Intersecting Solitons, Amoeba and Tropical Geometry

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1 Introduction

Solitons in Yang-Mills-Higgs theory in the Higgs phase (with **8** SUSY)

Elementary solitons: **Vortex** and **Domain wall** (**Kink**)

Vortices and Domain walls preserve **1/2** of SUSY : **1/2 BPS solitons**

Composite solitons in the **Higgs phase** : **1/4 BPS solitons**

Webs of domain **walls**, Magnetic **monopoles** with **vortices**,

Instantons inside a **Vortex** (**Web of Vortices**)

(Scherk-Schwarz twisted) **dimensional reduction** :

Web of Vortices $\longrightarrow$ all other **1/4** BPS composite solitons

Web of Vortices is most important among composite BPS solitons

Our purpose:

Study configurations of instantons and vortex sheets (**webs of vortices**)

In **8** SUSY $U(N_{\rm C})$ gauge theory with $N_{\rm F} = N_{\rm C}$ Higgs scalars

On $\mathbb{R}_t \times (\mathbf{C}^*)^2 \sim \mathbb{R}^{2,1} \times T^2$ (**5** dimensions) $\longrightarrow$ Dim. reduction

By using **Moduli Matrix** formalism

Use **amoeba** and **tropical geometry** to describe Webs of vortices

Results

1. **Vortex sheets**: zeros of a polynomial in the Moduli matrix
   Instanton positions: common zeros with another polynomial
2. Mathematical language of **amoeba** and **tropical geometry** are useful to visualize the web of vortices and to evaluate physical quantities.
3. **Moduli matrix** approach plays a crucial role to describe web of vortices.

2 Vortices and Instantons

SUSY $U(N_{\rm C})$ Gauge Theory with $N_{\rm F}$ Higgs fields

Higgs fields H as an $N_{\rm C} \times N_{\rm F}$ matrix, $\mu, \nu = 0, 1, 2, 3, 4$

$$\mathcal{L} = \mathrm{Tr}\left[-\frac{1}{2g^2}F_{\mu\nu}F^{\mu\nu} + \mathcal{D}_\mu H(\mathcal{D}^\mu H)^\dagger - \frac{g^2}{4}(HH^\dagger - c\mathbf{1}_{N_{\rm C}})^2\right]$$

Gauge coupling g for $U(N_{\rm C})$, Fayet-Iliopoulos (FI) parameter c

Coordinates of $(\mathbf{C}^*)^2$: $(x_1,\ y_1,\ x_2,\ y_2)$, $z_1 \equiv x_1 + iy_1,\ z_2 \equiv x_2 + iy_2$

Higgs Phase : **Walls**, **Vortices** are the only **elementary** solitons

Instantons, **monopoles**, **junctions** are **composite** solitons

Energy **lower bound** of static field configurations

$$E \;\geq\; -\frac{1}{g^2}\int \mathrm{Tr}\,(F\wedge F) - c\int \mathrm{Tr}\,F\wedge\omega = \frac{8\pi^2}{g^2} I + 2\pi c\, V$$

$\omega \equiv \frac{i}{2}(dz_1\wedge d\bar z_1 + dz_2\wedge d\bar z_2)$: the Kähler form on $(\mathbf{C}^*)^2$

Total instanton charge I, Instanton charge density $\mathcal{I}$

$$I \;\equiv\; \int \mathcal{I} \;\equiv\; -\frac{1}{8\pi^2}\int \mathrm{Tr}\,(F\wedge F) \;=\; \int ch_2$$

Vortex charge V, Vortex charge density $\mathcal{V}$

$$V \;\equiv\; \int \mathcal{V} \;\equiv\; -\frac{1}{2\pi}\int \mathrm{Tr}\,F\wedge\omega \quad = \; \int c_1\wedge\omega$$

Lower bound is saturated if the **BPS equations** are satisfied

$$F_{\bar z_1 \bar z_2} = 0, \quad \mathcal{D}_{\bar z_i} H = 0, \quad -2i(F_{z_1\bar z_1} + F_{z_2\bar z_2}) = \frac{g^2}{2}(HH^\dagger - c\mathbf{1}_{N_\mathrm{C}})$$

BPS equations contain at least instantons and intersecting vortex sheets

solutions to BPS eqs. preserve $1/4$ of SUSY $\longrightarrow$ **1/4 BPS** states

Solution of BPS equations

$F_{\bar{z}_1\bar{z}_2} = 0$: **integrability condition** for $\mathcal{D}_{\bar{z}_i}$ $W_{\bar{z}_i} = -iS^{-1}\partial_{\bar{z}_i}S$

Solution of the first 2 equations: $H = S^{-1}H_0$ with $\partial_{\bar{z}_i}H_0 = 0$

$N_{\rm C} \times N_{\rm F}$ matrix H_0 should be **holomorphic** : **Moduli Matrix**

Remaining BPS eq.(**Master eq.**): $\Omega \equiv SS^\dagger$, $\Omega_0 \equiv \frac{1}{c}H_0H_0^\dagger$

$$\partial_{\bar{z}_1}(\Omega\partial_{z_1}\Omega^{-1}) + \partial_{\bar{z}_2}(\Omega\partial_{z_2}\Omega^{-1}) = -\frac{g^2c}{4}\left(1_{N_{\rm C}} - \Omega_0\Omega^{-1}\right)$$

We consider $N_{\rm C} = N_{\rm F} = N$ case

Meissner effect in the **Higgs phase** (Higgs VEV):

Magnetic flux can penetrate superconducting (Higgs) phase as **Vortices**

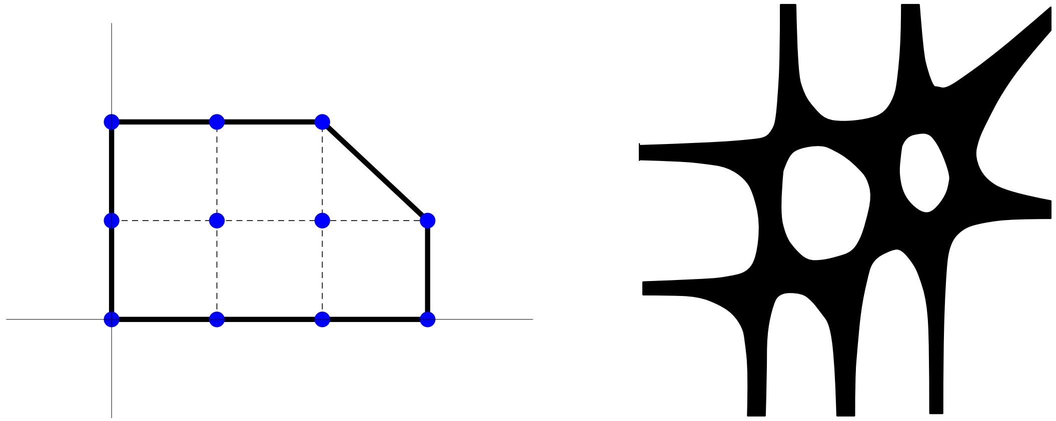
(Partial) **restoration of gauge symmetry** at the **core of vortex**

Vortex sheet in $z_1, z_2 \in ({\rm C}^*)^2$ can be defined by $\det H_0(z_1, z_2) = 0$

3 Webs of Vortex Sheets on $({\rm C}^*)^2$

Web of Vortices on $({\rm C}^*)^2 \simeq {\rm R}^2 \times T^2$: $y_i \sim y_i + 2\pi R_i, i = 1, 2$

$$P(u_1, u_2) \equiv \det H_0 = \sum_{(n_1,n_2)\in\mathbb{Z}^2} a_{n_1,n_2} u_1^{n_1} u_2^{n_2}, \quad u_i \equiv e^{\frac{z_i}{R_i}}$$



(a) Newton polytope (b) Amoeba

Figure 1: An example of amoeba; $P(u_1,u_2) = a_{0,0} + a_{1,0}u_1 + a_{2,0}u_1^2 + a_{3,0}u_1^3 + a_{0,1}u_2 + a_{1,1}u_1u_2 + a_{2,1}u_1^2u_2 + a_{3,1}u_1^3u_2 + a_{0,2}u_2^2 + a_{1,2}u_1u_2^2 + a_{2,2}u_1^2u_2^2$.

Newton polytope $\Delta(P) \subset \mathbf{R}^2$ of a Laurent polynomial $P(u_1,u_2)$

$$\Delta(P) = \textbf{conv. hull} \left\{ (n_1,n_2) \in \mathbb{Z}^2 \middle| \, a_{n_1,n_2} \neq 0 \right\}$$

a_{n_1,n_2} : moduli parameters for the webs of vortices

Amoeba of P : a projection of generic webs of vortices on x_1, x_2

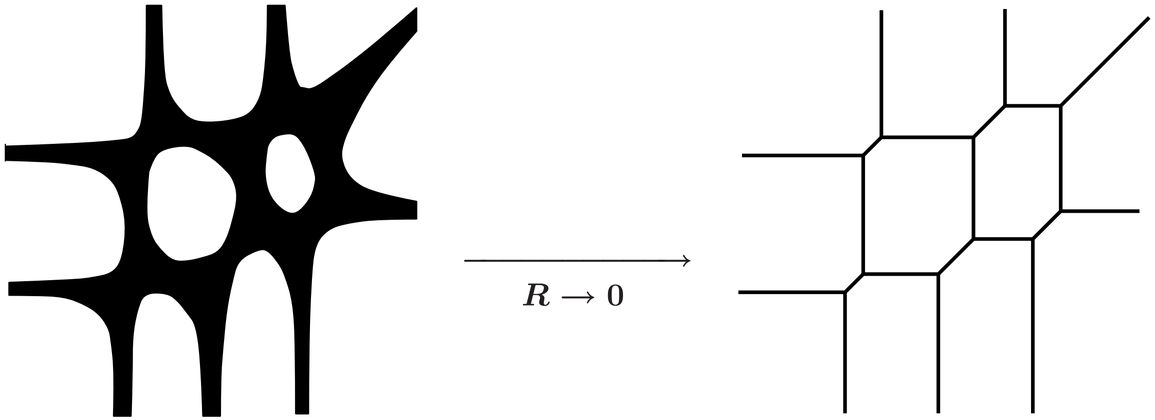
$$\mathcal{A}_P = \left\{ \big(R_1 \log|u_1|,\ R_2 \log|u_2|\big) \in \mathbb{R}^2 \ \middle| \ P(u_1,u_2) = 0 \right\}$$

Tenticles: asymptotic regions extending to infinity

Normals to the Newton polytope: semi-infinite cylinders of vortices

Internal lattice points of Newton polytope: holes (**vortex loops**)

Relation with Tropical Geometry



(a) Amoeba (b) Tropical variety

Figure 2: An example of the amoeba and corresponding tropical variety.

Amoeba is **smooth** even in the **thin wall** limit $l = 1/g\sqrt{c} \to 0$

Tropical limit: $R_1 = R_2 = R \to 0$ with fixed $r_{n_1,n_2} \equiv R \log |a_{n_1,n_2}|$

Amoeba degenerates into a set of lines ("spines"), called "tropical variety"

Skeleton (spine) of amoeba in $R \to 0$: position of domain walls

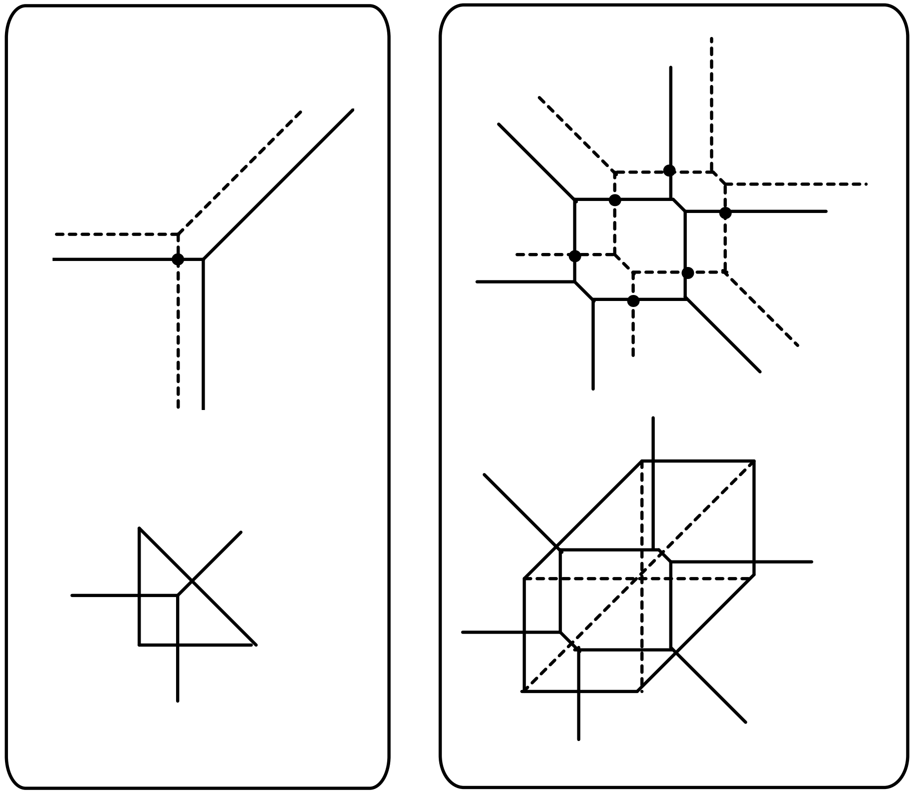


Figure 3: Intersection of one tropical variety and its shift and Newton polytope $\boldsymbol{\Delta(P)}$ (below). Number of intersection points is given by $\mathbf{2Area(\Delta(}\boldsymbol{P}\mathbf{))}$.

Intersection charge density becomes complex Monge-Ampère measure

$$I_{\rm intersection} = \int_X \mathcal{I} \frac{1}{8\pi^2} \int dd_c \log |P| \wedge dd_c \log |P| \;=\; \frac{1}{4\pi} \int_X dd_c \log |P|$$

Regularization : $P \longrightarrow P_1, P_2$ associated with the same Newton polytope

$$\frac{1}{8\pi^2} \int dd_c \log |P_1| \wedge dd_c \log |P_2| \;=\; \frac{1}{2} \#(X_1 \cdot X_2) = \mathrm{Area}(\Delta)$$

4 Instantons inside Non-Abelian Vortex Webs

$$H_0 = \begin{pmatrix} 1 & b(u_1, u_2) \\ 0 & P(u_1, u_2) \end{pmatrix}$$

$$P(u_1, u_2) = \sum a_{n_1,n_2} u_1^{n_1} u_2^{n_2}, \quad b(u_1, u_2) = \sum b_{n_1,n_2} u_1^{n_1} u_2^{n_2}$$

Vortex sheets are localized at $P(u_1, u_2) = 0$

Instanton number: computed from Ω with the correct boundary conditions

$$\Omega \equiv \begin{pmatrix} 1 + |b|^2 & b\overline{P} \\ P\bar{b} & \frac{\Omega_* - |P|^2}{1+|b|^2} + |P|^2 \end{pmatrix}$$

$$\bar{\partial}_{\bar{z}_1}(\Omega_* \partial_{z_1} \Omega_*^{-1}) + \bar{\partial}_{\bar{z}_2}(\Omega_* \partial_{z_2} \Omega_*^{-1}) = -\frac{g^2 c}{4}(1 - |P|^2 \Omega_*^{-1})$$

(Ω becomes a solution of the master equation if b is a constant)

$$\begin{aligned} I &= \frac{1}{8\pi^2} \int \left(dd_c \log |P| \wedge dd_c \log(1 + |b|^2) - dd_c \log |P| \wedge dd_c \log |P| \right) \\ &= I_{\text{instanton}} - I_{\text{intersection}} \end{aligned}$$

$$I_{\text{instanton}} = \frac{1}{8\pi^2} \int_{(\mathbf{C}^*)^2} dd_c \log |P| \wedge dd_c \log(1+|b|^2) = \frac{1}{4\pi} \int_X dd_c \log(1+|b|^2)$$

X : **zero locus** of P corresponding to the **vortex sheets**

instanton number is given by the **degree** of the map $b|_X : X \to \mathbf{C}P^1$

Distribution of topological charge:

Small instanton limit: $b_{n_1,n_2} \to \infty$ with fixed $b_{n_1,n_2}/b_{\tilde{n}_1,\tilde{n}_2}$

$dd_c \log(1 + |b|^2) \to dd_c \log |b|^2$: delta function on $b(u_1, u_2) = 0$

Instantons are localized at **common zeros** of $b(u_1, u_2)$ and $P(u_1, u_2)$

5 Conclusion

1. **Vortex sheets**: zeros of a polynomial in the Moduli matrix

   Instanton positions: common zeros with another polynomial

2. Mathematical language of **amoeba** and **tropical geometry** are useful to visualize the web of vortices and to evaluate physical quantities.

3. **Moduli matrix** approach plays a crucial role to describe web of vortices.