Slight Violation of the Alday-Maldacena Duality for a Wavy Circle

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IMM2 arXiv : 0803.1547
see also IMM arXiv : 0712.0159
IM8 arXiv : 0712.2316

I) Introduction

- Gauge / gravity correspondence (& matrix models) continue to be the central themes of string theory

- ① BDS's (Bern, Dixon, Smirnov) conjectured exponentiation a la Sudarkov of the all order planar n- gluon amplitudes for perturbative N=4 SYM,

$$\Rightarrow \quad e^{-D_{\sqcap}}$$

which is now known to be slightly violated at n=6, L=2 loop level.

BDKKRSVV 08031465
DHKS 08031466

② A new version of gauge-string duality by Alday-Maldacena : computation at strong coupling by the minimal surface of an AdS_5 string

$$\Rightarrow \quad e^{-\kappa A_{\sqcap}}$$

- ① and ② $\Rightarrow$ A fruitful assessment of the issue $D_{\sqcap} \overset{?}{\approx} \kappa A_{\sqcap}$ and its reformulation are called for today.

Hints in the strong coupling limit would be supplied by IMM2

- Note that $D_{\sqcap} = \kappa A_{\sqcap}$, if it were true, would give a resolution to the AdS-minimal surface problem.

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VI) The computation at a wavy circle

$$\triangle \equiv \triangle_0 - \mathcal{D}^2 + \mathcal{D}$$

see also Dobashi, Ito, Iwasaki 0805.3594

II) symbolically

$$\text{“ } \mathcal{A}_n(\mathbf{p}_1, \ldots, \mathbf{p}_n|\lambda) \;=\; \mathcal{A}_{\text{tree}}\, \mathcal{A}_{\text{IR}}\, \mathcal{A}_{\text{finite}} \text{ ”}$$

factorizes & exponentiates

To be more precise,

$$\mathcal{A}_n = g^{n-2} \sum_{L=0}^{\infty} a^L \sum_{\rho} \mathrm{Tr}(T^{a_{\rho(1)}} \cdots T^{a_{\rho(n)}}) A_n^{(L)}(\rho(1), \ldots, \rho(n))$$

$$\lambda = g^2 N$$

$$D = 4 - 2\epsilon$$

ρ : noncyclic perm

$$a = \frac{\lambda \mu^{2\epsilon}}{8\pi^2} (4\pi e^{-\gamma})^{\epsilon}$$

Define, with the help of MHV & N=4 SUSY,

$$M_n^{(L)}(\epsilon) \;\equiv\; \text{“ } A_n^{(L)}(\epsilon)/A_n^{(0)} \text{ ”} \; : \text{ scalar function}$$

$$\mathcal{A}_n = \mathcal{A}_n^{(0)} \sum_{L=0}^{\infty} a^L M_n^{(L)}(\epsilon)$$

$$\stackrel{?}{=} \mathcal{A}_n^{(0)} \exp\left[\sum_{\ell=1}^{\infty} a^\ell \left(f^{(\ell)}(\epsilon) M_n^{(1)}(\ell\epsilon) + c^{(\ell)} + o(\epsilon)\right)\right]$$

BDS conjecture

$$f^{(\ell)}(\epsilon) = f_0^{(\ell)} + \epsilon f_1^{(\ell)} + \epsilon^2 f_2^{(\ell)}$$

known

BES ↓

$$f(\lambda) \equiv 4 \sum_{\ell=1}^{\infty} a^\ell f_0^{(\ell)}$$ planar cusp anomalous dimension $\stackrel{\text{large}}{\sim} \sqrt{\lambda}$

III)

- $D_{\sqcap} = \oint_{\sqcap} \oint_{\sqcap} \frac{dy^{\mu} dy'_{\mu}}{(y-y')^{2+\epsilon}}$ is known to reproduce, after computation, the rep. of $M_n^{(1)}$

  in momentum space

$\sqcap$: polygon made of light-like segment

$=$ external momenta $\mathbf{p}_a$

Drummond, Korchemsky, Sokatchev; Brandhuber, Heslop, Travaglini

IV)

- AdS / CFT duality ; $\sqrt{\lambda} \equiv \sqrt{g^2 N} = \frac{R^2}{\alpha'}, \ \frac{1}{N} \sim g_s$ assumed

- compute the same gluon amplitude at strong coupling using tree level semiclassical string theory

- take T-duality in $\mu = 0, 1, 2, 3$ directions

$$\partial_a x^\mu = i \frac{R^2}{r^2} \epsilon_{ac} \partial_c y^\mu , \quad r = \frac{R^2}{z}$$

- T-dualized geometry, which is again AdS_5

$$ds^2 = R^2 \frac{dr^2 + dy_\mu dy^\mu}{r^2} , \quad r_{IR} = \frac{R^2}{z_{IR}} \to 0$$

- semiclassical string amplitudes

$$\sim \ (\text{prefactor}) \ e^{-S_E[y^\mu = y^\mu_{sa}, r = r_{cl}, k^I_\mu]}$$

V)

- work on the Euclidean worldsheet
- choose $\xi^1 = y_1$, $\xi^2 = y_2$
- The 1st ansatz ; $y_3 = 0$ $\cdots$ ①

$$S_{E,NG} = \frac{\sqrt{\lambda}}{2\pi}\int dy_1 dy_2 \sqrt{\det H} \ , \quad H_{ij} = \frac{1}{r^2}(\delta_{ij} - \partial_i y_0 \partial_j y_0 + \partial_i r \partial_j r)$$

recognize this as $f(\lambda) \overset{\lambda \text{ large}}{\sim} \sqrt{\lambda}$

- The 2nd ansatz ; $1 = y_\mu y^\mu + r^2 \Leftrightarrow Y^4 = 0$ $\cdots$ ②

IMM1, IM8

- ① and ② form AdS_3 ansatz, which contains the Alday-Maldacena rhombus solution.

- <u>Eq. of motion</u>

$$\delta y_0 \;:\; \partial_1\left(\frac{H_{22}}{r^2\sqrt{\det H}}\partial_1 y_0\right)+\partial_2\left(\frac{H_{11}}{r^2\sqrt{\det H}}\partial_2 y_0\right) \\ -\partial_1\left(\frac{H_{21}}{r^2\sqrt{\det H}}\partial_2 y_0\right)-\partial_2\left(\frac{H_{12}}{r^2\sqrt{\det H}}\partial_1 y_0\right)=0$$

$$\delta r \;:\; \partial_1\left(\frac{H_{22}}{r^2\sqrt{\det H}}\partial_1 r\right)+\partial_2\left(\frac{H_{11}}{r^2\sqrt{\det H}}\partial_2 r\right) \\ -\partial_1\left(\frac{H_{21}}{r^2\sqrt{\det H}}\partial_2 r\right)-\partial_2\left(\frac{H_{12}}{r^2\sqrt{\det H}}\partial_1 r\right)+\frac{2\sqrt{\det H}}{r}=0$$

- <u>linear approximation to NG</u>

eliminate r^2 through $S_{E,NG}$ and ②, linearize w.r.t. y_0

$\triangle y_0 = 0$, where $\triangle = \triangle_0 - \mathcal{D}^2 + \mathcal{D}$

$\triangle_0 = 4\partial\bar\partial,\ \mathcal{D} = z\partial + \bar z\bar\partial$

solution

$$y_0 = \sum_{k\geq 0} Re(\alpha_k z^k)\,\underline{{}_2F_1\!\left(\frac{k}{2},\frac{k-1}{2};k+1;z\bar z = x\right)}$$

$$(1+k\sqrt{1-x})(1-\sqrt{1-x})^k \ /\ x^k$$

VI)

- how to deal with the issue $D_\sqcap \overset{?}{\approx} \kappa A_\sqcap$ fruitfully in the strong coupling side

- explicit examples containing ∞ ly many parameters needed

  $\Rightarrow$ an infinitesimal deformation of the unit circle into an arbitrary curve on the plane

- circle solution (formal $n = \infty$ limit of lightlike n-gon)

AdS$_3$ ansatz
$$\begin{cases} y_3 = 0 \\ 1 = r^2 + y_\mu y^\mu = r^2 - y_0^2 + y_i^2 \end{cases}$$

now $y_0 = 0$

- $$L_{NG} = \frac{1}{r^2}\sqrt{1+(\partial_i r)^2}$$

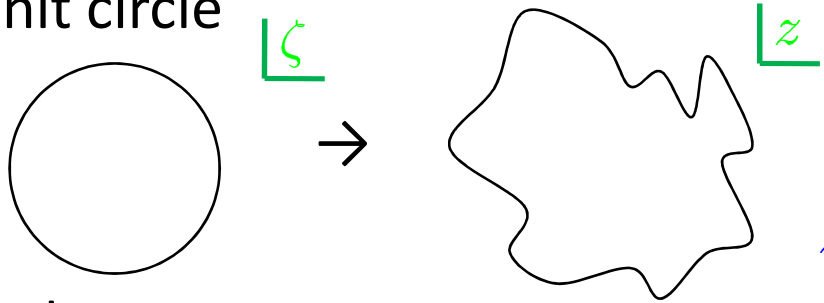
The only candidate to the solution which lie in these ansatze is

$r^2 = 1 - y_i^2$, which in fact solves IMM1

$$r(\partial_i r)(\partial_j r)(\partial_i \partial_j r) = (2 + r\partial^2 r)(1+(\partial_i r)^2)$$

- formulation

unit circle



bdd by Π

$z = y_1 + i y_2$

- consider the conformal map $z = H(\zeta)$
- find the shape of the minimal surface

$$r^2(z,\bar{z}) = 1 - \zeta\bar{\zeta} + a(\zeta,\bar{\zeta})$$

by solving the NG eq. for $a(\zeta,\bar{\zeta})$ subject to the b.c.

$$a|_{|\zeta|=1} = 0$$

action

- some simplification due to $\bar{\partial} z = 0$

- write $\partial z = 1 + \sum_{k=1}^{\infty} k h_k \zeta^{k-1} \equiv \partial H \equiv 1 + \partial h$

$$
\begin{aligned}
S_{NG}[a,h] &= \frac{\sqrt{\lambda}}{2\pi} \int d^2\zeta \frac{1}{r^2} \sqrt{|\partial H|^2 (|\partial H|^2 + 4 \partial r \bar{\partial} r)} \\
&= \frac{\sqrt{\lambda}}{2\pi} \int d^2\zeta \frac{|1+\partial h|^2 (1 - \zeta\bar{\zeta} + a + \frac{(\partial a - \bar{\zeta})(\bar{\partial} a - \zeta)}{|1+\partial h|^2})^{1/2}}{(1 - \zeta\bar{\zeta} + a)^{3/2}}
\end{aligned}
$$

- need to compute a=a(h) (at least) to the lowest order in h

- regularization needed

$$
0 = \partial \left(\frac{\partial \mathcal{L}}{\partial(\partial a)} \right) + \bar{\partial} \left(\frac{\partial \mathcal{L}}{\partial(\bar{\partial} a)} \right) - \frac{\partial \mathcal{L}}{\partial a} = \frac{1/4}{(1 - \zeta\bar{\zeta})^{3/2}} \triangle (a + \bar{\zeta} h + \zeta \bar{h}) + o(h^2)
$$

Eq. of motion

$$\triangle\psi = (\triangle_0 - \mathcal{D}^2 + \mathcal{D})\psi = 0,\ \triangle_0 = 4\partial\bar{\partial},\ \mathcal{D} = z\partial + \bar{z}\bar{\partial}$$

has appeared again in a-linearized problem

- The solution which satisfies the boundary condition is

$$a(\zeta,\bar{\zeta}) = 2\sum_{k=1}^{\infty} Re(h_k\zeta^{k-1})A_k(\zeta\bar{\zeta})$$

$$A_k(x) = F_{k-1}(x) - x$$

$$F_k(x) = \frac{(1+k\sqrt{1-x})(1-\sqrt{1-x})^k}{x^k}$$

- remaining procedure
    - substitute the solution into the regularized area
    - and evaluate it

- Results

$$\frac{D_\Pi}{2\pi} = \frac{L}{\lambda} - 2\pi - 4\pi \left[Q_\Pi^{(2)} - Q_\Pi^{(3,1)} - Q_\Pi^{(3,2)} \right] + 4\pi Q_\Pi^{(4)} + o(h^5)$$

$$\frac{A_\Pi}{2\pi} = \frac{L}{4\mu} - 1 - \frac{3}{2} \left[Q_\Pi^{(2)} - Q_\Pi^{(3,1)} - 4 Q_\Pi^{(3,2)} \right] + o(h^4)$$

$$Q_\Pi^{(2)} = \sum_{k=0}^{\infty} B_k |h_k|^2 \ , \quad B_k = \frac{k(k-1)(k-2)}{6}$$

$$Q_\Pi^{(3)} = \underset{\text{diagonal}}{\underset{\uparrow}{Q_\Pi^{(3,1)}}} + \underset{\text{off-diagonal}}{\underset{\uparrow}{Q_\Pi^{(3,2)}}} = \frac{1}{2} \sum_{i,j=0}^{\infty} c_{ij} (h_i h_j \bar{h}_{i+j-1} + \bar{h}_i \bar{h}_j h_{i+j-1})$$

$$c_{ij} = \frac{ij}{6} (i^2 + 3ij + j^2 - 6i - 6j + 7)$$

The red denote discrepancies

- comments on our result

- $\kappa_0 = \dfrac{8\pi}{3}$ in front of the bracket

  4 inside the bracket

  $\kappa_\square = 8$ $\quad \dfrac{\kappa_0}{\kappa_\square} = \dfrac{\pi}{3} \approx \dfrac{3.14}{3} \approx 1.05$ 5% discrepancy

- part in $D_\sqcap$ which is linear in $\bar{h}_\ell$ is consistent with the expression which is made of the Schwarzian derivative :

$$\oint_{\text{unit circle}} (\bar{z} - \bar{\zeta}) S_\zeta(z) \zeta^2 d\zeta \;, \quad S_\zeta(z) = \frac{z'''}{z'} - \frac{3}{2}\left(\frac{z''}{z'}\right)^2$$

- The planar nature of our wavy circle may have obscured some of the symmetry properties that this problem possesses.