Circular Wilson loop operator and master field

Shoichi Kawamoto (OCAMI, Osaka City University)

→ National Taiwan Normal University (from August)

With **T. Kuroki** (Rikkyo) and **A. Miwa** (Tokyo U., Komaba)

31 July 2008

Introduction

•BPS objects have played important roles in study of AdS/CFT correspondence

string(SUGRA) in a weakly curved background

$\updownarrow$

Strongly coupled gauge theory

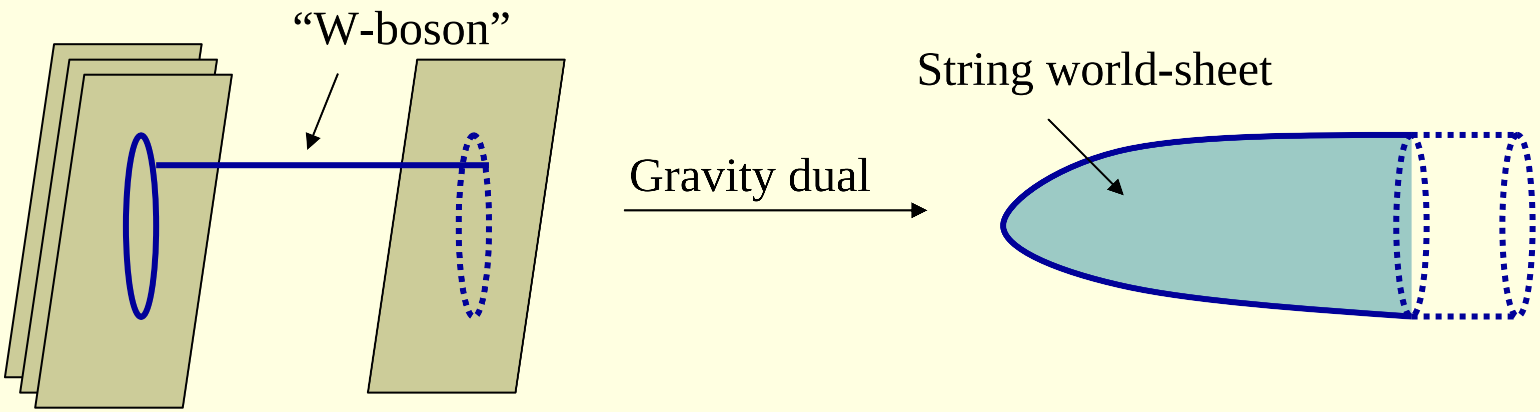
BPS objects will not get corrected by quantum effects

e.g. chiral primaries $I_{i_1 \cdots i_n} \mathrm{Tr}\left(\Phi^{i_1} \cdots \Phi^{i_n}\right)$

Today, we are going to look at different types of BPS objects.

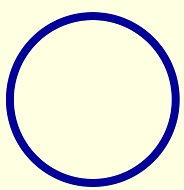
Introduction (continued)

In AdS/CFT correspondence, Wilson loops correspond to string (D-brane) world volume:

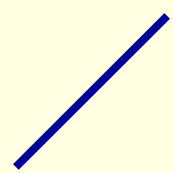


$$\left\langle W(C) \right\rangle = e^{-S_{\mathrm{string}}}$$

Wilson loop operators in N=4 super Yang-Mills may preserve some of supersymmetry.



Circular loop

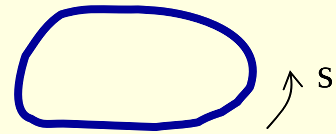


Straight line

Wilson loops in $\mathcal{N}$=4 super Yang-Mills

Wilson loop operators in SYM:

$$W(C) = \frac{1}{N} \mathrm{Tr} P \exp\left[\oint ds \left(iA_\mu(x(s))\dot{x}^\mu(s) + \Phi_i(y(s))\dot{y}^i(s) \right)\right]$$



In general, the existence of the loop breaks some of the symmetry of vacuum.

However, if the loop has a particular shape, it may preserve some of them.

SUSY transformation:

$$\delta_1 A_\mu = \overline{\Psi}\Gamma_\mu \varepsilon_1, \qquad \delta_2 A_\mu = \overline{\Psi}\Gamma_\mu (\Gamma_\nu x^\nu)\varepsilon_2,$$

$$\delta_1 \Phi_i = \overline{\Psi}\Gamma_i \varepsilon_1 \qquad \delta_2 \Phi_i = \overline{\Psi}\Gamma_i (\Gamma_\nu x^\nu)\varepsilon$$

Poincare SUSY superconformal

$$\delta W(C) = \frac{1}{N} \mathrm{Tr} P \int ds\, \overline{\Psi}\left(i\Gamma_\mu \dot{x}^\mu + \Gamma_i \dot{y}^i \right)\varepsilon\, e^{\oint iA+\Phi} \qquad \varepsilon = \varepsilon_1 + \Gamma_\mu x^\mu \varepsilon_2$$

BPS condition: $\left(i\Gamma_\mu \dot{x}^\mu + \Gamma_i \dot{y}^i \right)\varepsilon = 0$

$\dot{x}^2 = \dot{y}^2$ A half of ε locally solves the equation. (local BPS condition).

$\longrightarrow \quad \dot{y}^i = \theta^i |\dot{x}| \qquad \theta^2 = 1$

Supersymmetric Wilson loops

When θ is constant, we find two half-BPS solutions:

(Zaremobo, Drukker, …)

Straight line: $x^\mu(s) = (s,0,0,0)$

$\theta^i = \delta^{i5}$

$$\longrightarrow \quad \begin{aligned} &(i\Gamma_1 + \Gamma_5)\varepsilon_1 = 0, \\ &(i1 + \Gamma_{15})\varepsilon_2 = 0 \end{aligned}$$

Circular loop: $x^\mu(s) = (R\cos s, R\sin s, 0, 0)$

$\theta^i = \delta^{i5}$

$$\longrightarrow \quad \varepsilon_1 = iR\Gamma_{125}\varepsilon_2$$

These loops preserve $SL(2,R)\times SO(3) \subset SO(5,1)$ symmetry as well.

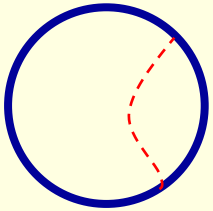
Circular Wilson loops and matrix model

Perturbative calculation of the expectation value of a circular Wilson loop.

$$\langle W(C)\rangle = 1 + \frac{1}{2N}\iint ds_1 ds_2\, \mathrm{Tr}\left\langle \left(iA_\mu \dot{x}^\mu(s_1) + \Phi_i\theta^i(s_1)|\dot{x}|\right)\left(iA_\nu \dot{x}^\nu(s_2) + \Phi_j\theta^j(s_2)|\dot{x}|\right)\right\rangle + \text{higher}$$

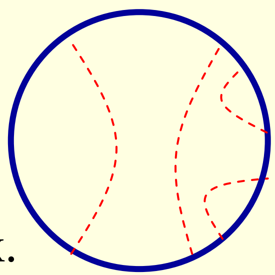
$$\mathcal{O}(g^2N) = \frac{N}{4}\iint ds_1 ds_2 \frac{g^2}{4\pi} \underbrace{\frac{-\dot{x}(s_1)\cdot\dot{x}(s_2) + |\dot{x}(s_1)||\dot{x}(s_2)|}{|\dot{x}(s_1)-\dot{x}(s_2)|^2}}_{=1/2}$$

$x(s) = (R\cos s, R\sin s, 0, 0)$



$$= \frac{N}{4}\frac{g^2}{4\pi^2}\frac{1}{2}\iint ds_1\, ds_2 = \frac{\lambda}{8}$$

$\mathcal{O}(\lambda^n)$:



$$= \frac{(\lambda/4)^n}{(2n)!} A_n$$

# of planar graphs

- Planar
- Ladder approx.

A_n can be calculated by Gaussian matrix model:

$$A_n = \left\langle \frac{1}{N}\mathrm{Tr}\, M^{2n}\right\rangle = \frac{1}{Z}\int dM\, \frac{1}{N}\mathrm{Tr}\, M^{2n}\, e^{-\frac{N}{2}\mathrm{Tr}\, M^2}$$

(Erickson-Semenoff-Zarembo)

$$\langle W(C)\rangle_{\mathrm{N=4\,SYM}} = \sum_{n=0}^{\infty}\frac{(\lambda/4)^n}{(2n)!}A_n = \left\langle \frac{1}{N}\mathrm{Tr}\, e^{\frac{\sqrt{\lambda}}{2}M}\right\rangle_{\text{Gaussian}} = \left\langle \frac{1}{N}\mathrm{Tr}\, e^{M}\right\rangle_{MM}$$

$$\langle \mathcal{O}\rangle_{MM} = \frac{1}{Z}\int dM\, \mathcal{O}\, e^{-\frac{2N}{\lambda}\mathrm{Tr}\, M^2}$$

Multiply wound loops and eigenvalue

If a circular loop goes around the same path *k* times

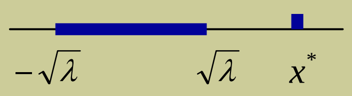
$\longrightarrow$ Consider the following partition function with a single trace operator

$$Z_k = \left\langle \frac{1}{N} \mathrm{Tr}\, e^{kM} \right\rangle_{MM} = \frac{1}{Z_0} \int \prod_{i=1}^{N} dx_i \, \frac{1}{N} \sum_i e^{k x_i} \, e^{-\frac{2N}{\lambda} \sum_i x_i^2} = \frac{1}{Z_0} \int \prod dx_i \, e^{-S_{eff}}$$

$$S_{eff} = \underbrace{\frac{2N}{\lambda} \sum_i x_i^2 - \sum_{i<j} \ln |x_i - x_j|^2}_{\mathrm{O}(\mathrm{N}^2)} - k x_N$$

O(k) ~ O(N)

Eigenvalue distribution



Define the eigenvalue distribution function $\rho(x) = \frac{1}{N} \sum_{i=1}^{N-1} \delta(x - x_i) + \frac{1}{N} \delta(x - x_N)$

O(N^2): semi-circle $\rho_0(x) = \frac{2}{\pi\lambda} \sqrt{\lambda - x^2}$

O(N): effective action for x_N $N^{-1} S(x_N) = \frac{x_N^2}{\lambda} - 2 \int_{-\sqrt{\lambda}}^{\sqrt{\lambda}} dx\, \rho(x) \ln\left| x - x_N \right| - \frac{k}{N} x_N$

$$\frac{\partial S_{eff}(x_N)}{\partial x_N} = 0 \quad \longrightarrow \quad x^* = \sqrt{\lambda} \sqrt{1 + \frac{k^2 \lambda}{16 N^2}} = \sqrt{\lambda} \sqrt{1 + \kappa^2} \qquad \kappa = \frac{k\sqrt{\lambda}}{4N}$$

(S.Yamaguchi)

Resolvent

To determine the eigenvalue distribution with having this loop operator, consider the expectation value of the resolvent:

$$R_k(z) = \left\langle \frac{1}{N}\mathrm{Tr}\frac{1}{z-M} \right\rangle_{kMM} \equiv \left\langle \frac{1}{N}\mathrm{Tr}\frac{1}{z-M}\frac{1}{N}\mathrm{Tr}\,e^{kM} \right\rangle_{MM}$$

$$= \left\langle \frac{1}{N}\mathrm{Tr}\frac{1}{z-M} \right\rangle_{MM}^{0} \left\langle \frac{1}{N}\mathrm{Tr}\,e^{kM} \right\rangle_{MM}^{0} + \frac{1}{N^2}\left\langle \frac{1}{N}\mathrm{Tr}\frac{1}{z-M}\ \frac{1}{N}\mathrm{Tr}\,e^{kM} \right\rangle_{MM}^{\mathrm{cyl}} + \cdots$$

By employing the loop equation (SD eq.),

$$\frac{1}{N^2}\left\langle \frac{1}{N}\mathrm{Tr}\frac{1}{z-M}\ \frac{1}{N}\mathrm{Tr}\,e^{kM} \right\rangle_{MM}^{\mathrm{cyl}} \cong \frac{k}{N^2}\left\langle \frac{1}{N}\mathrm{Tr}\frac{1}{z-M}e^{kM} \right\rangle_{MM}^{0}$$

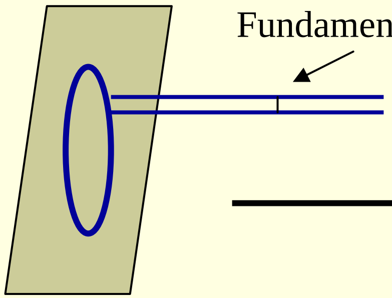
By Laplace transformation,

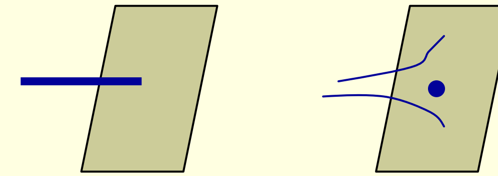
$$\frac{k}{N^2}\left\langle \frac{1}{N}\mathrm{Tr}\frac{1}{z-M}e^{kM} \right\rangle_{MM}^{0} = \frac{k}{N^2}\int_0^\infty dp\, e^{-pz}\left\langle \frac{1}{N}\mathrm{Tr}\,e^{(k+p)M} \right\rangle_{MM}^{0}$$

So 1/N correction (isolated pole) comes from a merged loop.

Note: to this order the resolvent is given by $R_k(z) = \frac{2}{\lambda}\left(z-\sqrt{z^2-\lambda}\right) + \frac{1}{N}\frac{1}{z-x^*}$

Wilson loops as D3-brane





A better description is given by D3-brane
(Giant Wilson loop)

Solution for the circular loop: **(Drukker-Fiol)**

Ansatz
$\eta(\rho), A_\psi(\rho)$

$$ds^2 = \frac{L^2}{\sin^2\eta}\left(d\eta^2 + \cos^2\eta d\psi^2 + d\rho^2 + \sinh^2\rho d\Omega_2^2\right) + L^2 d\Omega_5^2$$

$$S_{D3} = T_{D3}\int e^{-\phi}\sqrt{\det\left(g + 2\pi\alpha' F\right)} - T_{D3}\int P[C_4]$$

$$= 2N\int d\rho d\theta \frac{\sin\theta \sinh^2\rho}{\sin^4\eta}\left[\sqrt{\cos^2\eta\left(1+\eta'^2\right) + (2\pi\alpha')^2\frac{\sin^4\eta}{L^4}F_{\psi\rho}^2} - \left(\cos\eta + \eta'\sin\eta\frac{\sinh\rho - \cosh\rho\cos\theta}{\cosh\rho - \sinh\rho\cos\theta}\right)\right]$$

Solution of the equations of motion:

(***k***: number of strings (flux))

$$\sin\eta = \kappa^{-1}\sinh\rho$$

$$F_{\psi\rho} = \frac{ik\lambda}{8\pi N\sinh^2\rho} \qquad \left(\kappa = \frac{k\sqrt{\lambda}}{4N}\right)$$

Induced metric: $ds_{D3} = \frac{L^2}{\sin^2\eta}\left(\frac{1+\kappa^2}{1+\kappa^2\sin^2\eta}d\eta^2 + \cos^2\eta d\psi^2\right) + L^2\kappa^2 d\Omega_2^2$ $\quad \left(AdS_2 \times S^2\right)$

Boundary terms

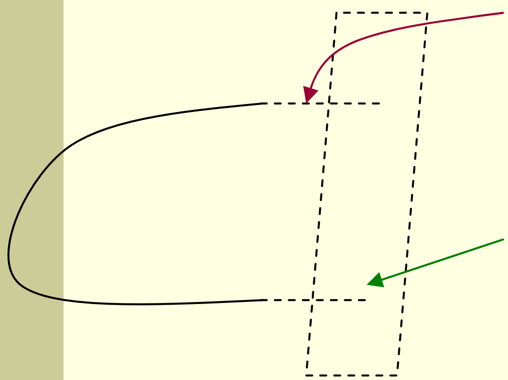
Needs to add the boundary terms $S_B = -\int \eta P_\eta - \int A_\psi \Pi$

η, A_ψ

$\updownarrow$

P_η, Π

(Drukker-Gross-Ooguri)



Near the boundary, the world-volume extends infinitely.

$\longrightarrow$ to cancel divergence (fundamental string)

At the boundary, need to fix P_η, Π (Dirichlet b.c.)

$\Pi = k$: number of fundamental string

Legendre transformation: $\delta\left(S + S_B\right) = \int (\text{e.o.m.}) - \int \left(\eta \delta P_\eta + A_\psi \delta \Pi\right)$

Now the action becomes the functional of P_η and Π

$$S + S_B\big|_{\text{sol}} = -2N\left(\kappa\sqrt{1+\kappa^2} + \sinh^{-1}\kappa\right)$$

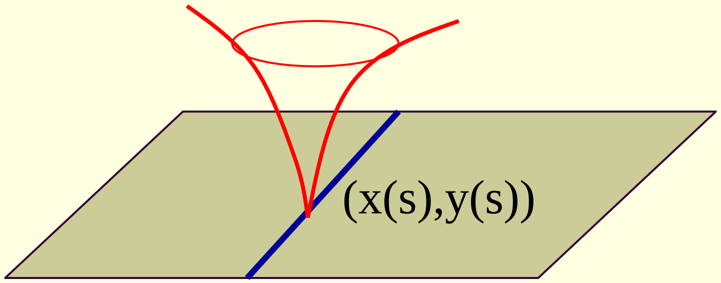
$\left\langle W(\text{circle})\right\rangle_{N=4\,\text{SYM}} = \left\langle \frac{1}{N}\operatorname{Tr} e^{kM}\right\rangle_{MM} \cong e^{S+S_B|_{\text{sol}}}$ **holds for large λ** (Drukker-Fiol)

Boundary condition for D3-Wilson loop

(S.K.-Kuroki-Miwa)

We have added the boundary terms, $S_B = -\int \Phi_i P_{\Phi_i} - \int A_\mu \Pi^\mu$ $\qquad \Phi_i = \frac{1}{2\pi\alpha'}\frac{L^2}{Y^i}$

At the boundary, we need to impose boundary conditions on P and Π.



Near boundary, D3 world-volume need to shrink down to be one dimensional.

$\longrightarrow$ Factorized S^2 part

We propose $\displaystyle \int_{S^2} \Pi^\mu \Big|_{\text{boundary}} = ik\,\dot{x}^\mu(s) \qquad \int_{S^2} P_{\Phi_i} \Big|_{\text{boundary}} = k\,\dot{y}^i(s)$

$$S_B = -k\int ds\left(iA_\mu \dot{x}^\mu(s) + \Phi_i \dot{y}^i(s)\right)$$

- Reparametrization invariant
- Charge balance condition of Callan-Maldacena string

Gauge theory resolvent from D3-brane

(S.K.-Kuroki-Miwa)

In the gauge theory side, 1/N correction (**an isolated pole**) appears as the contribution from a merged loop:

$$R^{\rm pole}(z) = \frac{1}{N}\left\langle \frac{1}{N}\mathrm{Tr}\frac{1}{z-M}e^{kM}\right\rangle_{MM}^{0} = \frac{1}{N}\int_0^\infty dp\, e^{-pz}\left\langle \frac{1}{N}\mathrm{Tr}\, e^{(k+p)M}\right\rangle_{MM}^{0}$$

We would like to evaluate the r.h.s by using D3-brane.

$\longrightarrow$ D3-brane with (k+p)-flux

$$\frac{k}{N^2}\int_0^\infty dp\, e^{-pz}\left\langle \frac{1}{N}\mathrm{Tr}\, e^{(k+p)M}\right\rangle_{MM} = \frac{k}{N^2}\int_0^\infty dp\, e^{-pz}\, e^{-S_{DBI}-S_{WZ}-S_B(k+p)}\Big|_{\text{classical sol.}}$$

$$S_B = -\int \eta P_\eta - \int A_\psi \Pi$$

$$S_\eta = -\eta_0 \int P_\eta = \frac{\sqrt{\lambda}}{2\pi\eta_0}\int_{S^2} P_\Phi = \frac{\sqrt{\lambda}}{\eta_0}(k+p)$$

$$S_\Pi = -\int d\psi\, i\Pi A_\psi = -i(k+p)\int d\psi\, A_\psi$$

$$\Phi = \sqrt{\Phi_i\Phi_i} = \frac{1}{2\pi\alpha'}\frac{L^2}{y}$$

$$\eta^2 P_\eta = -\frac{\sqrt{\lambda}}{2\pi}P_\Phi \qquad y \approx \eta$$

Gauge theory resolvent from D3-brane

$$S_\eta = -\eta_0 \int P_\eta = \frac{\sqrt{\lambda}}{2\pi\eta_0}\int_{S^2} P_\Phi = \frac{\sqrt{\lambda}}{\eta_0}(k+p)$$

$$S_\Pi = -\int d\psi\, i\Pi A_\psi = -i(k+p)\int d\psi\, A_\psi$$

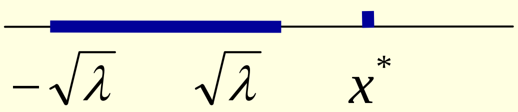
Perform Laplace transformation w.r.t. *p*

$$R^{\text{pole}}(z) = \frac{k}{N^2} e^{-S_{DBI}-S_{WZ}-S_B(k)} \left. \frac{1}{z - \dfrac{\sqrt{\lambda}}{\eta_0} + i\int d\psi A_\psi} \right|_{\text{classical sol.}}$$

$$= \frac{k}{N^2} e^{-S_{D3}} \frac{1}{z - x^*} \cong \frac{1}{N}\frac{1}{z-x^*}$$

$$i\int d\psi\, A_\psi = i\int d\psi d\rho\, F_{\psi\rho}$$
$$= -\frac{k\lambda}{4N}\int d\rho \frac{1}{\sinh\rho}$$
$$= -\sqrt{\lambda}\sqrt{1+\kappa^2} + \frac{\sqrt{\lambda}}{\eta_0} + \mathcal{O}(\eta_0)$$

Eigenvalue distribution



It reproduce the correct position of the isolated pole

$$x^* = \sqrt{\lambda}\sqrt{1+\frac{k^2\lambda}{16N^2}} = \sqrt{\lambda}\sqrt{1+\kappa^2} \qquad \kappa = \frac{k\sqrt{\lambda}}{4N}$$

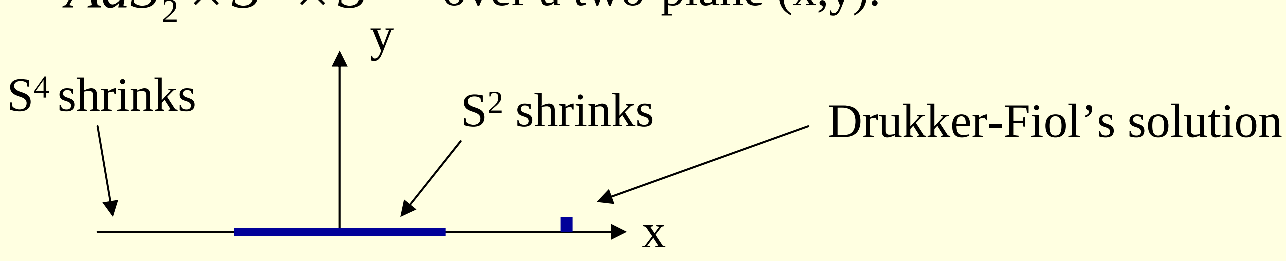
Conclusion

- The isolated eigenvalue of the matrix model is identified with the electric flux of the D3-brane solution.
- We propose boundary conditions for D3-brane solution.

Another approach:

•Bubbling Wilson loop (Yamaguchi, Lunin, Okuda-Trancanelli, …)

$AdS_2 \times S^2 \times S^4$ over a two-plane (x,y).



$$\left\langle \frac{1}{N}\mathrm{Tr}\, e^{pM} \frac{1}{N}\mathrm{Tr}\, e^{kM} \right\rangle_{MM} = \left\langle \frac{1}{N}\mathrm{Tr}\, e^{pM} \right\rangle_{MM} \left\langle \frac{1}{N}\mathrm{Tr}\, e^{kM} \right\rangle_{MM} + \left\langle \frac{1}{N}\mathrm{Tr}\, e^{pM} \frac{1}{N}\mathrm{Tr}\, e^{kM} \right\rangle_{MM}^{\text{cylinder}} + \cdots$$

$$= \int_0^\infty dp\, e^{-pz} \left\langle \frac{1}{N}\mathrm{Tr}\, e^{pM} \frac{1}{N}\mathrm{Tr}\, e^{kM} \right\rangle_{MM}$$

$$\left\langle \frac{1}{N}\mathrm{Tr} \frac{1}{z-M} \right\rangle_k = \left\langle \frac{1}{N}\mathrm{Tr} \frac{1}{z-M} \frac{1}{N}\mathrm{Tr}\, e^{kM} \right\rangle_{MM}$$

$$= \int_0^\infty dp\, e^{-pz} \left\langle \frac{1}{N}\mathrm{Tr}\, e^{pM} \frac{1}{N}\mathrm{Tr}\, e^{kM} \right\rangle_{MM}$$

$$Z = \Phi_1 + i\Phi_2,\ X = \Phi_3 + i\Phi_4$$