Black Holes in the Dilatonic Einstein-Gauss-Bonnet Theory in Various Dimensions – Asymptotically Flat Black Holes –

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1 Introduction

- **Check of the predictions of superstring theories**

The situation where the effects of quantum gravity become important

$\Rightarrow$ **Black holes (singularity)**
Early universe (singularity)

It is urgent to see whether and how these problems are resolved and if superstrings can give realistic models of particles and their interaction including gravity

Here we consider black holes.

- **We need dilaton!!**

Many studies of black holes have been performed by using low-energy effective theories inspired by string theories, which typically involve not only the metric but also the dilaton field (as well as several gauge fields).

There are studies of such solutions in Einstein theories with dilaton.

- **What about higher order corrections?**

It is known that there are correction terms of higher orders in the curvature to the lowest effective supergravity action coming from superstrings. The simplest correction is the Gauss-Bonnet (GB) term coupled to the dilaton field.

However, black holes in Einstein-GB theories have been studied much but WITHOUT DILATON!

In order to understand properties of black holes in string theories, we should include dilaton!

- **Another motivation:**

Many people consider the application to the calculation of shear viscosity in strongly coupled gauge theories using black hole solutions in five-dimensional Einstein-GB theory via AdS/CFT correspondence, but without dilaton. In order to see this in the context of superstrings, we should again include dilaton.

2 Dilatonic Einstein-GB theory

2.1 Basic equations

The action:

$$S = \frac{1}{2\kappa_D^2} \int d^D x \sqrt{-g} \left[R - \frac{1}{2} (\partial_\mu \phi)^2 + \alpha_2 e^{-\gamma\phi} R_{\rm GB}^2 \right],$$

R: **the scalar curvature,** $\quad \phi$: **a dilaton field,**
$R_{\rm GB}^2 = R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} - 4 R_{\mu\nu} R^{\mu\nu} + R^2$: **the GB combination,**
$\kappa_D^2 = 8\pi G_D$: **a** D**-dimensional gravitational constant,**
$\alpha_2 = \alpha'/8$: α' **is the Regge slope parameter,** $\quad \gamma = 1/2$.

Line element in D-dimensional static spacetime

$$ds_D^2 = -\left(k - \frac{2Gm}{r^{D-3}}\right) e^{-2\delta} dt^2 + \left(k - \frac{2Gm}{r^{D-3}}\right)^{-1} dr^2 + r^2 h_{ij} dx^i dx^j .$$

where $h_{ij}dx^i dx^j$ represents the line element of a $(D-2)$-dimensional hypersurface with constant curvature of signature k and volume Σ_k for $k = \pm 1, 0$.

Master equations:

$$\tilde{m}' \frac{D-2}{\tilde{r}^{D-4}} h - \frac{1}{4} B \tilde{r}^2 \phi'^2 - \frac{1}{2}(D-1)_4\, e^{-\gamma\phi} \frac{(k-B)^2}{\tilde{r}^2} + 2(D-2)_3\, \gamma e^{-\gamma\phi} B(k-B)(\phi'' - \gamma\phi'^2)$$
$$+(D-2)_3\, \gamma e^{-\gamma\phi} \phi' \frac{(k-B)[(D-3)k-(D-1)B]}{\tilde{r}} = 0\,,$$
$$\delta'(D-2)\tilde{r}h + \frac{1}{2}\tilde{r}^2\phi'^2 - 2(D-2)_3\, \gamma e^{-\gamma\phi}(k-B)(\phi'' - \gamma\phi'^2) = 0\,,$$
$$(e^{-\delta}\tilde{r}^{D-2}B\phi')' = \gamma(D-2)_3 e^{-\gamma\phi-\delta}\tilde{r}^{D-4}\Big[(D-4)_5\frac{(k-B)^2}{\tilde{r}^2} + 2(B' - 2\delta' B)B'$$
$$-4(k-B)BU(r) - 4\frac{D-4}{\tilde{r}}(B' - \delta' B)(k-B)\Big],$$

where we have defined

$$
\begin{aligned}
\tilde{r} &\equiv \frac{r}{\sqrt{\alpha_2}}, \quad \tilde{m} \equiv \frac{Gm}{\alpha_2^{(D-3)/2}}, \quad B \equiv k - \frac{2\tilde{m}}{\tilde{r}^{D-3}}, \\
h &\equiv 1 + 2(D-3)e^{-\gamma\phi}\Big[(D-4)\frac{k-B}{\tilde{r}^2} + \gamma\phi'\frac{3B-k}{\tilde{r}}\Big], \\
\tilde{h} &\equiv 1 + 2(D-3)e^{-\gamma\phi}\Big[(D-4)\frac{k-B}{\tilde{r}^2} + \gamma\phi'\frac{2B}{\tilde{r}}\Big], \\
U(r) &\equiv \frac{1}{2\tilde{h}}\Bigg[(D-3)_4\frac{k-B}{\tilde{r}^2 B} - 2\frac{D-3}{\tilde{r}}\Big(\frac{B'}{B} - \delta'\Big) - \frac{1}{2}\phi'^2 \\
&\quad + (D-3)e^{-\gamma\phi}\Bigg[(D-4)_6\frac{(k-B)^2}{\tilde{r}^4 B} - 4(D-4)_5\frac{k-B}{\tilde{r}^3}\Big(\frac{B'}{B} - \delta' - \gamma\phi'\Big) \\
&\quad - 4(D-4)\gamma\frac{k-B}{\tilde{r}^2}\Big(\gamma\phi'^2 + \frac{D-2}{\tilde{r}}\phi' - \Phi\Big) + 8\frac{\gamma\phi'}{\tilde{r}}\Big\{\Big(\frac{B'}{2} - \delta' B\Big)\Big(\gamma\phi' - \delta' + \frac{2}{\tilde{r}}\Big) \\
&\qquad - \frac{D-4}{2\tilde{r}}B'\Big\} + 4(D-4)\Big(\frac{B'}{2B} - \delta'\Big)\frac{B'}{\tilde{r}^2} - 4\frac{\gamma}{\tilde{r}}\Phi(B' - 2\delta' B)\Bigg]\Bigg], \\
\Phi &\equiv \phi'' + \Big(\frac{B'}{B} - \delta' + \frac{D-2}{\tilde{r}}\Big)\phi'.
\end{aligned}
$$

These equations have a symmetry under

$$
\phi \to \phi - \phi_\infty, \quad \tilde{r} \to e^{\frac{1}{2}\gamma\phi_\infty}\tilde{r}, \quad \delta \to \delta, \quad \tilde{m} \to e^{\frac{D-3}{2}\gamma\phi_\infty}\tilde{m}.
$$

$\Rightarrow$ **the asymptotic value of the dilaton field = 0**

Another shift symmetry

$$\delta \to \delta - \delta_\infty, \quad t \to e^{-\delta_\infty} t,$$

$\Rightarrow$ the asymptotic value of $\delta = 0$.

2.2 Boundary conditions

1. Asymptotic flatness at spatial infinity ($\tilde{r} \to \infty$):

$$\tilde{m}(\tilde{r}) \to \tilde{M} < \infty, \quad \delta(\tilde{r}) \to 0, \quad \phi(\tilde{r}) \to 0.$$

2. The existence of a regular horizon $\tilde{r}_H$:

$$2\tilde{m}_H = \tilde{r}_H^{D-3}, \quad |\delta_H| < \infty, \quad |\phi_H| < \infty.$$

3. The event horizon is the outermost one and the regularity of spacetime for $\tilde{r} > \tilde{r}_H$:

$$2\tilde{m}(\tilde{r}) < \tilde{r}^{D-3}, \quad |\delta(\tilde{r})| < \infty, \quad |\phi(\tilde{r})| < \infty.$$

Given the b.c. at the horizon, ϕ'_H is determined:

$$\begin{aligned}
&2C\gamma\Big[2(D-3) + (D-4)(3D-11)C + (D-4)C^2\Big\{(D-4)_5 + (D-2)(3D-11)\gamma^2\Big\} + 2(D-2)_5\, C^3\gamma^2\Big]\tilde{r}_H^2\phi_H'^2 \\
&+2\Big[(D-1)_2\,(D-4)C^2\Big\{2 + 2C - (D-4)_5\, C^2\Big\}\gamma^2 - \{1 + (D-4)C\}^2\,\{2(D-3) + (D-4)_5\, C\}\Big]\tilde{r}_H\phi'_H \\
&\quad +(D-1)_2\, C\Big[2(D-2) - 4(D-4)C - (D-4)^2(D+1)C^2\Big]\gamma = 0,
\end{aligned}$$

where we have defined

$$C = \frac{2(D-3)e^{-\gamma\phi_H}}{\tilde{r}_H^2}.$$

3 Non-dilatonic black hole solutions

$D=4$**: the GB term is total divergence and does not give any contribution.**

$D \geq 5$**: the field equations can be integrated to yield**

$$\bar{B} = 1 - \frac{2\bar{m}}{\tilde{r}^{D-3}}, \qquad \delta = 0,$$

$$\bar{m} = \frac{\tilde{r}^{D-1}}{4(D-3)_4}\left[-1 \pm \sqrt{1 + \frac{8(D-3)_4\bar{M}}{\tilde{r}^{D-1}}}\right],$$

$\bar{M}$**: an integration constant corresponding to the asymptotic value** $\bar{m}(\infty)$ **for the plus sign.**

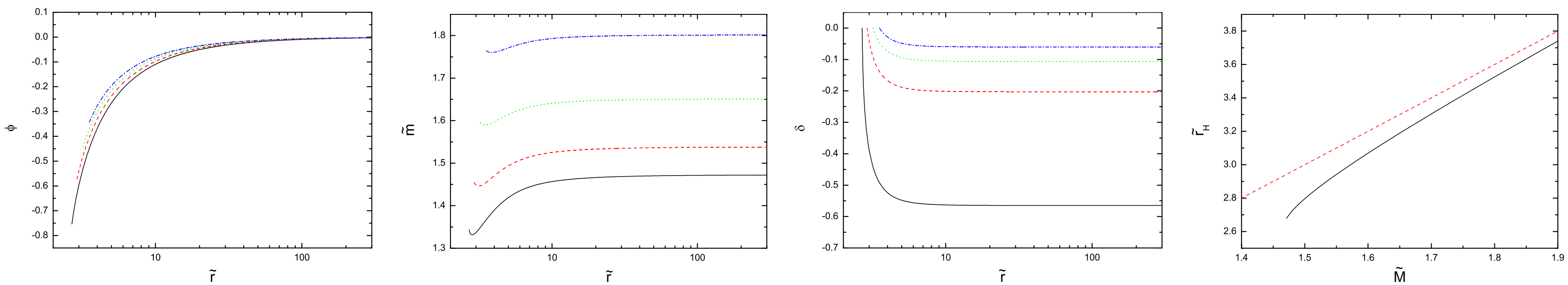
$\bar{M}$**-**$\tilde{r}_H$ **relation for the black hole without the dilaton field:**

$$\bar{M} = \frac{1}{2}\tilde{r}_H^{D-5}\left[\tilde{r}_H^2 + (D-3)_4\right].$$

Note that $\bar{M} \to 0$ **for** $D=4, \geq 6$**, but not for** $D=5$ **in the limit** $r_H \to 0$**.**

4 $D = 4$ black hole solutions

Give b. c. on ϕ_H and δ_H at the horizon $\Rightarrow$ $\phi'_H = \frac{1 \pm \sqrt{1-24C^2\gamma^2}}{2C\gamma\tilde{r}_H}$.
Only the smaller solution gives regular BH.
Use the shift symmetry to set the asymptotic value of the dilaton to zero.
Regular black hole solutions exist only for $\tilde{r}_H \geq 1.47126$.



dilaton　　mass function　　lapse function　　mass versus horizon radius

dilaton (solid line) and non-dilaton (dashed line)

Figure 1: **Black hole solutions in the four-dimensional Einstein-GB-dilaton system with $\gamma = \frac{1}{2}$. The behaviours are for four different radii of event horizon: $\tilde{r}_H = r_H/\sqrt{\alpha_2} = 2.68697$ (solid line), 2.90965 (dashed line), 3.19148 (dotted line) and 3.52851 (dash-dotted line). The right: mass versus horizon radius. The masses $\tilde{M}$ for these cases are found to be 1.47251, 1.53808, 1.65113, and 1.80161, respectively.**

5 $D = 5$ solutions

$$C\gamma(1+C+3C^2\gamma^2)\tilde{r}_H^2\phi_H'^2 - (1+C)(1+C-6C^2\gamma^2)\tilde{r}_H\phi_H' + 3C(3-2C-3C^2)\gamma = 0.$$

The discriminant of this equation is (for $\gamma = \frac{1}{2}$) always positive for $C > 0$
$\Rightarrow$ there is no bound on the value of $\tilde{r}_H$ for the reality of the solution
$\Rightarrow$ In contrast to the four-dimensional case, the regular black hole solutions exist for all $\tilde{r}_H > 0$.
Only the smaller solution gives regular BH.

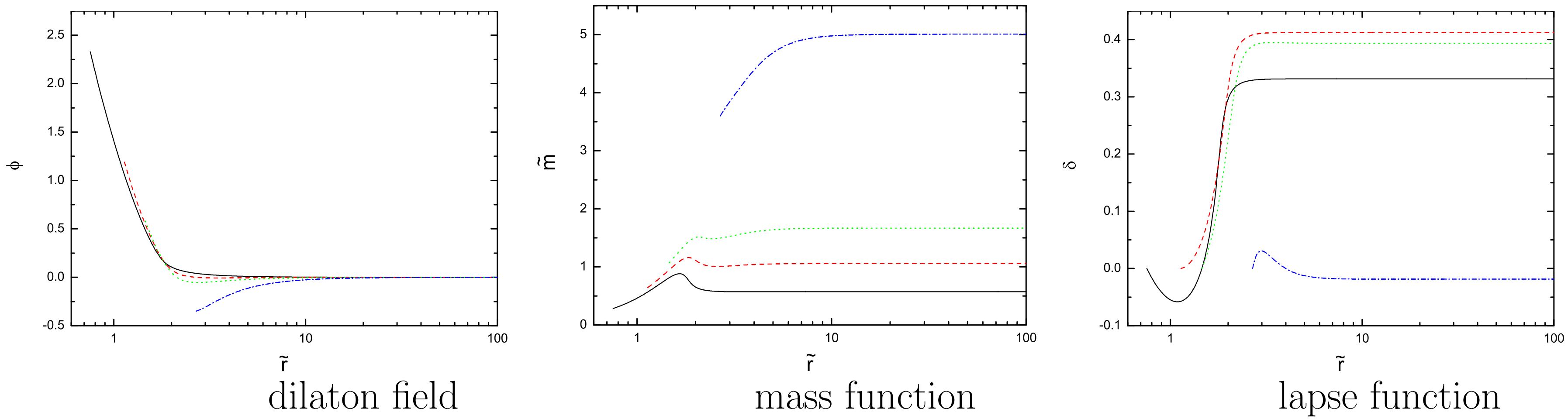
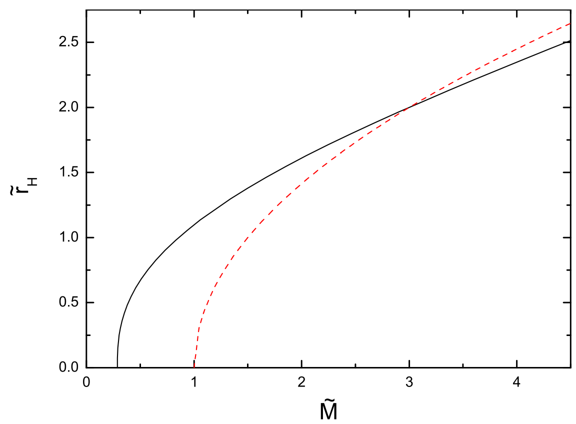


Figure 2: **Black hole solutions in the five-dimensional Einstein-GB-dilaton system for four different radii of event horizon: $\tilde{r}_H = r_H/\sqrt{\alpha_2} = 0.754129$ (solid line), 1.13599 (dashed line), 1.46193 (dotted line) and 2.68391 (dash-dotted line). The masses $\tilde{M}$ are 0.573328, 1.05972, 1.66924 and 5.0097, respectively.**

Properties:

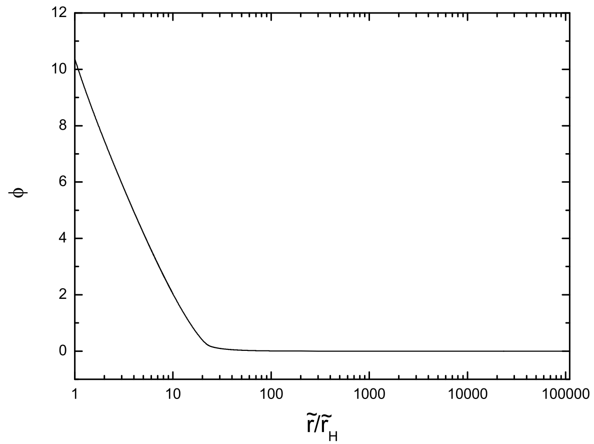
Regular black hole solutions exist for all $\tilde{r}_H > 0$.
4-dim.: solution disappears below certain radius.

The mass of the dilatonic black holes approaches a non-zero constant $\tilde{M} = 0.288185$ as $\tilde{r}_H \to 0$.
Similar to $D = 5$ non-dilatonic case: $\bar{M} = \frac{1}{2}(\tilde{r}_H^2 + 2) \to 1$ for $\tilde{r}_H \to 0$



mass versus horizon radius

dilaton (solid line) and non-dilaton (dashed line)



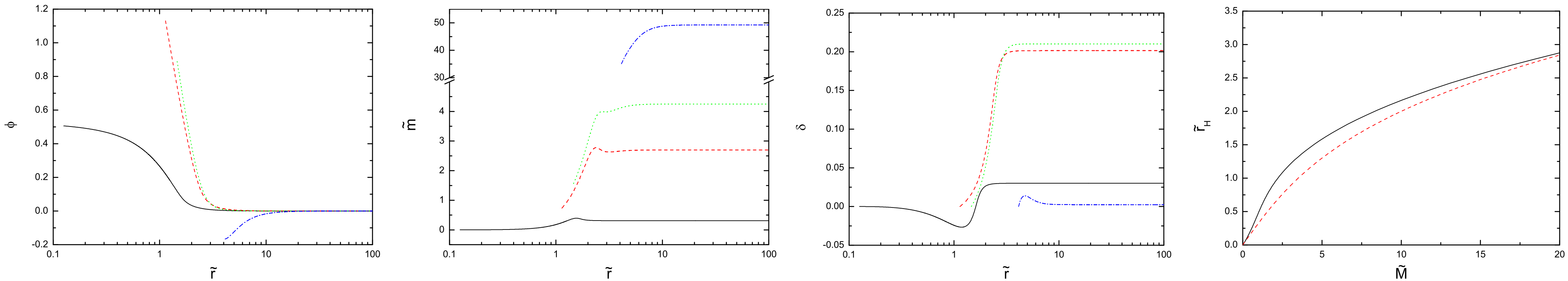
dilaton field

Right: The configuration of the dilaton field for the small black holes with $\tilde{r}_H = r_H/\sqrt{\alpha_2} = 0.0748464$. In the GB region the dilaton decays logarithmically and suddenly changes to the power decay $\sim r^{-2}$.

6 $D = 6$ solutions

$$C\gamma\Big[6+14C+4C^2(14\gamma^2+1)+48C^3\gamma^2\Big]\tilde r_H^2\phi_H'^2$$
$$+2\Big[40C^2\gamma^2(1+C-C^2)-(3+C)(1+2C)^2\Big]\tilde r_H\phi_H'+40C(2-2C-7C^2)\gamma=0.$$

The discriminant of this equation is (for $\gamma = \frac{1}{2}$) always positive for $C > 0$
$\Rightarrow$ The regular black hole solutions exist for all $\tilde r_H > 0$.
Only the smaller solution gives regular BH.
The mass of the black hole approaches 0 as $\tilde r_H \to 0$.



dilaton field | mass function | lapse function | mass versus horizon radius

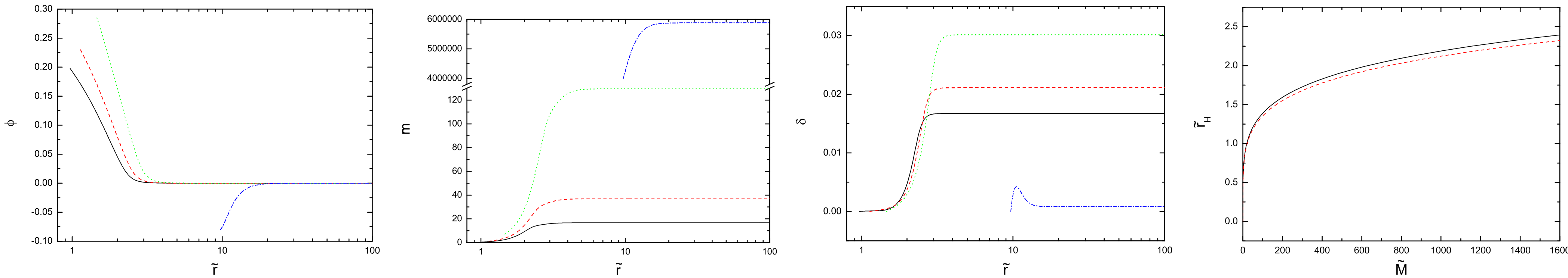
dilaton (solid) and non-dilaton (dashed)

Figure 3: **Black hole solutions in the six-dimensional Einstein-GB-dilaton system for four different radii of the event horizon: $\tilde r_H = r_H/\sqrt{\alpha_2} = 0.125367$ (solid line), 1.13596 (dashed line), 1.46199 (dotted line) and 4.12369 (dash-dotted line). The masses $\tilde M$ for these cases are found to be 0.311672, 2.6993, 4.25081 and 49.2744, respectively.**

7 $D = 10$ solutions

$$
C\gamma\Big[7+57C+6C^2(76\gamma^2+15)+1680C^3\gamma^2\Big]\tilde r_H^2\phi_H'^2
+\Big[432C^2\gamma^2(1+C-15C^2)-(1+6C)^2(7+15C)\Big]\tilde r_H\phi_H' - 72C(-4+6C+99C^2)\gamma = 0.
$$

The discriminant of this equation is (for $\gamma = \frac{1}{2}$) always positive for $C > 0$
$\Rightarrow$ **The regular black hole solutions exist for all $\tilde r_H > 0$.**
Only the smaller solution gives regular BH.
The mass of the black hole approaches 0 as $\tilde r_H \to 0$.



dilaton field — mass function — lapse function — mass versus horizon radius

dilaton (solid) and non-dilaton (dashed)

Figure 4: **Black hole solutions in the ten-dimensional Einstein-GB-dilaton system for four different radii of the event horizon: $\tilde r_H = r_H/\sqrt{\alpha_2} = 0.968549$ (solid line), 1.13596 (dashed line), 1.46194 (dotted line) and 9.68119 (dash-dotted line). The masses $\tilde M$ are 16.7172, 36.8633, 129.489 and 5.88035×10^6, respectively.**

Almost the same as those in the $D = 6$ case qualitatively.
The dilaton field ϕ monotonically increases for large black holes.

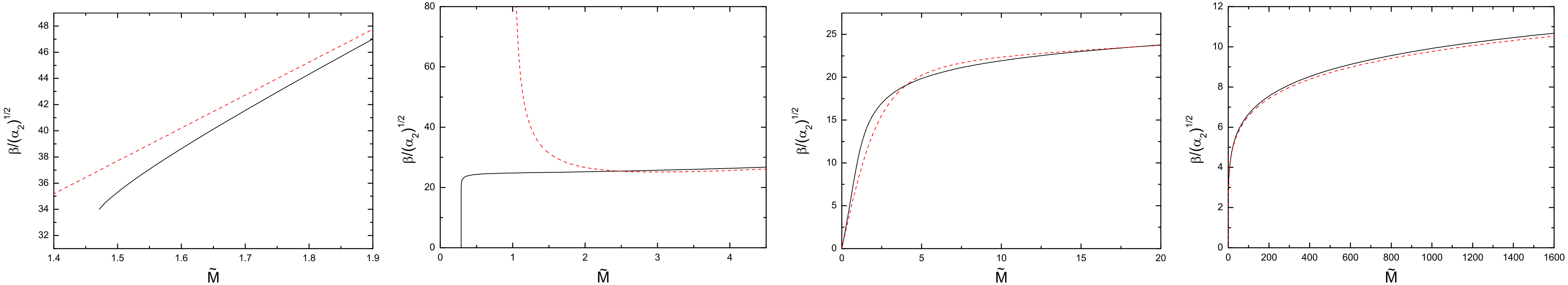
The mass of the black hole approaches zero for $\tilde{r}_H \to 0$.
This is again in agreement with the non-dilatonic case,

$$\bar{M} = \frac{1}{2}\tilde{r}_H^5(\tilde{r}_H^2 + 42).$$

8 Thermodynamics

Hawking temperature: $T_H = \dfrac{e^{-\delta_H}}{4\pi r_H}\left(D - 3 - \dfrac{2\tilde{m}'_H}{\tilde{r}_H^{D-4}}\right).$

$\tilde{M}$-β relations



4 dimensions 5 dimensions 6 dimensions 10 dimensions

Figure 5: **The mass-temperature diagram, where $\beta \equiv 1/T$. The dashed lines are the non-dilatonic case for comparison.**

$D = 4$: The GB term has the tendency to raise the temperature compared to the non-dilatonic solution (Schwarzschild black hole).

$D = 5$: Non-dilatonic case

The temperature increases as the mass of the black hole becomes small for large black holes

$\Rightarrow$ the heat capacity is negative.

Below the mass $\tilde{M} = 2.976072$, the temperature decreases as the mass becomes small.

The sign of the heat capacity changes at this mass. $\cdots$ Same as the Reissner-Nordström black hole solution: the second order phase transi-

tion.

As the black hole becomes small through Hawking radiation, the temperature becomes extremely law, and the solution cannot reach the singularity with zero horizon radius. This is favorable feature from the point of view of cosmic censorship hypothesis.

Dilatonic case:

Thermodynamic properties change drastically. The heat capacity is negative in all the mass range, and the temperature blows up at the singular solution. This is due to the nontrivial coupling between the dilaton field and the GB term and the resultant divergence of the dilaton field at the horizon.

$D \geq 6$**:**

The behavior of the temperature is qualitatively the same as that in the non-dilatonic case.

The dilaton field has tendency to lower the temperature for the large black hole, while it raises the temperature for small black hole.

The temperature diverges for the zero mass "solution" and the black hole continues evaporating.

In GR, the horizon radius of the black hole is related to entropy by $S = \pi r_H^2$.

With GB gravity, entropy is not obtained by a quarter of the area of the event horizon.

$$S = \frac{A_H}{4}\left[1 + 2(D-2)_3 \frac{\alpha_2 e^{-\gamma\phi_H}}{r_H^2}\right],$$

where A_H is the area of the event horizon.

M-S plots:

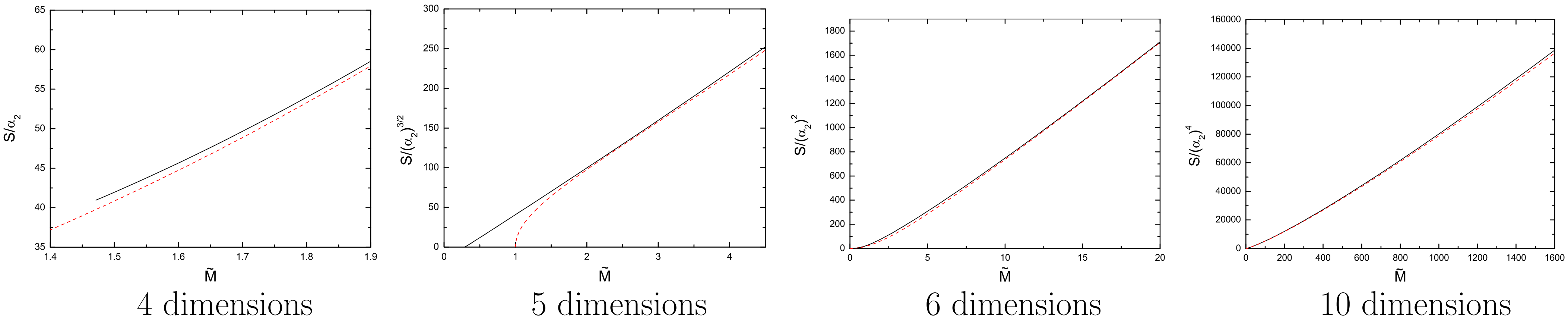


Figure 6: **The dashed lines are the non-dilatonic case for comparison.**

No qualitative difference between the dilatonic and the non-dilatonic cases.

The solution disappears at the nonzero finite mass for $D = 4$ (the dilatonic solution) and $D = 5$.

Entropy of the dilatonic black hole is always larger than that of the non-dilatonic black hole with the same mass.

9 Conclusions and discussions

$D = 5$**:**

The effects of the GB term is negligible for the large black holes ($r_H \gg \ell_s$), and the dilaton field decays with power.

For the small black holes ($r_H \leq \ell_s$), spacetime is divided into the GR region and the GB region with a sharp transition.

In the GB region the dilaton field behaves logarithmically and the effective energy density becomes negative. The regular black hole solutions exist for all horizon radius. In the zero horizon-radius limit the solution becomes singular. These properties are same as those of the non-dilatonic solutions.

$D \geq 6$**:**

For small black holes ($r_H \ll \ell_s$), the string effect extends just around their event horizons which are much smaller than the string scale.

Counter-intuitive: One naturally expects that the string effect extends to ℓ_s in any situation.

The regular solution exists for any horizon-radius.

In the zero horizon-radius limit, the mass of the solution approaches zero which is different from the lower dimensional cases.

Remaining problems:

1. **The global structures:**

   Our numerical analysis was limited to outer spacetime of the event horizon.

   The global structures of the solutions such as the existence of the inner horizon and (central or branch) singularity have not been clarified. This may be done by integrating field equations inward numerically.

2. **The ambiguity of the frames:**

   We have studied the solution in the Einstein frame.

   There is, however, a possibility that the properties of solutions changes drastically by transforming to the string frame. In particular, the conformal transformation may become singular.

3. **Stability:**

   The stability of our solutions is another important subject to study.

4. **Charged solution:**

   It would be also interesting to extend our analysis to dilatonic black holes (large and small) with charges.