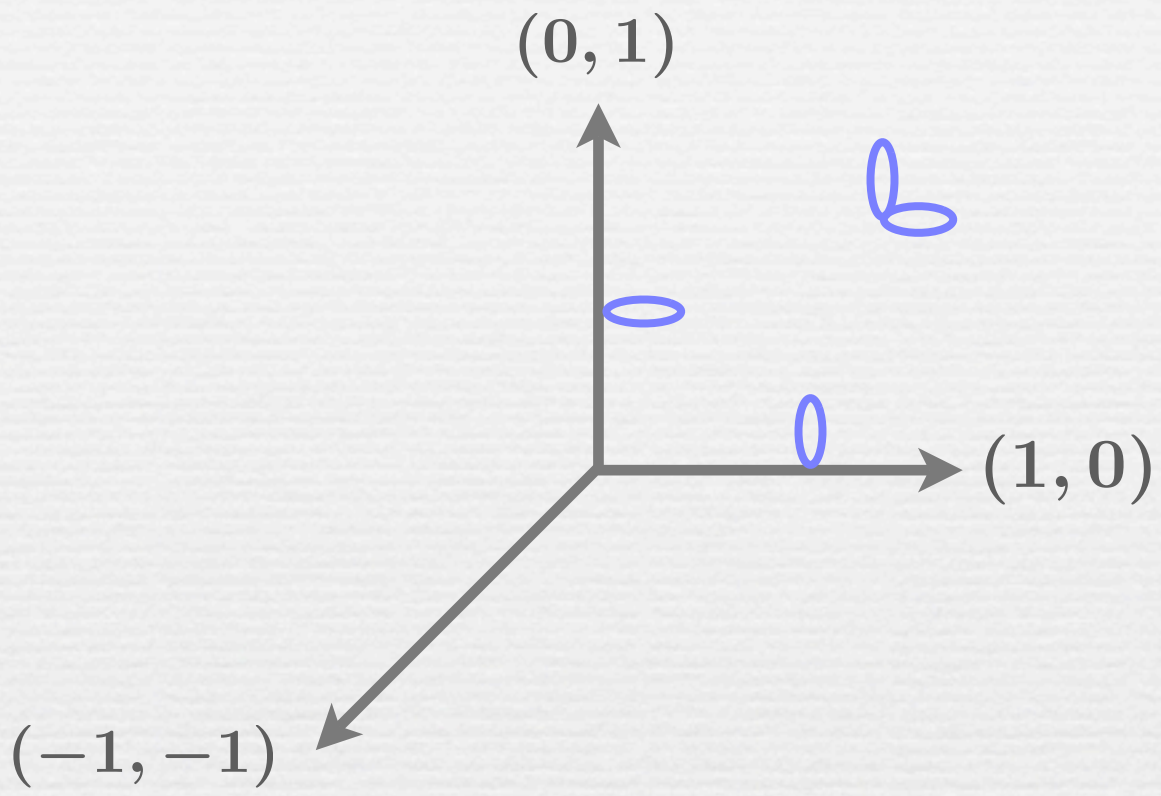
  . . . . .

- Toric Calabi-Yau manifold

- $\mathbb{C}^1$

$$z = |z|e^{i\theta} \qquad T^1(\theta) \text{ fibration over } \mathbb{R}(|z|)$$

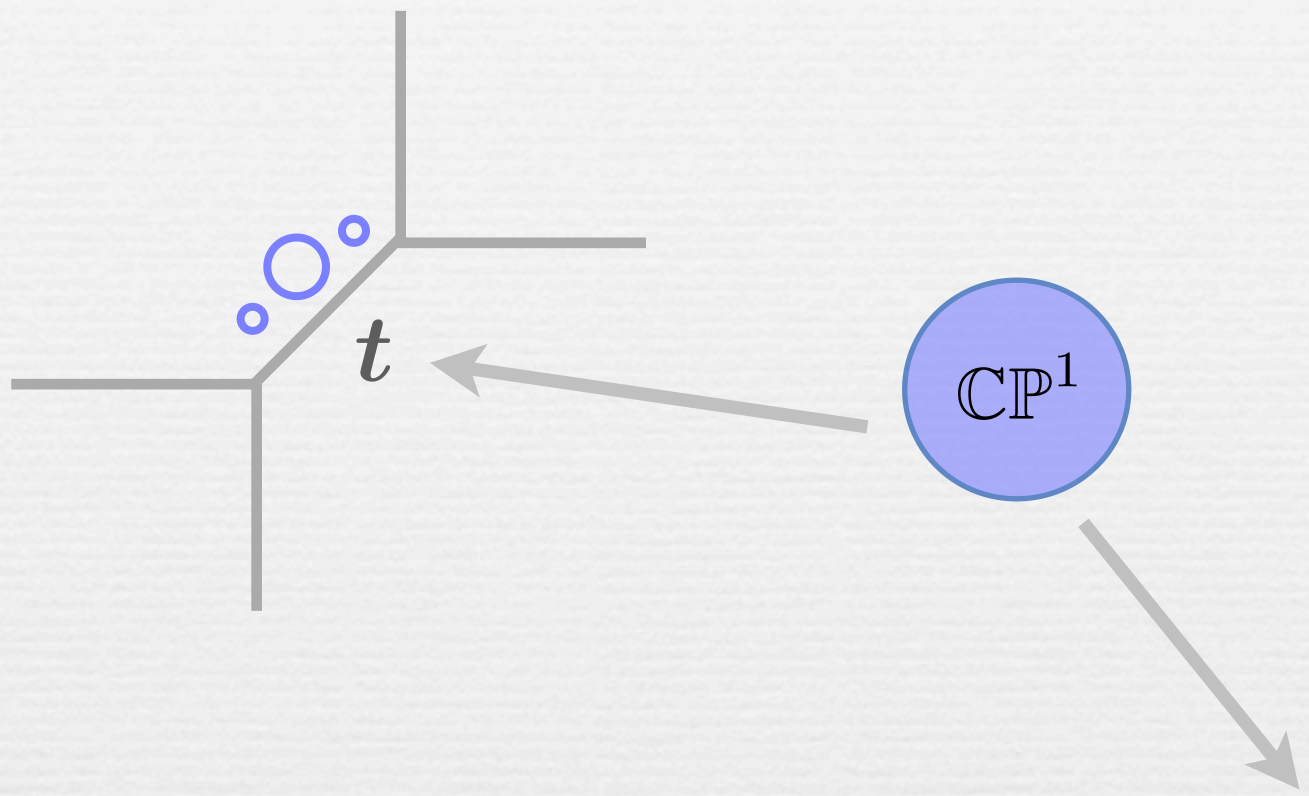
- $\mathbb{C}^3$



We are focusing on T^2-action

$$e^{\alpha r_\alpha + \beta r_\beta} : (z_1, z_2, z_3) \longrightarrow (e^{i\alpha} z_1, e^{-i\beta} z_2, e^{-i\alpha + i\beta} z_3)$$

- Resolved conifold $\mathcal{O}(-1)\oplus\mathcal{O}(-1)\to\mathbb{P}^1$



$$\det\begin{pmatrix} A_1 & A_2 \\ B_1 & B_2 \end{pmatrix}=0 \quad \rightarrow \quad \begin{pmatrix} A_1 & A_2 \\ B_1 & B_2 \end{pmatrix}\begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}=0$$

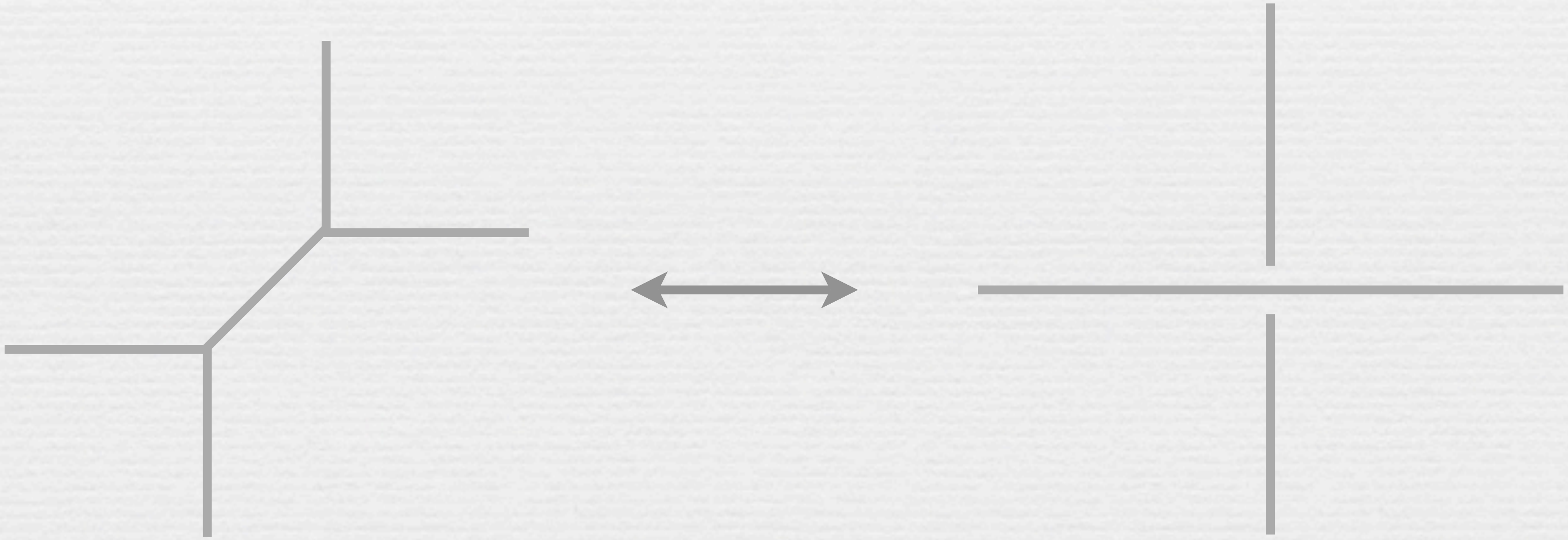
conifold (singular)

resolved conifold

$$\mathcal{O}(n) \longrightarrow \mathbb{CP}^1 \qquad z_S = \frac{1}{z_N}$$

$$\{\phi\} \qquad \{z\} \qquad \phi_S = (z_N)^n \phi_N$$

- **Geometric transition**



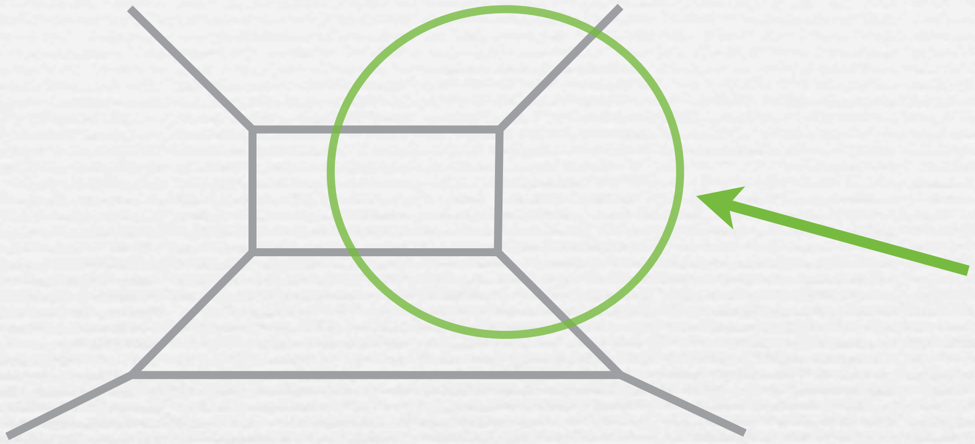
closed A-model on
resolved conifold

open A-model on
deformed conifold

→ Chern-Simons theory
[Witten, '93]

- Topological Vetex & toric Calabi-Yau manifolds

How to compute topological string amplitudes for toric Calabi-Yau manifolds ?



Locally they look like a conifold

↓

Geometric transition enable us to calculate these amplitudes using Chern-Simons theory

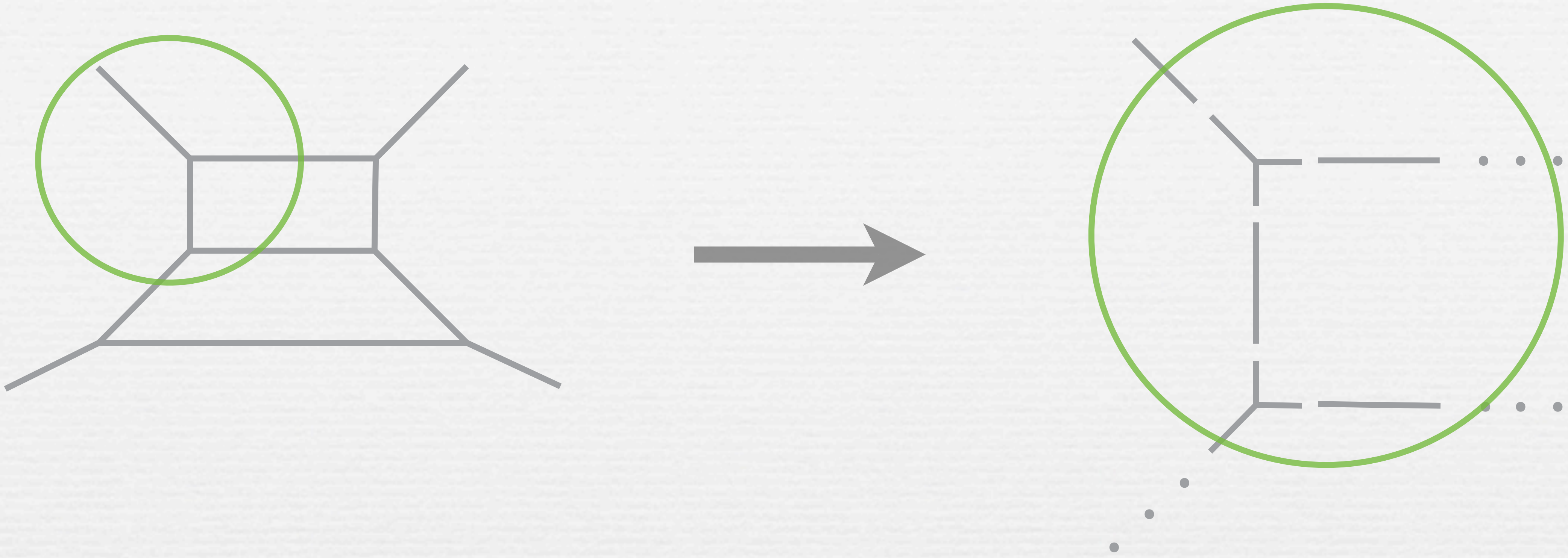
→ **Topological vertex [AMKV, '03]**

1. Decompose a toric web-diagram into vertices and propagators

2. Assign Young diagrams for each edges of these parts

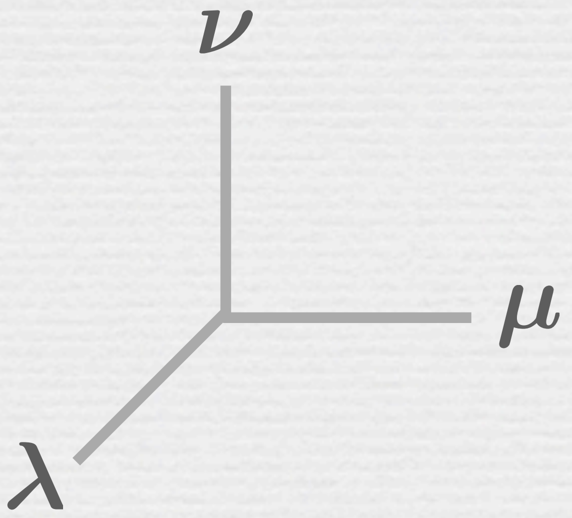
$$\mu = \{\mu_i \in \mathbb{Z}_{\geq 0} \,|\, \mu_1 \geq \mu_2 \geq \cdots\}$$

3. Glue them to get topological string partition function



▲ Decomposition of toric web-diagram

Parts

1. 

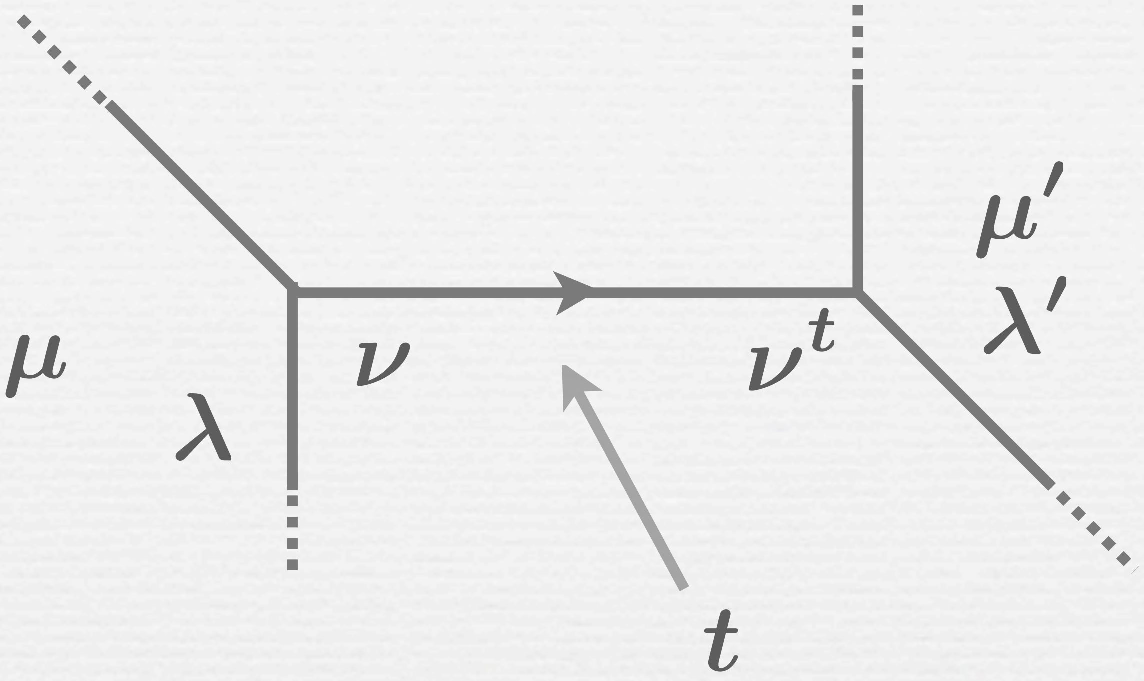
: trivalent vertex . . . local $\mathbb{C}^3$ patch $\longrightarrow$ $C_{\lambda,\mu,\nu}$

vertex function

2. 

: edes . . . $\mathbb{CP}^1$ $\longrightarrow$ $f_\mu(q)^n(-1)^{|\mu|}e^{-t|\mu|}\delta\mu, \nu^t$

framing factor & propagator



Gluing along a leg is done by the following procedure

$$Z = \cdots \sum_{\nu} C_{\lambda\mu\nu}(-1)^{|\nu|} e^{-t|\nu|} (f_{\nu})^{n} C_{\lambda'\mu'\nu^t} \cdots$$

vertex function propagator framing factro

$$C_{\lambda\mu\nu}(q) = q^{\kappa_\mu/2} s_{\nu^t}(q^{-\rho}) \sum_{\eta} s_{\lambda^t/\eta}(q^{-\nu-\rho}) s_{\mu/\eta}(q^{-\nu^t-\rho})$$

- Schur functions & free fermions

The modes of free fermions in 2-dim is given by

$$\left\{\psi_i, \psi_j^*\right\} = \delta_{i+j,0}, \quad \{\psi_i, \psi_j\} = \left\{\psi_i^*, \psi_j^*\right\} = 0, \quad i, j \in \mathbb{Z} + 1/2.$$

Bozonization of these fermions gives

$$\alpha_n = \sum_{j \in \mathbb{Z}+1/2} : \psi_{-j+n} \psi_j^* :$$

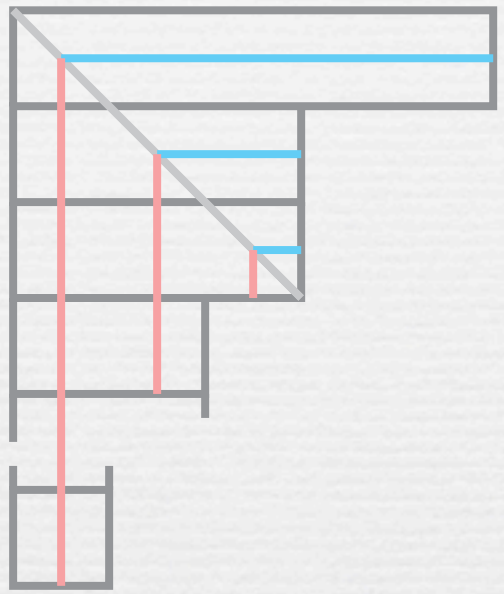
$$[\alpha_m, \alpha_n] = m\delta_{m+n,0}$$

$$\longrightarrow \quad V_{\pm}(x_i) = \exp\left[\sum_{i=1,n=1} \frac{x_i^n}{n} \alpha_{\pm}\right]$$

Vertex operators

Let us introduce the Frobenius coordinate

$$a_i(R) = R_i - i, \quad b_i(R) = R_i^t - i$$



$$\rightarrow \quad |R\rangle = (-1)^{r(r-1)/2+\sum_{j=1}^{r(R)} b_j} \prod_{j=1}^{r(R)} \psi_{-a_j-1/2}\psi^*_{-b_j-1/2}|0\rangle$$

They satisfies the completeness relation $\sum_{\mu} |\mu\rangle\langle\mu| = 1$

$$s_{R/Q}(x_i) = \langle R|V_-(x_i)|Q\rangle = \langle Q|V_+(x_i)|R\rangle$$

- Conventions

$$C_{\lambda\mu\nu}(q) = q^{\kappa_\mu/2} s_{\nu^t}(q^{-\rho}) \sum_\eta s_{\lambda^t/\eta}(q^{-\nu-\rho}) s_{\mu/\eta}(q^{-\nu^t-\rho})$$

$$q^{-\rho} \longleftrightarrow x_i = q^{i-\frac{1}{2}}$$

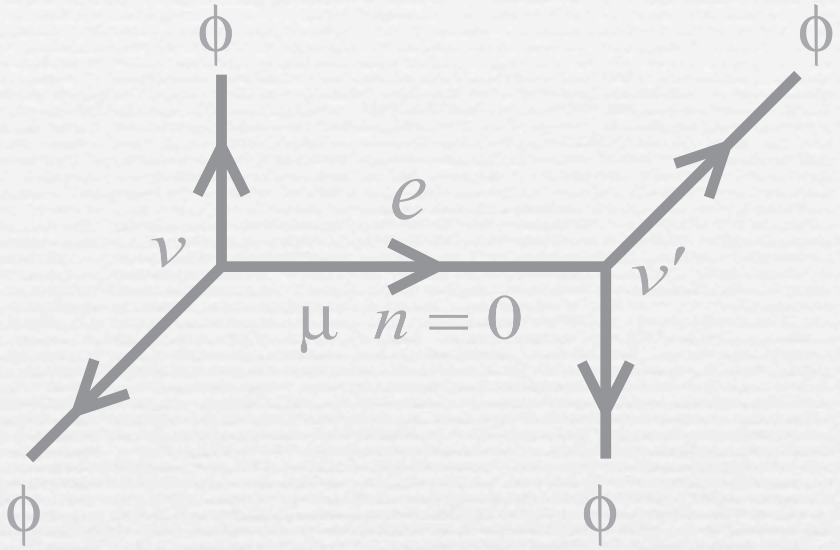
$$i = 1, 2, 3, \cdots$$

$$q^{-\mu-\rho} \longleftrightarrow x_i = q^{-\mu_i+i-\frac{1}{2}}$$

$$|\mu| = \sum_i \mu_i \qquad ||\mu||^2 = \sum_i {\mu_i}^2$$

$$\begin{aligned} \kappa_\mu &= \sum_i \mu_i(\mu_i + 1 - 2i) \\ &= \sum_i {\mu_i}^2 - \sum_j {\mu_j^t}^2 = -\kappa_{\mu^t} \end{aligned}$$

Ex: Conifold



$$Q = e^{-t}$$

$$q = e^{-\hbar}$$

$$Z = \sum_{\nu} C_{\phi\phi\nu}(-Q)^{|\nu|} C_{\phi\phi\nu^t}$$

$$= \sum_{\nu} s_{\nu}(q^{-\rho})(-Q)^{|\nu|} s_{\nu^t}(q^{-\rho})$$

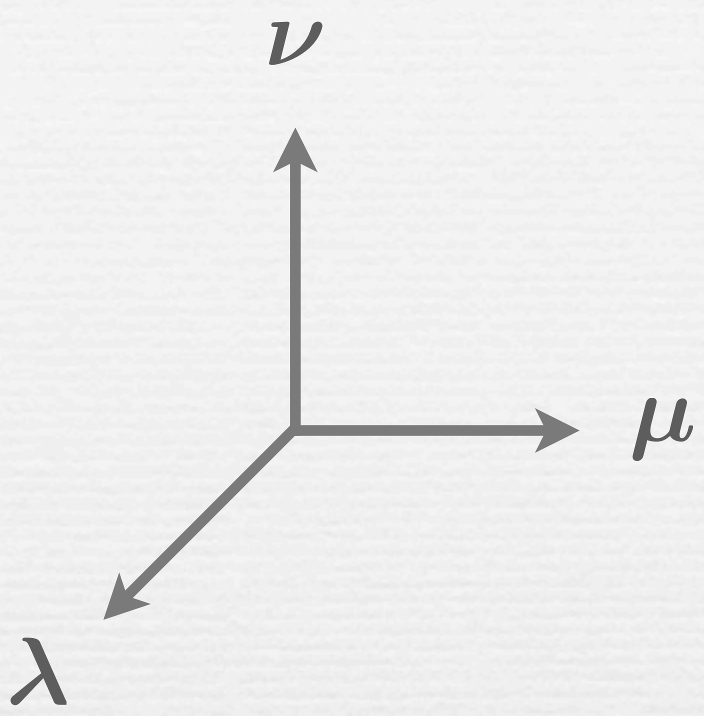
$$= \prod_{n=1}^{\infty} (1 - Qq^n)^n$$

Formulae

$$s_{\mu}(Qx) = Q^{|\mu|} s_{\mu}(x)$$

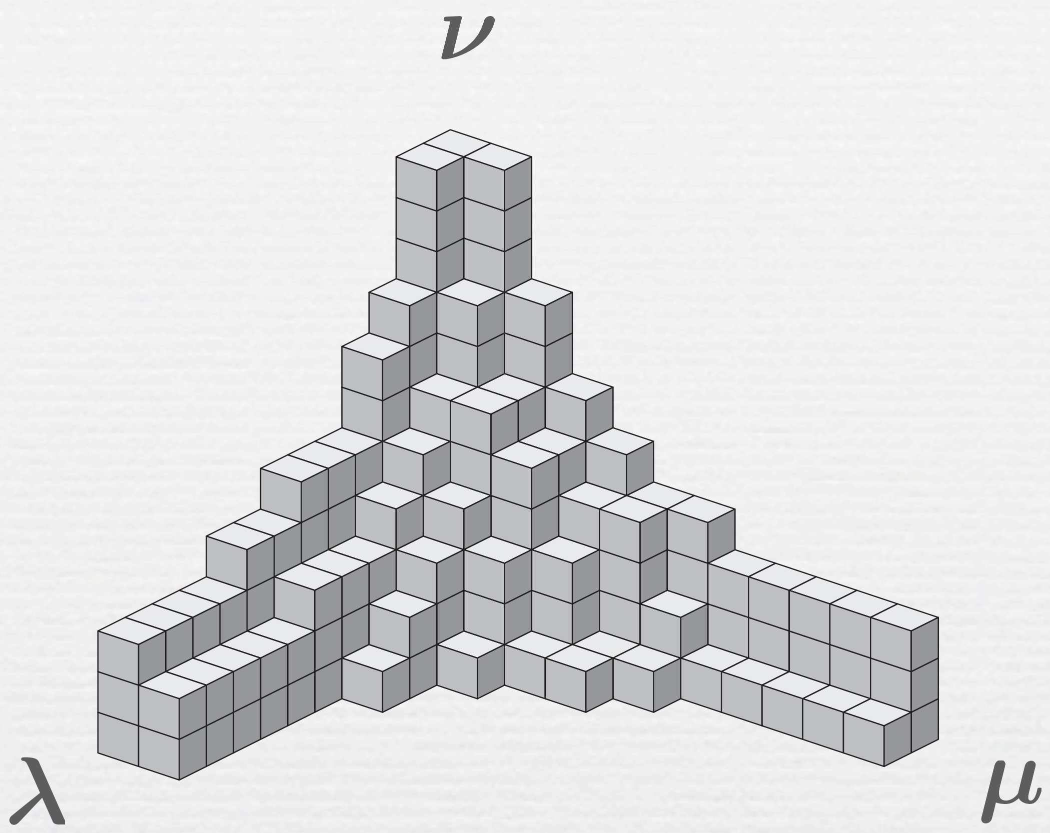
$$\sum_{\mu} s_{\mu^t}(x) s_{\mu}(y) = \prod_{i,j} (1 + x_i y_j)$$

- Duality to Crystal melting [Okounkov-Leshetikin-Vafa, '04]



$\mathbb{C}^3$ -patch

$\longleftrightarrow$



melting crystal corner

grand-canonical ensemble melting crystals

topological vertex !

$$Z_{\lambda,\mu,\nu} = \sum_{\text{crystals}} e^{-\hbar\#(\text{boxes})} \longrightarrow C_{\lambda,\mu,\nu}$$

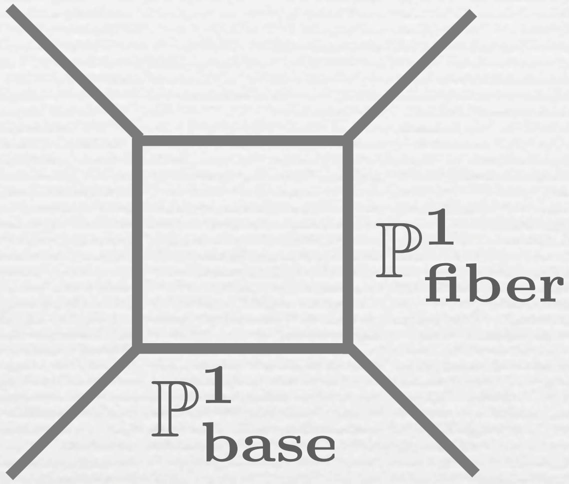
$$\frac{1}{k_B T} = \hbar$$

From the melting crystals picture, there exist cyclic symmetries

$$C_{\lambda,\mu,\nu}(q) = C_{\mu,\nu,\lambda}(q) = C_{\nu,\lambda,\mu}(q)$$

- **Geometric Engineering** [Iqbal-KashaniPoor, '04],[Eguchi-Kanno, '04]

Local Hirzhebruch : $K \to \mathbb{P}^1 \times \mathbb{P}^1$



$$Z = \sum_{\mu_1,\mu_2,\mu_3,\mu_4} {Q_F}^{|\mu_1|+|\mu_3|} \, {Q_B}^{|\mu_2|+|\mu_4|} \, q^{-\kappa_{\mu_1}/2+\kappa_{\mu_2}/2-\kappa_{\mu_3}/2-\kappa_{\mu_4}/2}$$

$$\times \, C_{\phi\,\mu_1\,\mu_4{}^t} \, C_{\phi\,\mu_2{}^t\,\mu_1{}^t} \, C_{\mu_2\,\phi\,\mu_3} \, C_{\phi\,\mu_4\,\mu_3{}^t}$$

$$= \sum_{\mu_2,\mu_4} {Q_B}^{|\mu_2|+|\mu_4|} \, q^{+\kappa_{\mu_2}/2-\kappa_{\mu_4}/2} \, K_{\mu_4\,\mu_2}(Q_F) \, K_{\mu_2{}^t\,\mu_4{}^t}(Q_F)$$

$$
\begin{aligned}
K_{\mu\nu} &= \sum_{\lambda} {Q_F}^{|\lambda|}\, q^{-\,\kappa_\lambda\,/2}\, C_{\phi\lambda\,\mu^t}\, C_{\nu^t\,\lambda^t\,\phi} \\
&= s_{\mu^t}(q^{-\rho})\, s_{\nu}(q^{-\rho}) \sum_{\lambda} {Q_F}^{|\lambda|}\, s_{\lambda}(q^{-\mu-\rho})\, s_{\lambda}(q^{-\,\nu^t\,-\rho}) \\
&= q^{\|\mu\|^2\,/2+||\nu^t||^2/2}\, \tilde{Z}_{\mu^t}(q)\, \tilde{Z}_{\nu}(q)\, \prod_{i,j=1}^{\infty} \frac{1}{1-Q_F\, q^{-\,\mu_i\,-\,{\nu^t}_j\,+i+j-1}}
\end{aligned}
$$

$$
s_{\mu}(q^{-\rho}) = q^{||\mu^t||^2/2} \prod_{s\in\mu} \left(1-q^{h_\mu(s)}\right)^{-1} = q^{||\mu^t||^2/2}\, \tilde{Z}_{\mu}(q)
$$

Let us extract the instanton part from the above partition function

$$
Z(Q_F, Q_B) = Z^{\text{pert.}}(Q_F) Z^{\text{inst.}}(Q_F, Q_B)
$$

$$
Z^{\text{pert.}}(Q_F) \equiv K_{\phi\phi}\,(Q_F)^2 = \left[\prod_{i,j=1}^{\infty} \frac{1}{1-Q_F\, q^{i+j-1}}\right]^2
$$

Then we obtain

$$Z^{\text{inst.}}(Q_F, Q_B) \sum_{\mu,\nu} Q_B{}^{|\mu|+|\nu|}\, q^{||\mu||^2+||\nu^t||^2}\, \tilde{Z}_\mu(q)\, \tilde{Z}_{\mu^t}(q)\, \tilde{Z}_\nu(q)\, \tilde{Z}_{\nu^t}(q) \left[\prod_{i,j=1}^{\infty} \frac{1 - Q_F\, q^{+i+j-1}}{1 - Q_F\, q^{-\mu_i - \nu^t{}_j + i + j - 1}} \right]^2$$

Identification with gauge theory parameters

$$Q_B = (\beta\Lambda)^4 \qquad Q_F = e^{-4\beta a} \qquad q = e^{-2\beta\hbar}$$

Under this identification, this topological string partition function is precisely the Nekrasov partition function of SU(2) gauge theory on $\mathbb{R}^4 \times S^1{}_\beta$

$$Z^{\text{inst.}} = \sum_{k=0}^{\infty} 4^k (e^{-\beta a}\beta\Lambda)^{4k}$$
$$\times \sum_{|\mu|+|\nu|=k} W_\mu^2(q) W_{\nu^t}^2(q) \prod_{n\in\mathbb{Z}} (1 - e^{-2\beta(2a+n\hbar)})^{-2C_n(\mu,\nu^t)}$$

$$W_\mu(q) = s_\mu(q^\rho)$$

2. Refined Topological Vertex

- Seiberg-Witten theory

Low energy effective action of $\mathcal{N}=2$ gauge theory is described by Seiberg-Witten prepotential

$$S_{\text{eff}} = \int \tau_{ij} F^i_{\mu\nu} F^{j\mu\nu} d^4x + \cdots$$

$$\tau_{ij} = \frac{\partial^2}{\partial a_i \partial a_j} \mathcal{F}(a, \Lambda)$$

Low energy effective theory is Higgsed $SU(N) \to U(1)^{N-1}$ by vev

$$\langle \Phi \rangle = \mathrm{diag}(a_1, \cdots, a_N)$$

$F^i_{\mu\nu}$ $i = 1, \cdots, N-1$ are Cartans $U(1)^{N-1}$

- **Nekrasov formulae** [Nekrasov, '02]

Nekrasov gave the generating function of Seiberg-Witten prepotential via instanton caluculus

$$Z^{\mathrm{Nek.}}(a,\Lambda,\hbar)=\exp\sum_{g=0}^{\infty}\hbar^{2g-2}\mathcal{F}_g(a,\Lambda)$$

$$\mathcal{F}_0(a,\Lambda)=\mathcal{F}^{\mathrm{SW}}(a,\Lambda)\,,\quad \mathcal{F}^{\mathrm{SW}}=\lim_{\hbar\to 0}\hbar^2\log Z^{\mathrm{Nek.}}$$

It is given by a specific sigma model on instanton moduli space

$$Z^{\mathrm{Nek.}}=\sum_{k=0}^{\infty}\Lambda^{2Nk}Z_k$$

$$Z_k=\int_{\mathcal{M}(N,k)}\mathcal{D}\mu\, e^{-Q\Psi}$$

In general, Nekrasov formula has two parameters $\hbar\longrightarrow\epsilon_1,\epsilon_2$

Explicit expression for Nekrasov partition function is

$$Z^{Nek.}(\vec{a},\Lambda,\epsilon_1,\epsilon_2)=\sum_{\{\mu_\alpha\}}e^{-\frac{1}{2}N|\vec{\mu}|(\epsilon_1+\epsilon_2)}\Lambda^{2N|\vec{\lambda}|}$$

$$\prod_{\alpha,\beta=1}^{N}\frac{1}{\prod_{s\in\mu_\alpha}(1-e^{-(a_{\mu_\alpha}+1)\epsilon_1+l_{\mu_\beta}\epsilon_2+a_\alpha-a_\beta})}$$

$$\frac{1}{\prod_{t\in\mu_\beta}(1-e^{a_{\mu_\beta}\epsilon_1-(l_{\mu_\alpha}+1)\epsilon_2+a_\alpha-a_\beta})}$$

Recall that the partition function is precisely the topological string partition function of SU(N)-geometry for $\epsilon_1=-\epsilon_2=\hbar$

$$Z^{\mathrm{Nek.}}(\vec{a},\Lambda,\epsilon_1=-\epsilon_2=\hbar)=Z^{\mathrm{A-model}}(t_i,\hbar)$$

So it is very natural to expect that there exist the extension of topological string which recover the Nekrasov’s partition function for $\epsilon_1\neq-\epsilon_2$

- Refinement in superstrings

Let us consider BPS state counting under non self-dual background in order to "refine" the generating function

$$\mathcal{F} = \sum_{\Sigma\in H_2(M,\mathbb{Z})} \sum_{n\in Z} \sum_{j_L,j_R} N_{\Sigma}^{(j_L,j_R)} \log\det{}_{(j_L,j_R)} \left(\Delta + m_{(\Sigma,n)}{}^2 + 2m_{(\Sigma,n)}\,\sigma_L\left(F_+ + F_-\right)\right)$$

$$= \sum_{\Sigma\in H_2(M,Z)} \sum_{k=1}^{\infty} \sum_{j_L,j_R} N_{\Sigma}^{(j_L,j_R)} (-1)^{-2(j_L+j_R)} e^{-k\,T_\Sigma} \frac{\left(\sum_{l=-j_L}^{j_L} (tq)^{-2kl}\right)\left(\sum_{m=-j_R}^{j_R} (t/q)^{-2km}\right)}{k\left(t^{k/2}-t^{-k/2}\right)\left(q^{k/2}-q^{-k/2}\right)}$$

$$q = e^{F_+} \qquad t = e^{F_-}$$

→ This is the refine partition function in the context of superstring theory.

- **Refined Topological Vertex**

It is very hard to find a guiding star for refining the topological vertex formalism

- **Awata-Kanno's idea**

The Macdonald functions are a multi-parameter extension of the Schur functions.

$$s_\mu(q^{-\rho}) = q^{\frac{||\mu^t||^2}{2}} \tilde{Z}_\mu(q)$$

$$\Big\downarrow$$

$$P_{\nu^t}(t^{-\rho};q,t) = t^{\frac{1}{2}||\nu||^2}\tilde{Z}_\nu(t,q), \quad \tilde{Z}_\mu(t,q) = \prod_{(i,j)\in\nu}(1-t^{\nu_j^t-i+1}q^{\nu_i-j})^{-1}$$

Macdonald function

$$\tilde{Z}_\mu(t,q) \xrightarrow{t\,\rightarrow\,q} \tilde{Z}_\mu(q)$$

So we may obtain a refined vertex by replacing the Schur functions with the specialized Macdonald functions !

[Awata-Kanno, '05, '08]

$$C_{\lambda\mu}{}^{\nu}(t,q) = f_\nu(t,q)^{-1} P_{\mu^t}(t^\rho;q,t)$$
$$\times \sum_\eta \left(\frac{q}{t}\right)^{\frac{|\eta|}{2}} (-1)^{|\lambda|+|\eta|} \iota P_{\lambda^t/\eta^t}(t^{\mu^t} q^\rho;t,q) P_{\nu/\eta}(t^\rho q^\nu;q,t)$$

↑

skew Macdonald function

The framing factor is [M.T. '07]

$$f_\nu(t,q) = (-1)^{|\nu|} t^{-\frac{||\nu^t||^2}{2}} q^{\frac{||\nu||^2}{2}}$$

For the geometric engineering Calabi-Yau's, the amplitudes computed using the vertex give the Nekrasov's partition functions [Awata-Kanno, '08].

Awata-Kanno's refined vertex possesses nice symmetries.

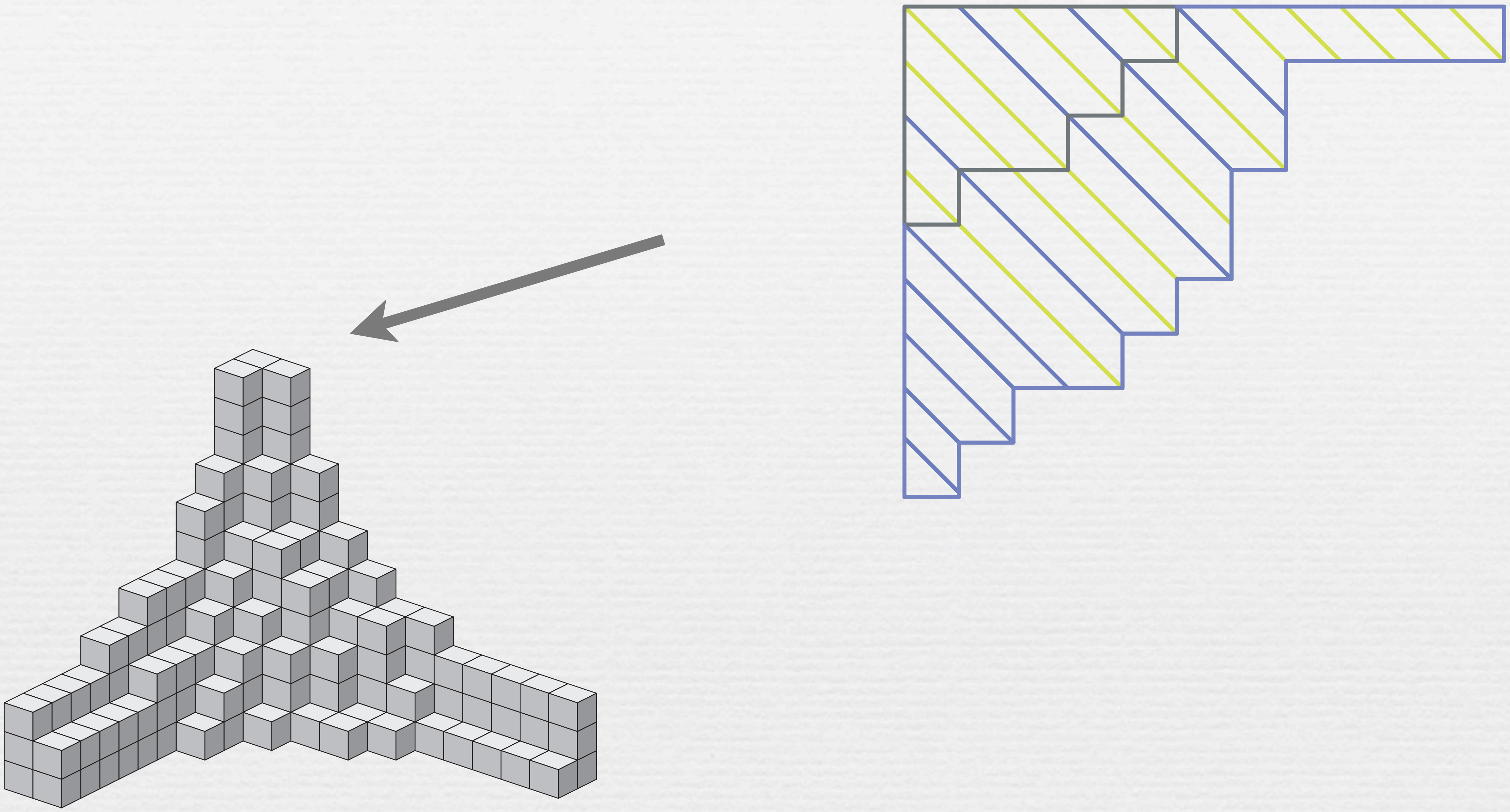
$$C_{\lambda\phi}{}^{\phi}(t,q) = C_{\phi\lambda}{}^{\phi}(t,q) = C_{\phi\phi}{}^{\lambda}(t,q)f_{\lambda}(t,q)$$

$$C_{\mu\phi}{}^{\nu}(t,q)f_{\nu}(t,q) = C_{\nu\phi}{}^{\mu}(t,q)f_{\mu}(t,q)$$

$$C_{\phi\lambda}{}^{\nu}(t,q)f_{\nu}(t,q) = C_{\phi\nu}{}^{\lambda}(t,q)f_{\lambda}(t,q)$$

But it doesn't have a cyclic symmetry.

- Iqbal-Kozcaz-Vafa’s idea

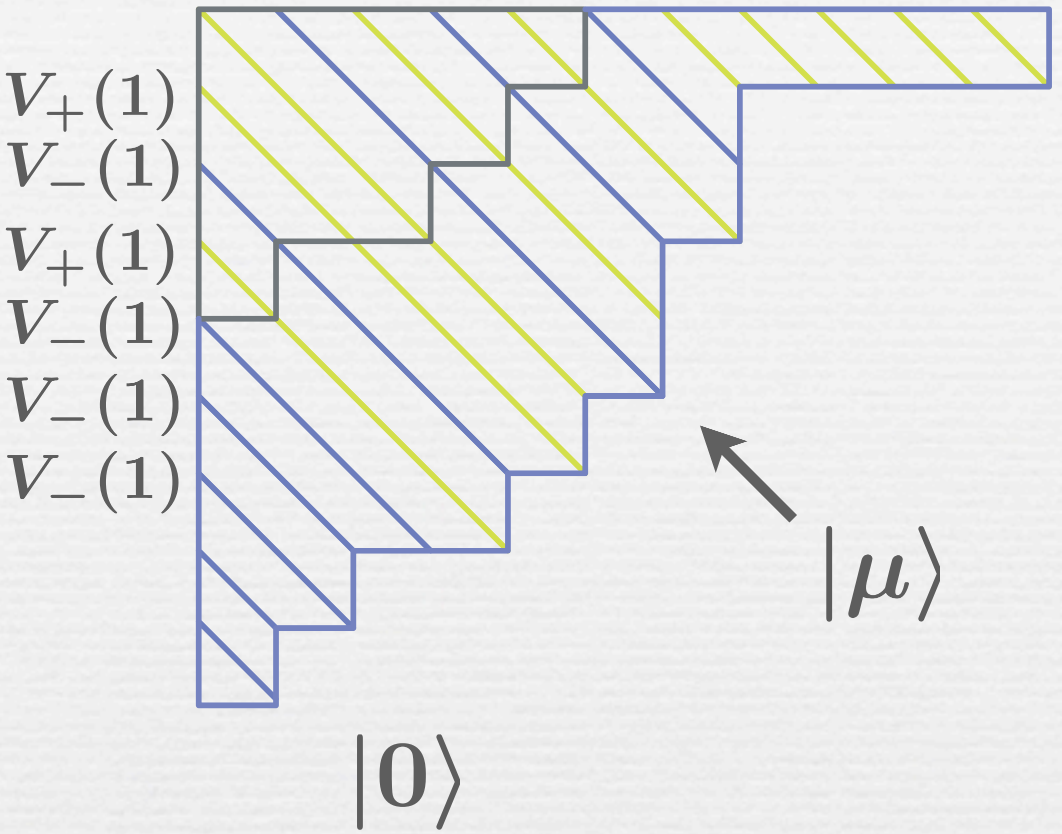


$$Z_{\lambda,\mu,\nu}(t,q) = \sum_{\text{crystals}} t^{\#(\text{blue})} q^{\#(\text{yellow})} \longrightarrow C_{\lambda,\mu,\nu}(t,q)$$

$$L_0|\mu\rangle = |\mu||\mu\rangle$$

$$V_-(1)|\mu\rangle = \sum_{\nu>\mu} |\nu\rangle$$

$$V_+(1)|\mu\rangle = \sum_{\nu<\mu} |\nu\rangle$$



$$\begin{aligned} Z_\nu &= \langle 0|q^{L_0}V_+q^{L_0}V_+t^{L_0}V_- \cdots q^{L_0}V_+t^{L_0}V_-t^{L_0}|0\rangle \\ &= \langle 0|\cdots V_+(q^{i-1}t^{-\nu_i^t})\cdots V_-(t^j q^{-\nu_j})\cdots|0\rangle \\ &= \prod_{i,j=1}^{\infty}(1-t^{j-\nu_i^t}q^{i-1-\nu_j}) \end{aligned}$$

Macdonald function !!

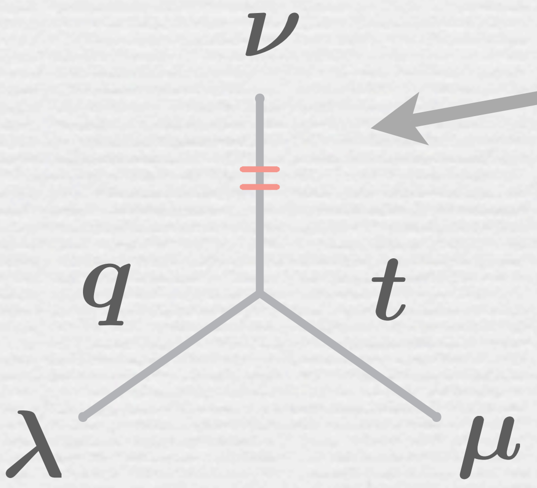
$$V_-(x)V_+(y) = \prod_{i,j=1}^{\infty}(1-x_iy_j)V_+(y)V_-(x)$$

[Iqbal-Kozcaz-Vafa]

$$C_{\lambda\mu\nu}(t,q) = \left(\frac{q}{t}\right)^{\frac{\|\mu\|^2+\|\nu\|^2}{2}} t^{\frac{\kappa_\mu}{2}} P_{\nu^t}(t^{-\rho};q,t) \sum_\eta \left(\frac{q}{t}\right)^{\frac{|\eta|+|\lambda|-|\mu|}{2}} s_{\lambda^t/\eta}(t^{-\rho}q^{-\nu}) s_{\mu/\eta}(t^{-\nu^t}q^{-\rho})$$

IKV's refined vertex breaks the cyclic symmetry for three legs of the vertex !

$$C_{\lambda,\mu,\nu}(t,q) \neq C_{\mu,\nu,\lambda}(t,q) \neq C_{\nu,\lambda,\mu}(t,q)$$



preferred direction

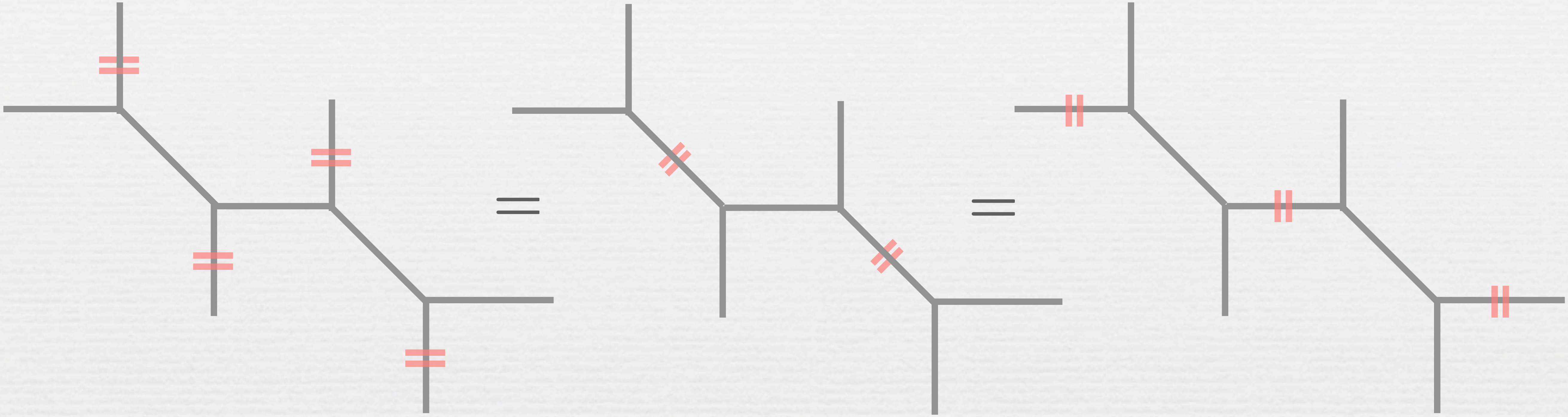
additional rule

take all the direction parallel (same $\mathbb{CP}^1$)

→ There exists the ambiguity about a choice of preferred direction when one construct the amplitudes using the refined vertex !

For the toric Calabi-Yau's corresponding to $\mathcal{N}=2$ gauge theory, it is proposed that the amplitudes are independent of the choices of their preferred directions !

Slicing invariance (conjecture !)

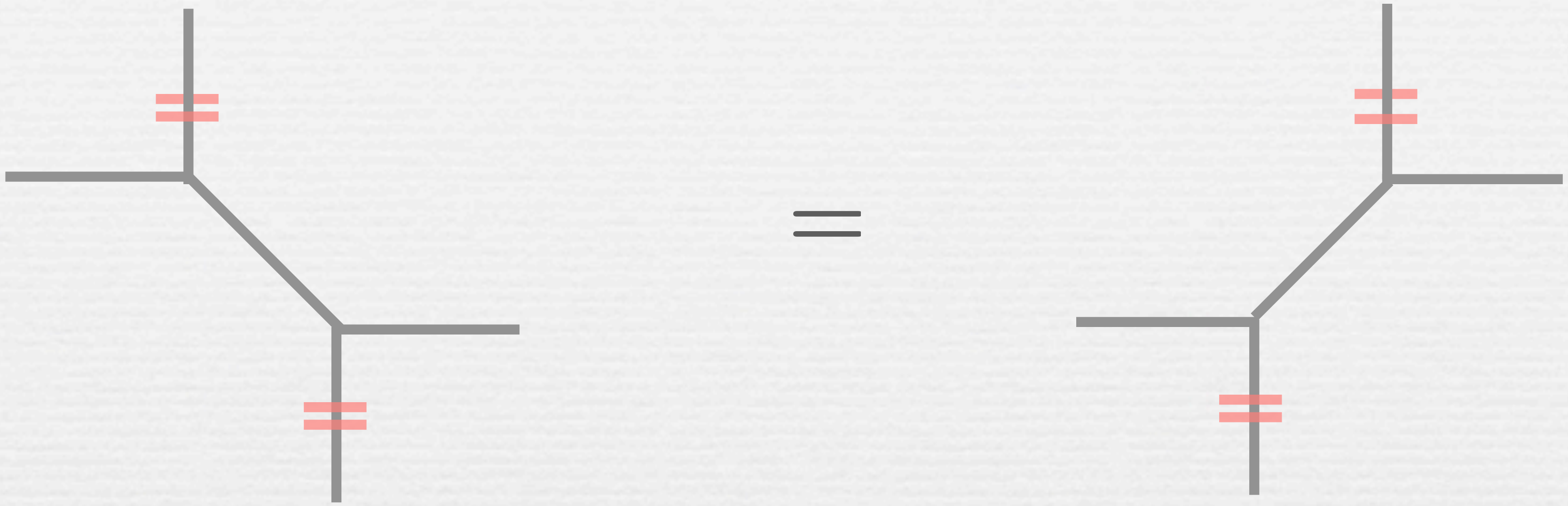


$$\sum_{\nu} P_{\nu}(-Q_1 t^{-\rho}; q, t) P_{\nu^t}(q^{-\rho}; t, q) \prod_{(i,j)\in\nu} (1 - Q_2 t^{i-1/2} q^{j-1/2-\nu_i})$$

$$= \prod_{i,j=1}^{\infty} \frac{1 - Q_1 t^{i-1/2} q^{j-1/2}}{1 - Q_1 Q_2 t^{i} q^{j-1}}$$

New formula for Macdonald function (For $Q_2 = 0$, see [Macdonald, '95])

Flop invariance [M.T, ‘08]



$$t \longrightarrow -t$$

- Refined Vertex & Hilbert Scheme [Iqbal-Kozcaz-Shabbir, '08]

- $\mathbf{Hilb}^n[\mathbb{C}^2]$ = Hilbert scheme of n points on $\mathbb{C}^2$

  = $\mathbb{C}^2$ with the choice of n points on it

- χ_y - genus

$$\chi_y(M) = \int_M \mathbf{ch}\Lambda_{-y}(T^*M)\mathbf{td}(M)$$

$$= \int_M \prod_{i=1}^{d} (1 - ye^{-x_i}) \frac{x_i}{1 - e^{-x_i}}$$

$$\Big\downarrow \text{ Localization formula under } U(1)^n \to M$$

$$= \sum_{p=1}^{n} \prod_{i=1}^{d} \frac{(1 - ye^{-w_{p,i}})}{1 - e^{-w_{p,i}}}$$

weights of torus action

fixed points of torus action

$T^*(\mathbf{Hilb}^n[\mathbb{C}^2]) = B_n \oplus B_n^*$

The torus action on $\mathbb{C}^2$ lifts to an action on $\mathbf{Hilb}^n[\mathbb{C}^2]$ and their weights had been obtained in [Nakajima, '95]. Thus we get

- $\chi_0(B_n)|_\nu = \tilde{Z}_\nu(t,q) \quad n = |\nu|$

ν : fixed point of torus action $(z_1, z_2) \to (tz_1, q^{-1}z_2)$

$$\longrightarrow \quad C_{\phi\phi\nu}(t,q)$$

The rest one-leg vertices are also interpreted as χ_y -genus

- $\chi_0(B_n)|_\mu = s_\mu(q^{-\rho}) \qquad (z_1, z_2) \to (q^{-1}z_1, qz_2)$

$$\longrightarrow \quad C_{\phi\mu\phi}(t,q)$$

- $\chi_0(B_n)|_\lambda = s_\lambda(t^{-\rho}) \qquad (z_1, z_2) \to (tz_1, t^{-1}z_2)$

$$\longrightarrow \quad C_{\lambda\phi\phi}(t,q)$$

$$\begin{tikzcd}
& \mathbf{Hilb}^{\bullet}[\mathbb{C}^2] & \\
& \mathbf{Hilb}^{\bullet}[\mathbb{C}^3] \arrow[u, swap, "{\begin{array}{l}(z_1,z_2)\to(tz_1,q^{-1}z_2)\\ z_3=0\end{array}}"] \arrow[r] \arrow[dl, "{\begin{array}{l}(z_1,z_2,z_3)\to\\ \quad (tz_1,q^{-1}z_2,t^{-1}qz_3)\end{array}}"] & \mathbf{Hilb}^{\bullet}[\mathbb{C}^2] \\
\mathbf{Hilb}^{\bullet}[\mathbb{C}^2] & &
\end{tikzcd}$$

$(z_2, z_3) \to (q^{-1}z_2, qz_3)$

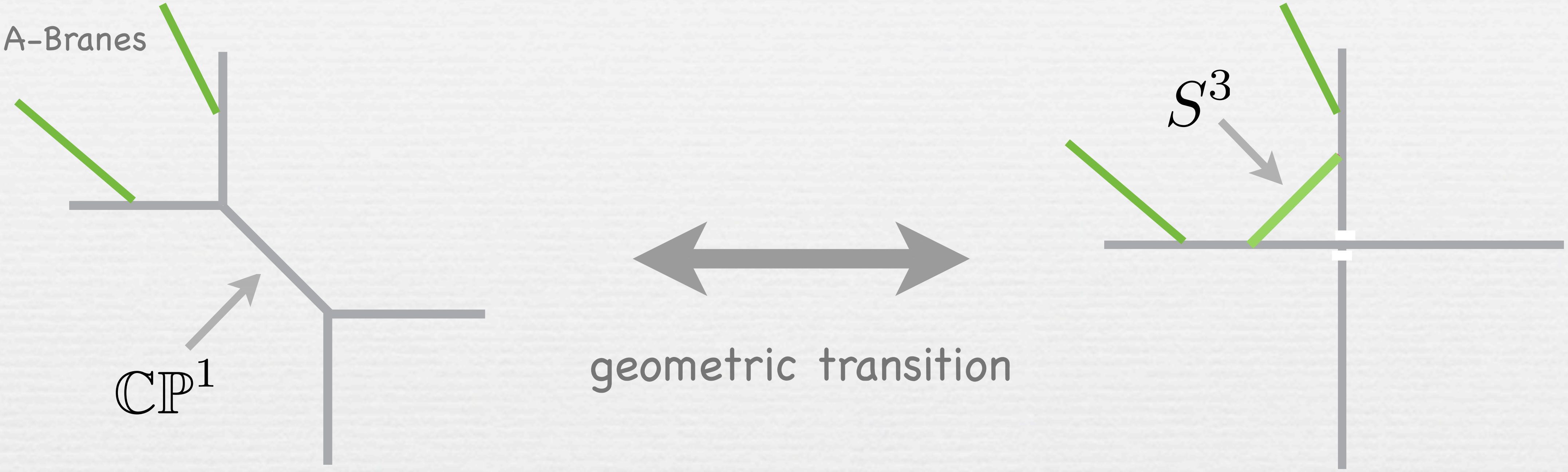
$z_1 = 0$

$(z_1, z_3) \to (tz_1, t^{-1}z_3)$

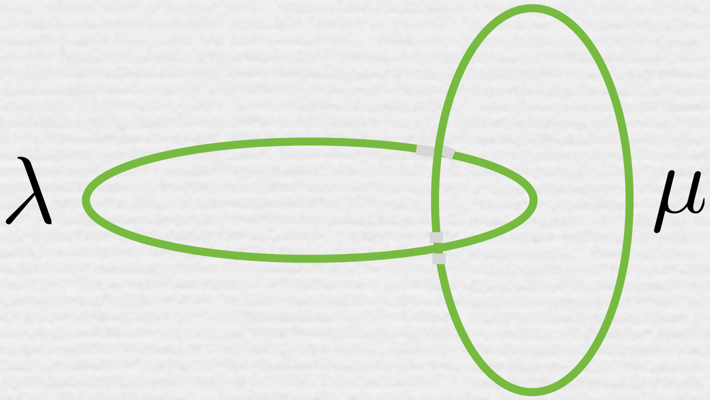
$z_2 = 0$

- **Homological Link Invariants**

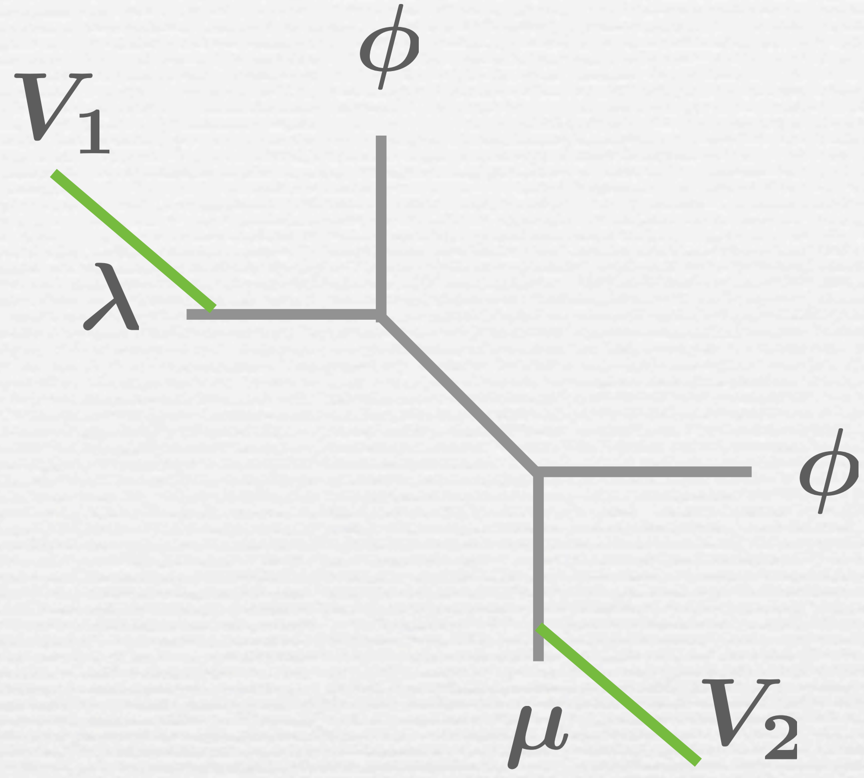
- Chern-Simons invariants in topological strings [Ooguri-Vafa]



Boundaries of worldsheets make Wilson loop in S^3



Hopf link



$$Z = \sum_{\lambda,\mu} Z_{\lambda,\mu}(q;Q)\,\mathrm{Tr}_\lambda V_1\,\mathrm{Tr}_\lambda V_2$$

$$\downarrow$$

$$Z_{\lambda,\mu}(q;Q) = \sum_{\nu} C_{\lambda,\nu,\phi}(q)(-Q)^{|\nu|}C_{\nu^t,\mu,\phi}(q)$$

$$\Big\downarrow \quad Q = q^N$$

$$\boxed{W_{\lambda\mu} = q^{\kappa_\mu/2}s_\lambda(q^{-\rho})s_\mu(q^{-\rho-\lambda}, q^{-N+\rho}) \prod_{(i,j)\in\lambda}(1 - q^{-N+i-j})}$$

Hopf link invariant !!

- Homological link invariants

Polynomial invariants of knots and links

$$\bar{\mathcal{P}}^{sl(N)}_{R_1,\cdots,R_k}(\mathbf{q}) = \sum_{i,j\in\mathbb{Z}} (-1)^j \mathbf{q}^i \dim \mathcal{H}^{sl(N),R_1,\cdots,R_k}_{i,j}(L)$$

Euler characteristic

Conjectural cohomology

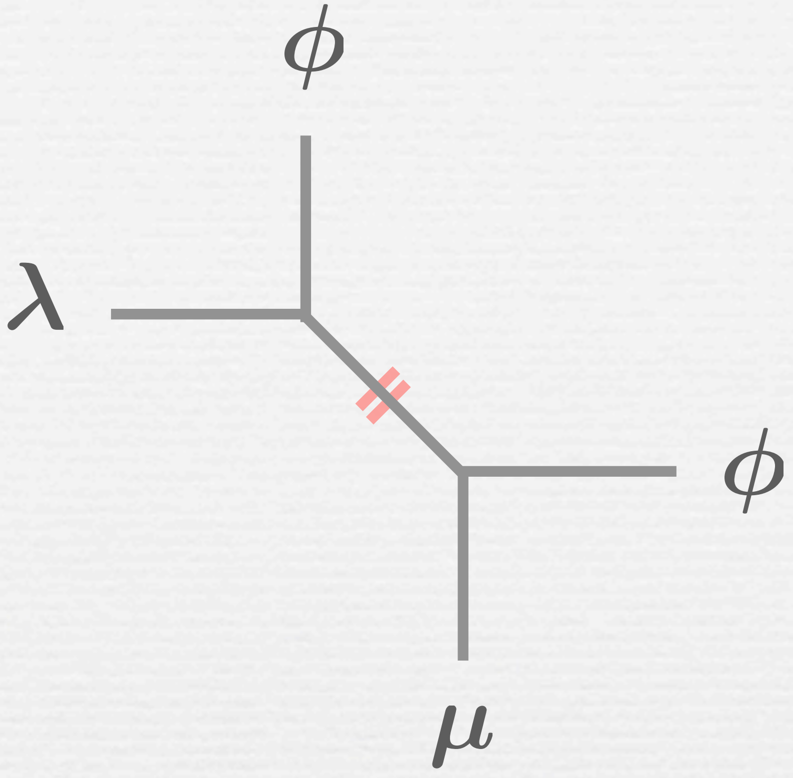
Homological invariant

$$\bar{\mathcal{P}}^{sl(N)}_{R_1,\cdots,R_k}(\mathbf{q},\mathbf{t}) = \sum_{i,j\in\mathbb{Z}} \mathbf{q}^i \mathbf{t}^j \dim \mathcal{H}^{sl(N),R_1,\cdots,R_k}_{i,j}(L)$$

Poincare characteristic

Mathematical theory of homological link invariant is formulated for some representations.

Gukov-Iqbal-Kozcaz-Vafa embedded it into refined topological strings expecting that it gives some insights into their formulation.



$$Z_{\boldsymbol{\lambda\mu}}(t,q,Q)=\sum_{\nu}(-Q)^{|\nu|}C_{\boldsymbol{\phi\mu\nu}}(t,q)C_{\boldsymbol{\lambda\phi\nu^t}}(q,t)$$

Recall the refined topological vertex

$$C_{\boldsymbol{\lambda\mu\nu}}(t,q)=\left(\frac{q}{t}\right)^{\frac{\|\mu\|^2+\|\nu\|^2}{2}}t^{\frac{\kappa_\mu}{2}}P_{\nu^t}(t^{-\rho};q,t)\sum_{\eta}\left(\frac{q}{t}\right)^{\frac{|\eta|+|\lambda|-|\mu|}{2}}s_{\lambda^t/\eta}(t^{-\rho}q^{-\nu})s_{\mu/\eta}(t^{-\nu^t}q^{-\rho})$$

From the partition function we get the superpolynomial

Gukov-Iqbal-Kozcaz-Vafa

$$\bar{\mathcal{P}}_{\lambda.\mu}(\mathbf{q},\mathbf{t},\mathbf{a})=\sum_{\nu}(-Q)^{|\nu|}t^{\frac{1}{2}||\nu||^2}q^{\frac{1}{2}||\nu^t||^2}\tilde{Z}_\nu(q,t)\tilde{Z}_{\nu^t}(t,q)s_\lambda(t^{-\rho}q^{-\nu^t})s_\mu(t^{-\rho}q^{-\nu^t})$$

- Superpolynomial proposal of Gukov-Iqbal-Kozcaz-Vafa

Superpolynomial [Gukov-Iqbal-Kozcaz-Vafa, '07]

$$\bar{\mathcal{P}}_{\lambda.\mu}(\mathbf{q},\mathbf{t},\mathbf{a}) = \sum_{\nu}(-Q)^{|\nu|}t^{\frac{1}{2}||\nu||^2}q^{\frac{1}{2}||\nu^t||^2}\tilde{Z}_{\nu}(q,t)\tilde{Z}_{\nu^t}(t,q)s_{\lambda}(t^{-\rho}q^{-\nu^t})s_{\mu}(t^{-\rho}q^{-\nu^t})$$
$$\times \prod_{i,j=1}(1-Qt^{1-1/2}q^{j-1/2})^{-1}(-1)^{|\lambda|+|\mu|}\left(Q^{-1}\sqrt{\frac{q}{t}}\right)^{\frac{|\lambda|+|\mu|}{2}}\left(\frac{q}{t}\right)^{|\lambda||\mu|}$$

New parameters

$$\sqrt{t} = \mathbf{q}, \quad \sqrt{q} = -\mathbf{t}\mathbf{q}, \quad Q = -\mathbf{t}/\mathbf{a}^2.$$

↓

$\mathbf{a} = \mathbf{q}^N$ (Large-N duality)

Homological link invariants for Hopf link

$$\bar{\mathcal{P}}^{sl(N)}_{R_1,\cdots,R_k}(\mathbf{q},\mathbf{t})$$

They must give homological invariants

- Example

Hopf link

$$\bar{\mathcal{P}}_{\Box,\Box}(\mathbf{q},\mathbf{t},\mathbf{a}) = \frac{1}{\mathbf{a}^2}\left(\frac{1-\mathbf{q}^2+\mathbf{q}^4\mathbf{t}^2}{(1-\mathbf{q}^2)^2} - \mathbf{a}^2\frac{1+\mathbf{q}^2\mathbf{t}^2-\mathbf{q}^2+\mathbf{q}^4\mathbf{t}^2}{(1-\mathbf{q}^2)^2} + \mathbf{a}^4\frac{\mathbf{q}^2\mathbf{t}^2}{(1-\mathbf{q}^2)^2}\right) \qquad (*)$$

$(*)$ gives the Kovanov-Rozansky invariant for Hopf link

$$\boxed{\bar{\mathcal{P}}_{\Box,\Box}(\mathbf{q},\mathbf{t},\mathbf{a}=\mathbf{q}^N) = \mathbf{q}^{-2N} KhR(2_1^2)}$$

Polynomial !

$$KhR_{N=3}(2_1^2) = 1 + \mathbf{q}^2 + \mathbf{q}^4 + \mathbf{q}^4\mathbf{t}^2 + 2\mathbf{q}^6\mathbf{t}^2 + 2\mathbf{q}^8\mathbf{t}^2 + \mathbf{q}^{10}\mathbf{t}^2$$

$$KhR_{N=4}(2_1^2) = 1 + \mathbf{q}^2 + \mathbf{q}^4 + \mathbf{q}^4\mathbf{t}^2 + \mathbf{q}^6 + 2\mathbf{q}^6\mathbf{t}^2 + 3\mathbf{q}^8\mathbf{t}^2 + 3\mathbf{q}^{10}\mathbf{t}^2$$

$$+2\mathbf{q}^{12}\mathbf{t}^2 + \mathbf{q}^{14}\mathbf{t}^2$$

- BPS state counting in non-self dual backgrouond

- Non-SUSY attractor and OSV-SV conjecture

. . . . .

Summary

- We review topological vertex method of A-model caluculation
- We also see the proposal of refined topological vertex
- We apply the refined vertex for homological link invariants

Future Directions

- More applications of refined topological vertex and BPS states counting
- Rigorous definition of refined vertex and it's mathematical structure

Fin