Domain wall solitions and Hopf algebraic translational symmetries in noncommutative field theories

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1.Introduction

- We believe that noncommutative field theories are important subjects for studying Planck scale physics, especially quantum gravity.

- Recently, it was pointed out that noncommutative field theories would have nontrivial symmetries, which have Hopf algebraic structure.

  Moyal plane: Chaichian, Kulish, Nishijima, Tureanu (2004),etc.
  Noncommutative gravity: Aschieri, Dimitrijevic, Meyer, Schupp, Wess (2005), etc.
  SU(2) noncommutative spacetime: Freidel, Livine (2005), etc.

- In order for quantum field theories to possess Hopf algebraic symmetries, we have to include braiding (nontrivial statistics).

  SU(2) noncommutative spacetime: Freidel, Livine (2005)
  Moyal plane: Balachandran, Mangano, Pinzul, Vaidya (2006)
  General case: Y.S, Sasakura (2007)

- In addition to the importance of the Hopf algebraic symmetries, the braiding can recover the unitarity and renormalization.

- We want to study more physical aspects of Hopf algebraic symmetry.

⟹ How is "symmetry breaking" of Hopf algebraic symmetry?

- We study a domain wall soliton in three dimensional noncommutative field theory in Lie-algebraic noncommutative space-time

$$[x^i, x^j] = 2i\kappa\epsilon^{ijk}x_k \quad .$$

- It is interesting to consider a domain wall soliton in the Lie-algebraic noncommutative space-time because

1. What is the generator of a one-parameter family of domain wall solutions which comes from a Hopf algebraic translational symmetry?

2. Is the moduli field on a domain wall massless?

Contents

2. Review of noncommutative field theory in the Lie-algebraic noncommutativity

$$[\hat{x}^i, \hat{x}^j] = 2i\kappa\epsilon^{ijk}\hat{x}_k, \quad (i, j, k = 0, 1, 2)$$

Imai, Sasakura (2000), Freidel, Livine (2005)

<u>Commutation relation</u>

$$[\hat{x}^i, \hat{x}^j] = 2i\kappa\epsilon^{ijk}\hat{x}_k,$$
$$[\hat{P}^i, \hat{x}^j] = -i\eta^{ij}\sqrt{1+\kappa^2\hat{P}^2} + i\kappa\epsilon^{ijk}\hat{P}_k,$$
$$[\hat{P}^i, \hat{P}^j] = 0.$$

Lorentz invariance and Jacobi identity are satisfied.

These operators can be identified with Lie algebra of ISO(2,2)

$$\hat{x}_i = \kappa(\hat{J}_{-1,i} - \frac{1}{2}\epsilon_i{}^{jk}\hat{J}_{jk}),$$
$$\hat{P}_i = \hat{P}_{\mu=i},$$

$$\left(\hat{M}_i = \kappa(\hat{J}_{-1,i} + \frac{1}{2}\epsilon_i^{jk}\hat{J}_{jk}), \right)$$

with the constraint

$$1 + \kappa^2\hat{P}^\mu\hat{P}_\mu = 0$$
$$(\mu = -1, 0, 1, 2)$$

⇒ SL(2,R) group momentum space

where ISO(2,2) Lie alg. is given by

$$[\hat{J}_{\mu\nu}, \hat{J}_{\rho\sigma}] = -i(\eta_{\mu\rho}\hat{J}_{\nu\sigma} - \eta_{\mu\sigma}\hat{J}_{\nu\rho} - \eta_{\nu\rho}\hat{J}_{\mu\sigma} + \eta_{\nu\sigma}\hat{J}_{\mu\rho}),$$
$$[\hat{J}_{\mu\nu}, \hat{P}_\rho] = -i(\eta_{\mu\rho}\hat{P}_\nu - \eta_{\nu\rho}\hat{P}_\mu),$$
$$[\hat{P}_\mu, \hat{P}_\nu] = 0,$$

Scalar field

$$\phi(x) = \int dg \tilde{\phi}(g) e^{iP(g)\cdot x}$$

$$g = P^{-1}(g) - i\kappa P^i(g)\tilde{\sigma}_i \in SL(2,R)$$

$$\tilde{\sigma}_0 = \sigma_2,\ \tilde{\sigma}_1 = i\sigma_3,\ \tilde{\sigma}_2 = i\sigma_1$$

$$P^{-1}(g) = \pm\sqrt{1+\kappa^2 P^i(g) P_i(g)}$$

From now on, we pay attention to the positive sign for simplicity.

Star product

$$e^{iP(g_1)\cdot x} \star e^{iP(g_2)\cdot x} = e^{iP(g_1 g_2)\cdot x} \ ,$$

where

$$P^i(g_1 g_2) = P^i(g_1)\sqrt{1+\kappa^2 P(g_2)^2} + \sqrt{1+\kappa^2 P(g_1)^2}\, P^i(g_2) - \kappa \epsilon^{ijk} P_j(g_1) P_k(g_2) \ .$$

Thus coproduct of the translational operator is given by

$$\Delta(P^i) = P^i \otimes \sqrt{1+\kappa^2 P^l P_l} + \sqrt{1+\kappa^2 P^l P_l} \otimes P^i - \kappa \epsilon^{ijk} P_j \otimes P_k$$

→ Hopf algebraic structure!

Thus, the momentum conservation is deformed.

Action

$$S = \int d^3x \left[-\frac{1}{2}(\partial^i \phi \star \partial_i \phi)(x) - \frac{1}{2} m^2 (\phi \star \phi)(x) - \frac{\lambda}{4}(\phi \star \phi \star \phi \star \phi)(x) \right]$$

- At the quantum level, deformed momentum conservation is violated in the non-planar diagrams.

Imai, Sasakura (2000)

- To keep the momentum conservation law, we have to introduce braiding such that

$$\boxed{\psi(\tilde{\phi}_1(g_1)\tilde{\phi}_2(g_2)) = \tilde{\phi}_2(g_2)\tilde{\phi}_1(g_2^{-1}g_1g_2)},$$

which was discovered in three dimensional quantum gravity.

Freidel, Livine (2005)

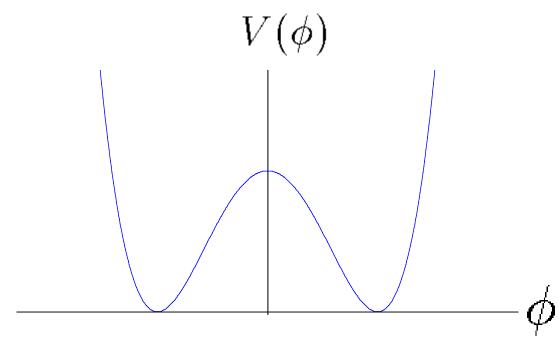
3. Derrick's theorem in the noncommutative ϕ^4 theory

Action

$$S = \int d^3x \left[-\frac{1}{2}(\partial^i\phi \star \partial_i\phi)(x) + \frac{1}{2}m^2(\phi\star\phi)(x) - \frac{\lambda}{4}(\phi\star\phi\star\phi\star\phi)(x) - \frac{m^4}{4\lambda}\right]$$

Equation of motion for $\phi(x)$

$$\partial^2\phi(x) + m^2\phi(x) - \lambda(\phi\star\phi\star\phi)(x) = 0$$



- Star product is included.
- Translational symmetry is not clear.

- Consider only one spatial direction and define

$$P = \frac{1}{\kappa}\sinh(\kappa\theta) \quad \text{, where} \quad -\infty < \theta < \infty.$$

$$\phi(x) = \int dg\,\tilde{\phi}(g)e^{iP(g)x} = \int \frac{d\theta}{2\pi}\tilde{\phi}(\theta)e^{\frac{i}{\kappa}\sinh(\kappa\theta)x}$$

usual sum!

$$e^{iP(g_1)x} \star e^{iP(g_2)x} = e^{iP(g_1g_2)x}$$

$$\longrightarrow \quad e^{\frac{i}{\kappa}\sinh(\kappa\theta_1)x} \star e^{\frac{i}{\kappa}\sinh(\kappa\theta_2)x} = e^{\frac{i}{\kappa}\sinh(\kappa(\theta_1+\theta_2))x}$$

- Next, we define $$h(x) = \int \frac{d\theta}{2\pi} \tilde{\phi}(\theta) e^{i\theta x}.$$

- Equation of motion becomes

$$\frac{1}{\kappa^2} \sin^2(\kappa\partial) h(x) + m^2 h(x) - \lambda h^3(x) = 0.$$

- no star product
- translation invariant

- To analyze this, we consider an action for $h(x)$

$$S_h = \int dx \left[-\frac{1}{2\kappa^2} \sin(\kappa\partial)h(x) \sin(\kappa\partial)h(x) + \frac{1}{2} m^2 h^2(x) - \frac{\lambda}{4} h^4(x) - \frac{m^4}{4\lambda} \right].$$

- Expanding with κ , the energy is

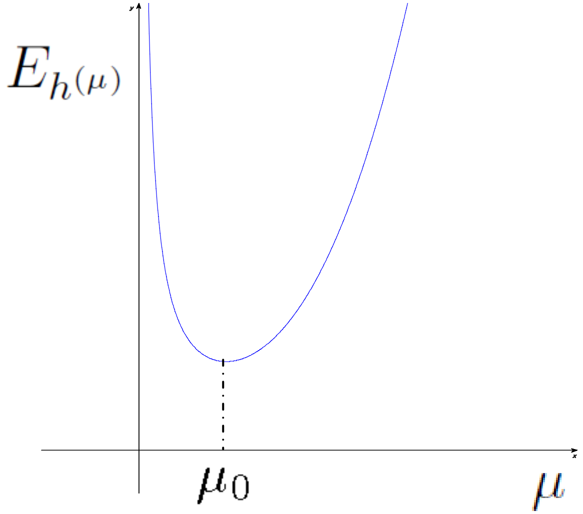
$$E_h = -S_h = \int dx \left[\frac{1}{2} \left(\sum_{n=1}^{\infty} \kappa^{2n-2} C_n \, \partial^n h(x) \partial^n h(x) \right) + V\left(h(x)\right) \right],$$

where $C_n = 2^{n-1}/(n!(2n-1)!!) > 0$

$$V\left(h(x)\right) = -\tfrac{1}{2} m^2 h^2(x) + \tfrac{\lambda}{4} h^4(x) + \tfrac{m^4}{4\lambda} \geq 0.$$

- Rescaling $x^i \to x'^i = \mu x^i \ (0 < \mu < \infty)$ and defining $h^{(\mu)}(x) = h(\mu x)$, the energy for $h^{(\mu)}(x)$ becomes

$$E_{h^{(\mu)}} = \int dx \left[\frac{1}{2}\left(\sum_{n=1}^{\infty} \kappa^{2n-2} C_n (\partial^n h^{(\mu)}(x))^2\right) + V(h^{(\mu)}(x))\right]$$
$$= \int dx' \frac{1}{\mu}\left[\frac{1}{2}\left(\sum_{n=1}^{\infty} \kappa^{2n-2} C_n (\partial'^n h(x'))^2\right) + V(h(x'))\right]$$
$$= \frac{1}{\mu} E_0 + \sum_{n=1}^{\infty} \mu^{2n-1} E_{2n},$$



where
$$E_0 = \int dx\, V\,(h(x))\,,$$
$$E_{2n} = \frac{C_n}{2} \int dx\, (\partial^n h(x))^2.$$

- Since all the E_0 and E_{2n} are non-negative, $E_{h^{(\mu)}}$ takes a minimal value at a positive finite μ.

Domain wall solitons may exist.

4. Domain wall solitons and the moduli fields in Lie algebraic noncommutative spacetime

4.1 The moduli space of the domain wall and the generator

- Equation of motion for $h(x)$ is given by

$$\frac{1}{\kappa^2}\sin^2(\kappa\partial)h(x) + m^2 h(x) - \lambda h^3(x) = 0.$$

This equation is invariant under the usual translation $x \to x + a$

- We can obtain the perturbative solutions as follows.

$$h(x) = h_0(x) + \kappa^2 h_2(x) + \kappa^4 h_4(x) + \cdots$$

$$h_0(x) = \tanh(x+a),$$

$$h_2(x) = \frac{2(x+a)}{3\cosh^2(x+a)} - \frac{4\tanh(x+a)}{3\cosh^2(x+a)}$$

$$h_4(x) = \frac{134(x+a)}{45\cosh^2(x+a)} - \frac{8(x+a)}{3\cosh^4(x+a)} - \frac{40\tanh(x+a)}{9\cosh^2(x+a)} - \frac{4(x+a)^2\tanh(x+a)}{9\cosh^2(x+a)} + \frac{52\tanh(x+a)}{9\cosh^4(x+a)}.$$

where

$$m^2 = 2, \lambda = 2.$$

- Solutions of $\phi(x)$ are in principle given from $h(x+a)$.

$$h(x) = \int \frac{d\theta}{2\pi} \underline{\tilde{\phi}(\theta) e^{i\theta x}}$$

$$\downarrow$$

$$\phi(x) = \int \frac{d\theta}{2\pi} \underline{\tilde{\phi}(\theta)} e^{\frac{i}{\kappa} \sinh(\kappa\theta) x}$$

$$h(x+a) = \int \frac{d\theta}{2\pi} \underline{\tilde{\phi}(\theta) e^{i\theta a}} e^{i\theta x}$$

$$\downarrow$$

$$T_a \phi(x) = \int \frac{d\theta}{2\pi} \underline{\tilde{\phi}(\theta) e^{i\theta a}} e^{\frac{i}{\kappa} \sinh(\kappa\theta) x}$$

$$= e^{ia\hat{\theta}} \phi(x)$$

where $\hat{\theta} = \frac{1}{\kappa} \sinh^{-1}(\kappa \hat{P})$

Thus, $\hat{\theta}$ is the generator of a one-parameter family of domain wall solutions.

In the usual case ($\kappa \to 0$), the generator of a one-parameter family of domain wall solutions is given by $\hat{\theta} = \hat{P}$.

4.2 The moduli field from the Hopf algebraic translational symmetry

- Let us assume $\phi^a_{sol}(x^1)$ is a general solution of the one dimensional equation of motion for $\phi(x)$. We expand $\phi^a_{sol}(x^1)$ with respect to a as

$$\phi^a_{sol}(x^1) = \phi_{sol}(x^1) + \underline{a}\, g(x^1) + \cdots,$$

where $g(x^1)$ should satisfy the following equation,

$$\partial^2 g(x^1) + m^2 g(x^1) - 3\lambda\phi_{sol}(x^1) \star \phi_{sol}(x^1) \star g(x^1) = 0.$$

- In order to obtain an equation for a moduli field, we replace a to a moduli field $a(x_0, x_2)$

$$\phi(x) = \phi_{sol}(x^1) + \underline{a(x^0, x^2)} \star g(x^1) + \cdots.$$

This should satisfy the three dimensional equation of motion for $\phi(x)$.

- Inserting $\phi(x) = \phi_{sol}(x^1) + a(x^0, x^2) \star g(x^1)$ into the equation of motion,

and taking the first order of $a(x_0, x_2)$, we obtain

$$\partial^2(a(x^0,x^2) \star g(x^1)) + m^2(a(x^0,x^2) \star g(x^1)) \\ - 3\lambda\phi_{sol}(x^1) \star \phi_{sol}(x^1) \star a(x^0,x^2) \star g(x^1) = 0,$$

where we have used the braiding property

$$\phi_1(x) \star \phi_2(x) = \phi_2(x) \star \phi_1(x).$$

- Using the equation for $g(x^1)$, we obtain

$$(g(x^1) - \partial^2 g(x^1)) \star \partial^2 a(x^0, x^2) = 0 \quad .$$

- Since $g(x^1) - \partial^2 g(x^1) \neq 0$, we obtain

$$\boxed{\partial^2 a(x_0, x_2) = 0.}$$

Thus we find that the moduli field, which propagates on the domain wall, is massless.

5. Summary

- We studied the domain wall solution and its moduli in the Lie-algebraic noncommutative space-time.

- We found that the generator of a one-parameter family of the domain wall solutions is given by $\hat{\theta} = \frac{1}{\kappa} \sinh^{-1}(\kappa \hat{P})$

- We checked the moduli field on the domain wall is massless.

Question

- A scalar field $\phi(x)$ is not a c-number in braided quantum field theory. Can we interpret the classical solutions with the braid statistics physically?