2008/07/28 YITP Workshop

Lattice formulation of 2D SQCD with exact supersymmetry

(2次元 $\mathcal{N} = (2, 2)$ SQCD の超対称性を保つ格子定式化)

Fumihiko Sugino

(Okayama Institute for Quantum Physics)

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1 Introduction

◇ **Lattice Formulation**: **Standard method for nonperturbative study of QFT**

[Wilson]

Fewer relevant operators to be fine-tuned in the continuum limit

→ Practically easier to construct the continuum theory

⇒ Desirable to possess more symmetries at the level of lattice theory

◇ **Extension to SUSY Theories**

A part of supercharges can be preserved on the lattice:

2D Wess-Zumino model [Sakai-Sakamoto, Kikukawa-Nakayama, Catterall]

pure SYM models [Kaplan et al, Ishii et al] ← orbifolding,

[F.S., Catterall] ← TFT approach

SYM + matter fields [Endre-Kaplan, Matsuura] ← orbifolding,

[F.S.] This Talk ← TFT approach

Here, we construct a lattice model for

2D $\mathcal{N} = (2,2)$ SQCD (SYM + (anti-)fundamental matters)

with $G = U(N)$ (or $SU(N)$)

2D regular lattice

compact gauge fields

general matter superpotential,

keeping one of the supercharge Q.

2 SYM part of the Lattice Action

4D $\mathcal{N}=1$ SYM $\Rightarrow$ (dim. red.) $\Rightarrow$	2D $\mathcal{N}=(2,2)$ SYM
$A_\mu \quad (\mu=0,1)$	$A_\mu \Rightarrow U_\mu(x)$ (link variables on the lattice)
A_2, A_3	$\phi(x), \bar{\phi}(x)$ (site variables)

Fermions : 4-component Majorana spinor

$\Psi(x) = \left(\psi_0(x), \psi_1(x), \chi(x), \frac{1}{2}\eta(x)\right)^T$ (site variables)

Q-SUSY on the lattice

For admissible gauge fields ($||1-U_{01}(x)|| < \epsilon$)

$$
\begin{aligned}
&QU_\mu(x) = i\psi_\mu(x)U_\mu(x) \\
&Q\psi_\mu(x) = i\psi_\mu(x)\psi_\mu(x) + iD_\mu\phi(x) \\
&Q\phi(x) = 0 \\
&Q\bar{\phi}(x) = \eta(x), \quad Q\eta(x) = [\phi(x), \bar{\phi}(x)] \\
&Q\chi(x) = iD(x) + \frac{i}{2}\widehat{\Phi}(x), \quad QD(x) = -\frac{1}{2}Q\widehat{\Phi}(x) - i[\phi(x), \chi(x)],
\end{aligned}
\tag{2.1}
$$

where $D_\mu\phi(x) = U_\mu(x)\phi(x+\hat{\mu})U_\mu(x)^{-1} - \phi(x)$ (covariant difference) ,

$\widehat{\Phi}(x) = \frac{-i(U_{01}(x)-U_{10}(x))}{1-\frac{1}{\epsilon^2}||1-U_{01}(x)||^2} \sim 2F_{01}$

$\Rightarrow Q^2 =$ (infinitesimal gauge tr. with the parameter $\phi(x)$)

Lattice Action: Q-exact form $\Rightarrow$ Exact Q-SUSY

For admissible gauge fields ($||1 - U_{01}(x)|| < \epsilon$ for $\forall x$),

$$
\begin{aligned}
S_{\mathrm{SYM}}^{(\mathrm{lat})} &= Q \frac{1}{g_0^2} \sum_x \mathrm{tr}\left[\chi(x)\left\{-\frac{i}{2}\widehat{\Phi}(x) + iD(x)\right\} + \frac{1}{4}\eta(x)[\phi(x), \bar{\phi}(x)] - i\sum_\mu \psi_\mu(x) D_\mu \bar{\phi}(x)\right] \\
&= \frac{1}{g_0^2}\sum_x \mathrm{tr}\left[\frac{1}{4}\widehat{\Phi}(x)^2 + \sum_\mu D_\mu\phi(x) D_\mu\bar{\phi}(x) + i\chi(x) Q\widehat{\Phi}(x) + i\sum_\mu \psi_\mu(x) D_\mu \eta(x)\right. \\
&\quad + \frac{1}{4}[\phi(x), \bar{\phi}(x)]^2 - \chi(x)[\phi(x), \chi(x)] - \frac{1}{4}\eta(x)[\phi(x), \eta(x)] \\
&\quad \left. - \sum_\mu \psi_\mu(x)\psi_\mu(x)\left(\bar{\phi}(x) + U_\mu(x)\bar{\phi}(x+\hat{\mu})U_\mu(x)^{-1}\right) - D(x)^2\right],
\end{aligned}
\tag{2.2}
$$

For other cases, $S_{\mathrm{SYM}}^{(\mathrm{lat})} = +\infty$. (i.e. The Boltzmann weight is zero.)

Note

Without the admissibility and the denominator of $\widehat{\Phi}$, the configurations

$$
U_{01}(x) = \begin{pmatrix} \pm 1 & & \\ & \ddots & \\ & & \pm 1 \end{pmatrix} \qquad \text{(up to gauge tr.)} \tag{2.3}
$$

for $\forall x$ give the vacua of the action.

To get the target theory, we should consider excitations around the single vacuum $U_{01}(x) = 1$.

The admissibility and the denominator of $\widehat{\Phi}$ smoothly remove the degenerated vacua $U_{01}(x)^2 = 1, U_{01}(x) \neq 1$. [Lüscher],

c.f. $f(t) = \begin{cases} e^{-c/t} & t \geq 0 \\ 0 & t < 0 \end{cases}$ with $c > 0$ is smooth and infinitely differentiable w.r.t. $t \in \mathrm{R}$

$\Rightarrow$ Q-SUSY is preserved.
(Take the traceless part of the numerator of $\widehat{\Phi}$ for $G = SU(N)$ case)

The Q-SUSY forbids the mass term $\phi\bar{\phi}$ appearing in the radiative correction.
$\Rightarrow$ The continuum theory can be constructed without any fine-tuning.
(Checked in the lattice perturbation. Computer simulation will give the nonperturbative check.)

3 Matter part of the Lattice Action

Φ_{+I} : Dim. Red. of 4D $\mathcal{N}=1$ chiral superfield (fundamental repre.)
(Flavors: $I=1,\cdots,n_+$)
Φ_{-I} : Dim. Red. of 4D $\mathcal{N}=1$ chiral superfield (anti-fundamental repre.)
(Flavors: $I=1,\cdots,n_-$)
The continuum theory is

$$\mathcal{L}_{\rm mat} = \left[\sum_{I=1}^{n_+} \Phi_{+I}^\dagger e^{V-\widetilde{V}_{+I}} \Phi_{+I} + \sum_{I=1}^{n_-} \Phi_{-I} e^{-V+\widetilde{V}_{-I}} \Phi_{-I}^\dagger\right]\Bigg|_{\theta\theta\bar\theta\bar\theta} \quad (3.1)$$

$$\mathcal{L}_{\rm pot} = W(\Phi_+,\Phi_-)|_{\theta\theta} + \overline{W}(\Phi_+^\dagger,\Phi_-^\dagger)\Big|_{\bar\theta\bar\theta} \quad (3.2)$$

where $\widetilde{V}_{\pm I} \equiv 2\theta_R\bar\theta_L\widetilde{m}_{\pm I} + 2\theta_L\bar\theta_R\widetilde{m}^*_{\pm I}$: twisted masses.

Note

Two kinds of fermion mass can be introduced.

- Complex mass terms ($\subset W, \overline{W}$):
  $m_I\left(\psi_{-IL}\psi_{+IR} - \psi_{-IR}\psi_{+IL}\right) + m_I^*\left(\bar\psi_{+IR}\bar\psi_{-IL} - \bar\psi_{+IL}\bar\psi_{-IR}\right)$
- Twisted mass terms ($\not\subset W, \overline{W}$):
  $\widetilde{m}_{+I}\bar\psi_{+IL}\psi_{+IR} + \widetilde{m}^*_{+I}\bar\psi_{+IR}\psi_{+IL} + \widetilde{m}_{-I}\psi_{-IR}\bar\psi_{-IL} + \widetilde{m}^*_{-I}\psi_{-IL}\bar\psi_{-IR}$

◇ **Flavor symmetry of $\mathcal{L}_{\rm mat}$: $U(n_+) \times U(n_-)$ ($\widetilde{m}_{\pm I} = \widetilde{m}^*_{\pm I} = 0$ case)**
⇒ $U(1)^{n_+} \times U(1)^{n_-}$ (generic $\widetilde{m}_{\pm I}$, $\widetilde{m}^*_{\pm I}$ case)

◇ **Let us consider the latticization, introducing the forward (backward) covariant differences $D_\mu (D^*_\mu)$:**

$$
\begin{aligned}
D_\mu \Phi_{+I}(x) &\equiv U_\mu(x)\Phi_{+I}(x+\hat{\mu}) - \Phi_{+I}(x) \\
D^*_\mu \Phi_{+I}(x) &\equiv \Phi_{+I}(x) - U_\mu(x-\hat{\mu})^{-1}\Phi_{+I}(x-\hat{\mu}) \\
D_\mu \Phi_{-I}(x) &\equiv \Phi_{-I}(x+\hat{\mu})U_\mu(x)^{-1} - \Phi_{-I}(x) \\
D^*_\mu \Phi_{-I}(x) &\equiv \Phi_{-I}(x) - \Phi_{-I}(x-\hat{\mu})U_\mu(x-\hat{\mu}) \\
&\;\;\vdots
\end{aligned}
\tag{3.3}
$$

and

$$
D^S_\mu \equiv \frac{1}{2}\left(D_\mu + D^*_\mu\right), \qquad D^A_\mu \equiv \frac{1}{2}\left(D_\mu - D^*_\mu\right). \tag{3.4}
$$

Q-SUSY on the lattice [Consider the case $n_+ = n_- \equiv n$]

$$
\begin{aligned}
&Q\phi_{+I}(x) = -\psi_{+IL}(x), \quad Q\psi_{+IL}(x) = -(\phi(x) - \widetilde{m}_{+I})\phi_{+I}(x) \\
&Q\psi_{+IR}(x) = \left(D_0^S + iD_1^S\right)\phi_{+I}(x) + F_{+I}(x) + \sum_\mu \frac{r}{2} D_\mu^A \phi_{-I}(x)^\dagger \quad \leftarrow \text{Wilson term } (r > 0) \\
&QF_{+I}(x) = (\phi(x) - \widetilde{m}_{+I})\psi_{+IR}(x) + \left(D_0^S + iD_1^S\right)\psi_{+IL}(x) + \sum_\mu \frac{r}{2} D_\mu^A \bar{\psi}_{-IR}(x) \\
&\qquad -i\frac{1}{2}\left(\psi_0(x)U_0(x)\phi_{+I}(x+\hat{0}) + U_0(x-\hat{0})^{-1}\psi_0(x-\hat{0})\phi_{+I}(x-\hat{0})\right) \\
&\qquad +\frac{1}{2}\left(\psi_1(x)U_1(x)\phi_{+I}(x+\hat{1}) + U_1(x-\hat{1})^{-1}\psi_1(x-\hat{1})\phi_{+I}(x-\hat{1})\right) \\
&\qquad -i\sum_\mu \frac{r}{2}\left(\psi_\mu(x)U_\mu(x)\phi_{-I}(x+\hat{\mu})^\dagger - U_\mu(x-\hat{\mu})^{-1}\psi_\mu(x-\hat{\mu})\phi_{-I}(x-\hat{\mu})^\dagger\right), \\
&\vdots. \qquad (3.5)
\end{aligned}
$$

Note

$$
Q\bar{\psi}_{-IR}(x) = -(\phi(x) - \widetilde{m}_{-I})\phi_{-I}(x)^\dagger. \qquad (3.6)
$$

$\Rightarrow$

$$
Q^2 F_{+I}(x) = (\phi(x) - \widetilde{m}_{+I})F_{+I}(x) - (\widetilde{m}_{+I} - \widetilde{m}_{-I}) \sum_{\mu=0}^{1} raD_\mu\phi_{-I}(x)^\dagger. \qquad (3.7)
$$

When $\widetilde{m}_{+I} = \widetilde{m}_{-I} (\equiv \widetilde{m}_I)$,

$$\Rightarrow Q^2 = \text{(infinitesimal gauge tr. with the parameter } \phi(x)) + \text{(infinitesimal flavor rotation with the parameter } \widetilde{m}_I) \tag{3.8}$$

$$\delta\Phi_{\pm I} = \mp \widetilde{m}_I \Phi_{+I}, \quad \delta\Phi^\dagger_{\pm I} = \pm \widetilde{m}_I \Phi^\dagger_{+I}$$

Lattice Action: Q-exact form

$$\begin{aligned}
S^{(\text{lat})}_{\text{mat}} &= \sum_{I=1}^{n} \left[S^{(\text{lat})}_{\text{mat},+I} + S^{(\text{lat})}_{\text{mat},-I} \right] \\
S^{(\text{lat})}_{\text{mat},+I} &= Q \sum_x \Bigg[\frac{1}{2} \bar\psi_{+IL}(x) \left\{ \left(D^S_0 + iD^S_1 \right) \phi_{+I}(x) - F_{+I}(x) + \sum_\mu \frac{r}{2} D^A_\mu \phi_{-I}(x)^\dagger \right\} \\
&\quad + \frac{1}{2} \left\{ \left(D^S_0 - iD^S_1 \right) \phi_{+I}(x)^\dagger - F_{+I}(x)^\dagger + \sum_\mu \frac{r}{2} D^A_\mu \phi_{-I}(x) \right\} \psi_{+IR}(x) \\
&\quad + \frac{1}{2} \bar\psi_{+IR}(x) (\bar\phi(x) - \widetilde{m}^*_{+I}) \phi_{+I}(x) - \frac{1}{2} \phi_{+I}(x)^\dagger (\bar\phi(x) - \widetilde{m}^*_{+I}) \psi_{+IL}(x) \\
&\quad + i \phi_{+I}(x)^\dagger \chi(x) \phi_{+I}(x) \Bigg], \qquad (3.9) \\
S^{(\text{lat})}_{\text{mat},-I} &= Q \sum_x \Bigg[\frac{1}{2} \left\{ \left(D^S_0 + iD^S_1 \right) \phi_{-I}(x) - F_{-I}(x) + \sum_\mu \frac{r}{2} D^A_\mu \phi_{+I}(x)^\dagger \right\} \bar\psi_{-IL}(x) \\
&\quad + \frac{1}{2} \psi_{-IR}(x) \left\{ \left(D^S_0 - iD^S_1 \right) \phi_{-I}(x)^\dagger - F_{-I}(x)^\dagger + \sum_\mu \frac{r}{2} D^A_\mu \phi_{+I}(x) \right\}
\end{aligned}$$

$$
+\frac{1}{2}\psi_{-IL}(x)(\bar{\phi}(x)-\widetilde{m}^{*}_{-I})\phi_{-I}(x)^{\dagger}-\frac{1}{2}\phi_{-I}(x)(\bar{\phi}(x)-\widetilde{m}^{*}_{-I})\bar{\psi}_{-IR}(x)
- i\phi_{-I}(x)\chi(x)\phi_{-I}(x)^{\dagger}\Big], \tag{3.10}
$$

Superpotential terms: (i: gauge group index)

$$
S^{(\mathrm{lat})}_{\mathrm{pot}} = Q\sum_{x}\sum_{I}\sum_{i=1}^{N}\Big[-\frac{\partial W}{\partial\phi_{+Ii}(x)}\psi_{+IRi}(x)-\frac{\partial W}{\partial\phi_{-Ii}(x)}\bar{\psi}_{-IRi}(x)
-\bar{\psi}_{+ILi}(x)\frac{\partial\overline{W}}{\partial\phi^{*}_{+Ii}(x)}-\psi_{-ILi}(x)\frac{\partial\overline{W}}{\partial\phi^{*}_{-Ii}(x)}\Big] \tag{3.11}
$$

Note

Due to the Wilson term, the flavor symmetry of $S_{\rm mat}^{(\rm lat)}$ is down to $U(1)^n$ (diagonal subgroup of $U(1)^n \times U(1)^n$).

$\Rightarrow$ The lattice action is Q-SUSY invariant when $\widetilde{m}_{+I} = \widetilde{m}_{-I} (\equiv \widetilde{m}_I)$

We will focus on this case.

(We can still choose $\widetilde{m}^*_{+I}, \widetilde{m}^*_{-I}$ freely!)

4 $U(1)_A$ Anomaly

◇ $U(1)_A$-symmetry with the charges:

$$\begin{aligned} +2 &: \phi \\ +1 &: \psi_\mu, \quad \psi_{\pm IL}, \quad \bar{\psi}_{\pm IR} \\ -1 &: \chi, \quad \eta, \quad \psi_{\pm IR}, \quad \bar{\psi}_{\pm IL} \\ -2 &: \bar{\phi}, \\ 0 &: \text{the others} \end{aligned} \tag{4.1}$$

is realized in the lattice action when all the twisted masses are zero.
In particular, the Wilson terms are consistent with the $U(1)_A$-symmetry.
Since $U(1)_A$ transforms the left-handed fermions and the right-handed fermions differently, it can be anomalous at the quantum level.

Note

- The gaugino fields (ψ_μ, χ, η) belong to the adjoint representation and do not contribute to the anomaly.
- $U(1)_A$ is not anomalous when $n_+ = n_-$. ⇒ consistent with our lattice formulation

$U(1)_A$-WT identity:

$$\partial_\mu \left\langle j_\mu^{U(1)_A}(x) \right\rangle = \left\langle \sum_{I=1}^{n} \left(\mathcal{M}_{+I}(x) + \mathcal{M}_{-I}(x) \right) \right\rangle , \tag{4.2}$$

with

$$\begin{aligned} \mathcal{M}_{+I}(x) &= 2\widetilde{m}_I \left(\phi_{+I}(x)^\dagger \bar{\phi}(x) \phi_{+I}(x) + \bar{\psi}_{+IL}(x) \psi_{+IR}(x) \right) \\ &\quad -2\widetilde{m}^*_{+I} \left(\phi_{+I}(x)^\dagger \phi(x) \phi_{+I}(x) + \bar{\psi}_{+IR}(x) \psi_{+IL}(x) \right) \end{aligned} \tag{4.3}$$

$$\begin{aligned} \mathcal{M}_{-I}(x) &= 2\widetilde{m}_I \left(\phi_{-I}(x) \bar{\phi}(x) \phi_{-I}(x)^\dagger + \psi_{-IR}(x) \bar{\psi}_{-IL}(x) \right) \\ &\quad -2\widetilde{m}^*_{-I} \left(\phi_{-I}(x) \phi(x) \phi_{-I}(x)^\dagger + \psi_{-IL}(x) \bar{\psi}_{-IR}(x) \right) . \end{aligned} \tag{4.4}$$

We can investigate the general case of $n_+ \neq n_-$, if the fields $\Phi_{+I}, \bar{\Phi}_{+I}$ $(I = n_+ + 1, \cdots, n)$ and $\Phi_{-I'}, \bar{\Phi}_{-I'}$ $(I' = n_- + 1, \cdots, n)$ are decoupled by sending $\widetilde{m}^*_{+I} \to \infty$ $(I = n_+ + 1, \cdots, n)$, $\widetilde{m}^*_{-I'} \to \infty$ $(I' = n_- + 1, \cdots, n)$.

Regarding $U(1)_A$-anomaly, we can check that the decoupling is achieved in the lattice perturbation,

and the anomalous WT-identity for n_+ fundamentals and n_- anti-fundamentals is correctly obtained :

$$\partial_\mu \left\langle j_\mu^{U(1)_A}(x) \right\rangle = -\frac{1}{\pi}(n_+ - n_-)\mathrm{tr}\, F_{01}(x) + \left\langle \sum_{I=1}^{n_+} \mathcal{M}_{+I}(x) + \sum_{I=1}^{n_-} \mathcal{M}_{-I}(x) \right\rangle . \qquad (4.5)$$

The anomaly arises from the fermion mass terms via

$$\sum_{I=1}^{n} \mathrm{tr}\, F_{01}(x)\, (ra)^2 \int_{-\pi/a}^{\pi/a} \frac{d^2 q}{(2\pi)^2} \hat{q}^2 \times \left(\hat{q}^2 \cos(aq_0)\cos(aq_1) - 2\bar{q}_0 \cos(aq_0) - 2\bar{q}_1^2 \cos(aq_1) \right) \left(\Delta_{+I}(q)^2 - \Delta_{-I}(q)^2 \right), \qquad (4.6)$$

where the lattice spacing a is introduced,

and $\bar{q}_\mu = \frac{1}{a}\sin(aq_\mu)$, $\hat{q}_\mu = \frac{2}{a}\sin(aq_\mu/2)$,

$$\Delta_{\pm I}(q) = \frac{1}{\bar{q}^2 + \left(\frac{ra}{2}\hat{q}^2\right)^2 + \widetilde{m}_I \widetilde{m}_{\pm I}^*} . \qquad (4.7)$$

Note

The decoupling is not completely trivial, because the holomorphic parts $\widetilde{m}_I$ are kept finite.

The Q-supersymmetry plays an important role to achieve the decoupling.

5 Summary

◇ We have presented a lattice formulation of 2D $\mathcal{N} = (2,2)$ SQCD with exactly keeping Q-SUSY.

- Gauge Group $G = U(N)$ (or $SU(N)$), Compact link variables $U_\mu(x)$
- In order to resolve the doubling of the matters, the lattice action is constructed in the case of the same number of the fundamental matters and anti-fundamental matters $(n_+ = n_-(\equiv n))$.
- Regarding the $U(1)_A$-anomaly, the case $n_+ \neq n_-$ is achieved by decoupling in the lattice perturbation.
  $\Rightarrow$ $n_+ \neq n_-$ case from the beginning?, Ginsparg-Wilson?
- In the $G = U(N)$ case, the FI and topological ϑ-term for the overall $U(1)$ can be introduced:
  $S^{(\mathrm{lat})}_{\mathrm{FI},\,\vartheta} = Q\kappa \sum_x \mathrm{tr}\ (-i\chi(x)) - \frac{\vartheta - 2\pi i\kappa}{2\pi} \sum_x \mathrm{tr}\ \ln U_{01}(x) \sim \int d^2x\, \mathrm{tr}\left(\kappa D - i\frac{\vartheta}{2\pi}F_{01}\right)$
  (Gauge field configurations of nontrivial topology can be taken into account by the admissibility condition with $0 < \epsilon < 1$.)

◇ Applications

- **Computer simulation for SUSY breaking** (c.f. [Kanamori's Talk])
- **Lattice formulation of gauged linear sigma-models (⊃ SQCD models)**
  ⇒ Numerical analysis of phases of Calabi-Yau nonlinear sigma models
  [Witten, Hanany-Hori, Hori-Tong]