On Tomboulis's Proof of Quark Confinement

Hiroshi Suzuki

Theoretical Physics Labolatory
RIKEN

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This PDF file, containing many hyperlinks to appendix, is intended to be "almost" self-contained. If you are intersted in this topic, download this file from the workshop website

References

- E. T. Tomboulis, “Confinement for all values of the coupling in four-dimensional SU(2) gauge theory,” arXiv:0707.2179 [hep-th].
- K. R. Ito and E. Seiler, arXiv:0711.4930 [hep-th].
- E. T. Tomboulis, arXiv:0712.2620 [hep-th].
- T. Kanazawa, Master thesis, Department of Physics, University of Tokyo, 8 January 2008.
- K. R. Ito and E. Seiler, arXiv:0803.3019 [hep-th].
- T. Kanazawa, arXiv:0805.2742 [hep-lat].

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Assertion

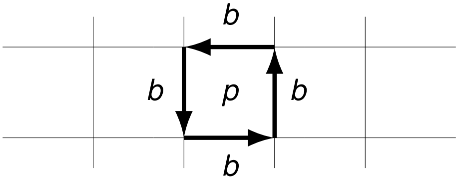
Tomboulis's assertion

Proof of the area law in 4-dimensional SU(2) lattice Yang-Mills theory, for any finite bare coupling β

- Partition function on periodic lattice Λ

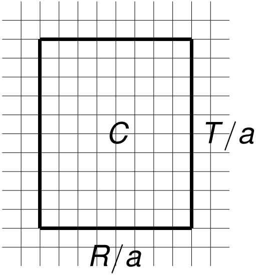
$$Z_\Lambda(\beta) \equiv \int \prod_{b\in\Lambda} dU_b \exp\left(\frac{\beta}{2}\sum_{p\subset\Lambda} \mathrm{Re}\,\mathrm{Tr}_{1/2}\, U_p\right) \qquad \beta \equiv \frac{4}{g_0^2}$$

- Plaquette variables $U_p \equiv U_{x,\mu} U_{x+\mu,\nu} U^\dagger_{x+\nu,\mu} U^\dagger_{x,\nu}$



Area law

- Area law of the Wilson loop (area $A_c = RT/a^2$)



$$\langle W(C)\rangle \equiv \left\langle \frac{1}{2}\mathrm{Tr}_{1/2}\prod_{b\in C} U_b \right\rangle \sim \exp\left(-\hat{\sigma} A_C\right) \quad (\hat{\sigma}\text{: string tension})$$

- Linear confining potential $V(R) \sim \hat{\sigma} R/a^2$

Caveat 1

- The assertion is highly non-trivial,
  - cf. 3-dimensional U(1) lattice gauge theory (Göpfert-Mack)

Caveat 1

However, it does not necessarily mean the quark confinement in continuum Yang-Mills theory

- The continuum theory is defined as a weak coupling limit

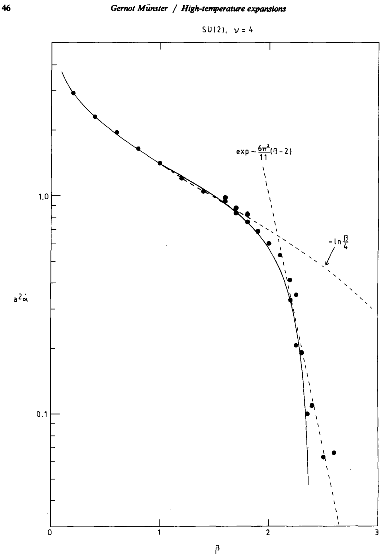
$$a = \frac{1}{\Lambda_L} e^{-\beta/(8b_0)} \to 0 \qquad \beta = \frac{4}{g_0^2} \to \infty \qquad b_0 \equiv \frac{11}{24\pi^2}$$

- One must prove the expected scaling ($\hat{\sigma}/a^2 \to$ const.)

$$\hat{\sigma} \propto e^{-\beta/(4b_0)} \qquad \text{for } \beta \to \infty$$

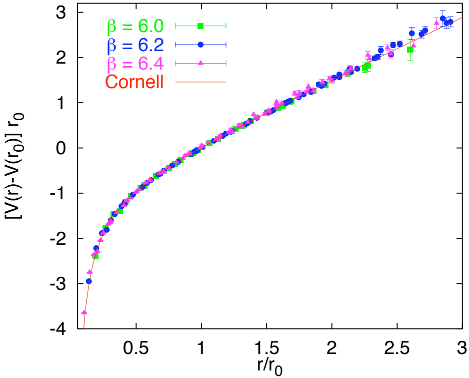
▸ cf.

Numerically... (Creutz ’80, SU(2))



and the continuum limit is. . .

- SU(3) (Bali-Schilling-Wachter '97)



Caveat 2

Caveat 2

The proof is incomplete. . .

- A serious leap of logic (pinned down by Kanazawa)
- At present, it is not clear how to remedy this point
- It is even not clear whether his way of argument will be useful in the future. . .

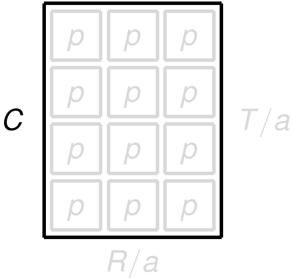
Underlying idea

- Area law is almost obvious for small β ("strong coupling") (Wilson '74)

$$\langle W(C) \rangle = \frac{1}{Z_\Lambda(\beta)} \int \prod_{b \in \Lambda} dU_b \, W(C) \, \exp\left(\frac{\beta}{2} \sum_{p \subset \Lambda} \mathrm{Tr}_{1/2} \, U_p \right)$$

$$\int dU \, U_{ab} = 0$$

$$\int dU \, U_{ab} U^\dagger_{cd} = \tfrac{1}{2} \delta_{ad} \delta_{bc}$$



- For $\beta \ll 1$,

$$\langle W(C) \rangle \simeq \left(\frac{\beta}{2 \cdot 2} \right)^{A_C} \Longrightarrow \hat{\sigma} = -\log(\beta/4) > 0$$

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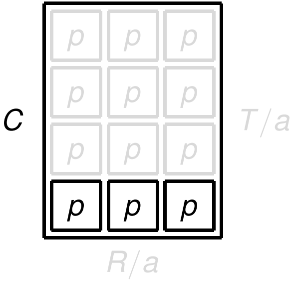

- For $\beta \ll 1$,

$$\langle W(C)\rangle \simeq \left(\frac{\beta}{2\cdot 2}\right)^{Ac} \Longrightarrow \hat{\sigma} = -\log(\beta/4) > 0$$

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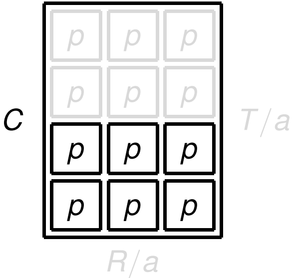

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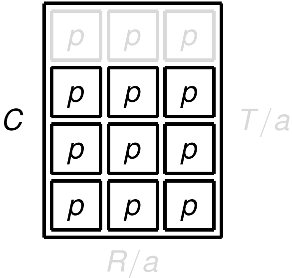

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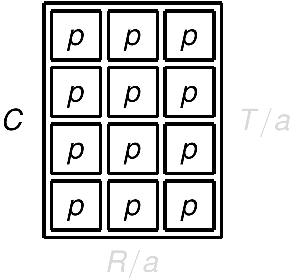

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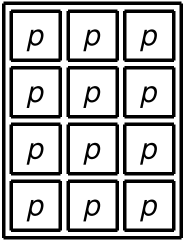

T/a

R/a

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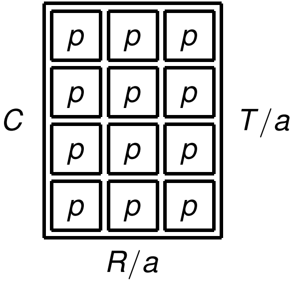

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Basic reasoning

- Strong coupling expansion for $\beta \ll 1$

$$\widehat{\sigma} = -\log(\beta/4) + \frac{\beta^2}{24} - \frac{5\beta^4}{288} + \frac{1117\beta^6}{414720} - \frac{3253\beta^8}{2654208} + \cdots$$

  has a finite radius of convergence (Osterwalder-Seiler, Kotecký-Preiss)

- Cannot extrapolate large β (“weak coupling") physics to small β (“strong coupling”) physics?
- Wilsonian renormalization group from short distance to long distance?
  - Exact treatment is difficult...
- Approximate coarse-graining (block-spin) scheme?
  - Migdal-Kadanoff transformation

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Steps in the proof

1. Migdal-Kadanoff (MK) transformation (approximate RG)
   - Upper and lower bounds for $Z_\Lambda(\beta)$
   - Interpolation parameters α_m that provides an exact expression for $Z_\Lambda(\beta)$
2. Twisted partition function $Z_\Lambda^{(-)}(\beta)$
   - Repeat the above steps using interpolation parameters α_m^+
3. Prove that it is possible to take $\alpha_n = \alpha_n^+$ for $n \gg 1$. The ratio (vortex free energy)

$$\frac{Z_\Lambda^{(-)}(\beta)}{Z_\Lambda(\beta)}$$

   is given by a convergent strong coupling expansion
4. Tomboulis-Yaffe inequality $\Rightarrow$ Area law

Migdal-Kadanoff (MK) transformation ('76, '77)

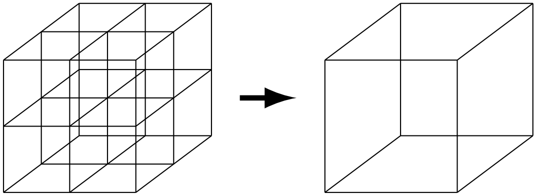
- Approximate block-spin (coarse-graining) transformation

$$a \to ba \to b^2 a \to b^3 a \to \cdots \qquad 2 \le b \in \mathbb{N}$$

$$\Lambda \to \Lambda^{(1)} \to \Lambda^{(2)} \to \Lambda^{(3)} \to \cdots$$

number of plaquettes $\quad |\Lambda| \to |\Lambda^{(1)}| \to |\Lambda^{(2)}| \to |\Lambda^{(3)}| \to \cdots$

- 3-dimensional $b = 2$ case



MK transformation (cont'd)

- The action changes as: $A_p(U,0) \equiv \frac{\beta}{2}\,\mathrm{Tr}_{1/2}\,U$

$$A_p(U,0) \to A_p(U,1) \to A_p(U,2) \to A_p(U,3) \to \cdots$$

- Character expansion: $\chi_j(U) \equiv \mathrm{Tr}_j\,U\ (j = 0, 1/2, 1, 3/2, \ldots)$

$$\exp(A_p(U,n)) = \sum_j d_j F_j(n) \chi_j(U) \qquad d_j \equiv 2j+1$$

$$\equiv F_0(n)\left[1 + \sum_{j\neq 0} d_j c_j(n) \chi_j(U)\right] \equiv F_0(n) f_p(U,n)$$

detail

- In terms of expansion coefficients

$$F_0(0) \to F_0(1) \to F_0(2) \to F_0(3) \to \cdots$$
$$c_j(0) \to c_j(1) \to c_j(2) \to c_j(3) \to \cdots$$

MK transformation (cont'd)

- The partition function changes as

$$Z_{n-1}(\{c_j(n-1)\}) \equiv \int \prod_{b\in\Lambda^{(n-1)}} dU_b \prod_{p\subset\Lambda^{(n-1)}} f_p(U_p, n-1)$$

$$\to F_0(n)^{|\Lambda^{(n)}|} Z_n(\{c_j(n)\})$$

$$\equiv F_0(n)^{|\Lambda^{(n)}|} \int \prod_{b\in\Lambda^{(n)}} dU_b \prod_{p\subset\Lambda^{(n)}} f_p(U_p, n)$$

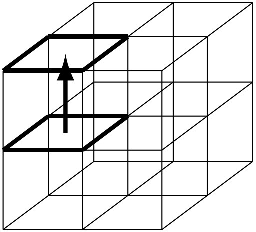
- For the MK transformation with $r = 1$, it can be seen that

$$f_p(U, n) > 0 \qquad \text{if } f_p(U, 0) > 0$$

MK transformation (cont'd)

Definitely,

- Step 1: plaquette move

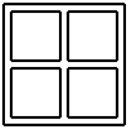


- For example, $(1, 2)$-plaquettes (totally b^2) are moved as

$$(x_1, x_2, x_3, x_4) \Rightarrow (x_1, x_2, ba, ba) \qquad a \leq x_3, x_4 \leq ba$$

MK transformation (cont'd)

- Step 2: 2-dimensional integrations (this step is exact)



- Under these steps, coefficients change according to ▸ detail

$$F_0(n-1) \to F_0(n) = (\widehat{F}_0(n))^{b^2} \qquad (\geq 1 \because \text{DL} \text{ ▸ detail})$$

$$c_j(n-1) \to c_j(n) = \left(\frac{\widehat{F}_j(n)}{\widehat{F}_0(n)}\right)^{b^2 r} \qquad (0 \leq c_j(n) \leq 1)$$

$$\widehat{F}_j(n) \equiv \int dU\,[f_p(U, n-1)]^{b^2} \frac{1}{d_j} \chi_j(U) \qquad (\geq 0 \because \text{DL} \text{ ▸ detail})$$

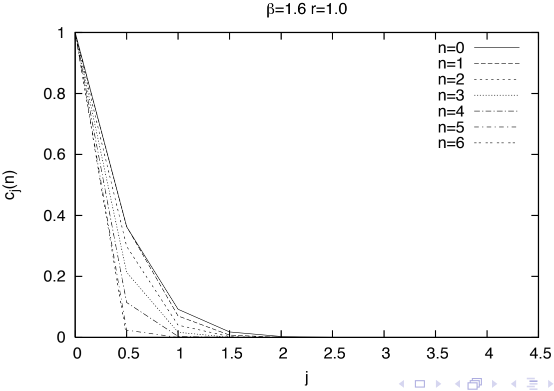
- Tomboulis introduced the deformation $0 < r < 1$

Flow of the MK transformation

- When $r = 1$, for $d \leq 4$, regardless of the initial β (Ito '85, Müller-Schiemann '88)

$$c_j(n) \to 0 \qquad \text{as } n \to \infty$$

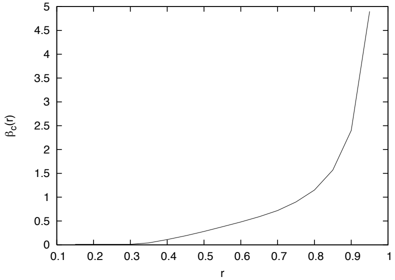
- Couplings flow to the strong coupling limit ($c_j \sim \beta^{2j}$)



Flow of the MK transformation (cont'd)

- When $0 < r < 1$, for $d = 4$, there exists a critical $\beta_c(r)$ such that ▸ detail

$$c_j(n) \to 1 \quad \text{as } n \to \infty \quad \text{if } \beta_c(r) < \beta$$
$$c_j(n) \to 0 \quad \text{as } n \to \infty \quad \text{if } \beta < \beta_c(r)$$



- Convergence to the strong coupling limit can be rigorously proved, if $r = 1 - 1/b$ and b is sufficiently large (Kanazawa '08)

Upper and lower bounds for the partition function

- MK transformed partition function provides an upper bound for the partition function in a one-step before

$$Z_{n-1}(\{c_j(n-1)\}) \leq F_0(n)^{|\Lambda^{(n)}|} Z_n(\{c_j(n)\})$$

- The proof is not difficult (see Tomboulis, Appendix A §4 for $r = 1$ case and note that $c_j(n)$ with $0 < r < 1$ are larger than those with $r = 1$)
- The lower bound

$$1 \leq Z_{n-1}(\{c_j(n-1)\})$$

follows from the facts that

$$\frac{\partial Z_{n-1}(\{c_j\})}{\partial c_k} > 0 \qquad Z_{n-1}(\{c_j = 0\}) = 1$$

- The former can be proved in a similar way to the usual proof of the reflection positivity (RP)

Interpolation by α

- Upper and lower bounds for the partition function

$$1 \leq Z_{n-1}(\{c_j(n-1)\}) \leq F_0(n)^{|\Lambda^{(n)}|} Z_n(\{c_j(n)\})$$

- Define

$$\widetilde{Z}_n(\alpha, t) = F_0(n)^{h(\alpha,t)|\Lambda^{(n)}|} Z_n(\{\alpha c_j(n)\}) \qquad 0 \leq \alpha \leq 1 \quad t \in \mathbb{R}$$

- The function $h(\alpha, t)$ specifies a way of interpolation, ex. $h(\alpha, t) = \exp\left(-t\frac{1-\alpha}{\alpha}\right)$

$$\frac{\partial h}{\partial \alpha} > 0 \quad \frac{\partial h}{\partial t} < 0 \quad h(0, t) = 0 \quad h(1, t) = 1$$

- Thus (since $\partial Z_n(\{c_j(n)\})/\partial c_k(n) \geq 0$ and $F_0(n) \geq 1$)

$$1 = \widetilde{Z}_n(0, t) \leq Z_{n-1}(\{c_j(n-1)\}) \leq \widetilde{Z}_n(1, t)$$

Exact expression for $Z_\Lambda(\beta)$

- For $n = 1$, setting $t = t_1$,

$$1 = \widetilde{Z}_1(0, t_1) \leq Z_\Lambda(\beta) \leq \widetilde{Z}_1(1, t_1)$$

- Thus, there exists $\alpha = \alpha_1(t_1)$ such that

$$\begin{aligned} Z_\Lambda(\beta) &= \widetilde{Z}_1(\alpha_1(t_1), t_1) \\ &= F_0(1)^{h(\alpha_1(t_1),t_1)|\Lambda^{(1)}|} Z_1(\{\alpha_1(t_1)c_j(1)\}) \end{aligned}$$

- For $n = 2$, setting $t = t_2$,

$$1 = \widetilde{Z}_2(0, t_2) \leq Z_1(\{\alpha_1(t_1)c_j(1)\}) \leq Z_1(\{c_j(1)\}) \leq \widetilde{Z}_2(1, t_2)$$

- Thus, there exists $\alpha = \alpha_2(t_2)$ such that

$$\begin{aligned} Z_1(\{\alpha_1(t_1)c_j(1)\}) &= \widetilde{Z}_2(\alpha_2(t_2), t_2) \\ &= F_0(2)^{h(\alpha_2(t_2),t_2)|\Lambda^{(2)}|} Z_2(\{\alpha_2(t_2)c_j(2)\}) \end{aligned}$$

Exact expression for $Z_\Lambda(\beta)$ (cont'd)

- Repeating these,

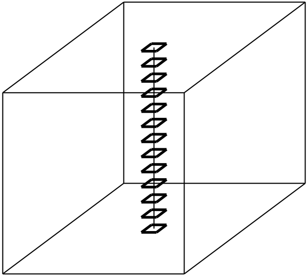
$$Z_\Lambda(\beta) = \left[\prod_{m=1}^{n} F_0(m)^{h(\alpha_m(t_m),t_m)|\Lambda^{(m)}|}\right] Z_n(\{\alpha_n(t_n)c_j(n)\})$$

- Note

$$\alpha_n(t_n)c_j(n) \ll 1 \qquad \text{for } n \gg 1$$

Center vortex and twisted partition function

- Vortex $\mathcal{V}$ is a $(1,2)$ plaquette and its translations along 3-4 directions (a closed surface in the dual lattice)



- Twisted partition function $-1 \in$ center of SU(2)

$$Z_{\Lambda}^{(-)}(\beta) \equiv \int \prod_{b \in \Lambda} dU_b \exp\left(\frac{\beta}{2}\left[\sum_{p \subset \mathcal{V}} \mathrm{Tr}_{1/2}(-U_p) + \sum_{p \subset \Lambda \setminus \mathcal{V}} \mathrm{Tr}_{1/2} U_p\right]\right)$$

MK transformation for $Z_\Lambda^{(-)}(\beta)$

- For MK transformed ones, we define

$$Z_n^{(-)}(\{c_j(n)\}) \equiv \int \prod_{b\in\Lambda^{(n)}} dU_b \prod_{p\subset\mathcal{V}} \left[1 + \sum_{j\neq 0} (-1)^{2j} d_j c_j(n) \chi_j(U_p)\right] \times \prod_{p\subset\Lambda^{(n)}\backslash\mathcal{V}} \left[1 + \sum_{j\neq 0} d_j c_j(n) \chi_j(U_p)\right]$$

- MK transformations applied to $Z_\Lambda^{(-)}(\beta)$ give rise to the sequences

$$F_0(0) \to F_0(1) \to F_0(2) \to F_0(3) \to \cdots$$
$$c_j(0) \to c_j(1) \to c_j(2) \to c_j(3) \to \cdots$$

that are identical to that for the untwisted one $Z_\Lambda(\beta)$

Exact expression for $Z_\Lambda^{(-)}(\beta)$

- $Z_n^{(-)}$ provides the upper bound for $Z_{n-1}^{(-)}$ (Tomboulis, Appendix A §4) but since $\partial Z_n^{(-)}(\{c_j\})/\partial c_k$ can be negative, we cannot repeat the argument in Step 1
- Nevertheless

$$Z_n^{+}(\{c_j\}) \equiv \frac{1}{2}\left(Z_n(\{c_j\}) + Z_n^{(-)}(\{c_j\})\right)$$

satisfies $\partial Z_n^{+}(\{c_j\})/\partial c_k > 0$ (The proof that uses a topological property of the center vortex is interesting; see Tomboulis, Appendix A §2)

- Therefore

$$Z_\Lambda^{+}(\beta) = \left[\prod_{m=1}^{n} F_0(m)^{h(\alpha_m^{+}(t_m^{+}),t_m^{+})|\Lambda^{(m)}|}\right] Z_n^{+}(\{\alpha_n^{+}(t_n^{+})c_j(n)\})$$

Ratio $Z^+_\Lambda(\beta)/Z_\Lambda(\beta)$

- Ratio of twisted and untwisted partition functions

$$\frac{Z^+_\Lambda(\beta)}{Z_\Lambda(\beta)} = \frac{\left[\prod_{m=1}^{n} F_0(m)^{h(\alpha^+_m(t^+_m),t^+_m)|\Lambda^{(m)}|}\right] Z^+_n(\{\alpha^+_n(t^+_n)c_j(n)\})}{\left[\prod_{m=1}^{n} F_0(m)^{h(\alpha_m(t_m),t_m)|\Lambda^{(m)}|}\right] Z_n(\{\alpha_n(t_n)c_j(n)\})}$$

- If the original lattice Λ is sufficiently large, it is possible to take t_m for each t^+_m such that detail

$$h(\alpha^+_m(t^+_m), t^+_m) = h(\alpha_m(t_m), t_m) \qquad \text{for } m = 1, 2, 3, \dots, n$$

- Then “bulk energies” are cancelled and we have

$$\frac{Z^+_\Lambda(\beta)}{Z_\Lambda(\beta)} = \frac{Z^+_n(\{\alpha^+_n(t^+_n)c_j(n)\})}{Z_n(\{\alpha_n(t_n)c_j(n)\})}$$

return

Crucial step

- Now, Tomboulis proves that, if n is sufficiently large, there exists t^* such that

$$\alpha^* \equiv \alpha_n(t^*) = \alpha_n^+(t^*)$$

- Proof: Introduce a function ($0 \leq \lambda \leq 1$)

$$\Psi(\lambda, t) \equiv h(\alpha_n(t), t) + \frac{1}{\log F_0(n)} \frac{1}{|\Lambda^{(n)}|} \Big[(1-\lambda)\log Z_n^+(\{\alpha_n^+(t_n^+) c_j(n)\}) + \lambda \log Z_n^+(\{\alpha_n(t) c_j(n)\}) - \log Z_{n-1}^+\Big]$$

- Motivation for $\Psi(\lambda, t) = 0$:

$$\Psi(\lambda = 0, t) = 0 \text{ has the solution } t = t_n$$

$$\Psi(\lambda = 1, t) = 0 \Longrightarrow \text{The solution } t \text{ is } t^*$$

Crucial step (cont'd)

- The solution $t(\lambda)$ of $\Psi(\lambda, t(\lambda)) = 0$ satisfies

$$\frac{dt(\lambda)}{d\lambda} = -\frac{\partial\Psi/\partial\lambda}{\partial\Psi/\partial t}(\lambda, t(\lambda))$$

- So $t^* = t(1)$ will be obtained by iteratively solving

$$t(\lambda) = t_n - \int_0^\lambda d\lambda' \, \frac{\partial\Psi/\partial\lambda}{\partial\Psi/\partial t}(\lambda', t(\lambda'))$$

if

$$\frac{\partial\Psi(\lambda, t)}{\partial t} \neq 0 \qquad \text{for } 0 \leq \lambda \leq 1$$

- In fact, for $n \gg 1$ (that is for $c_j(n) \ll 1$) detail

$$\frac{\partial\Psi(\lambda, t)}{\partial t} < 0 \qquad \text{for } 0 \leq \lambda \leq 1$$

QED

Vortex free energy

- We now have (setting $\alpha^* = \alpha_n(t^*) = \alpha_n^+(t^*)$)

$$\frac{Z_\Lambda^+(\beta)}{Z_\Lambda(\beta)} = \frac{Z_n^+(\{\alpha^* c_j(n)\})}{Z_n(\{\alpha^* c_j(n)\})} \Longrightarrow \frac{Z_\Lambda^{(-)}(\beta)}{Z_\Lambda(\beta)} = \frac{Z_n^{(-)}(\{\alpha^* c_j(n)\})}{Z_n(\{\alpha^* c_j(n)\})}$$

  and $\alpha^* c_j(n) \ll 1$

- The right hand side is given by a convergent series (the strong coupling cluster expansion) (Münster '81)

$$\log \frac{Z_n^{(-)}(\{\alpha^* c_j(n)\})}{Z_n(\{\alpha^* c_j(n)\})} = -2L_3^{(n)} L_4^{(n)} \exp\left(-\widehat{\rho}(n) L_1^{(n)} L_2^{(n)}\right)$$

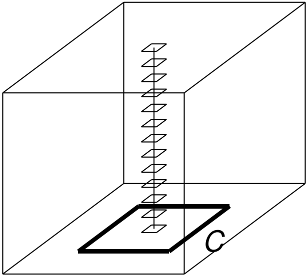
  where the 't Hooft string tension is

$$\widehat{\rho}(n) = -\log(\alpha^* c_{1/2}(n)) - 4(\alpha^* c_{1/2}(n))^4 + \cdots > 0$$

Tomboulis-Yaffe inequality

- Tomboulis-Yaffe inequality ('85)

$$\langle W(C) \rangle \leq 2 \left\{ \frac{1}{2} \left(1 - \frac{Z_\Lambda^{(-)}(\beta)}{Z_\Lambda(\beta)} \right) \right\}^{A_C / L_1 L_2}$$



Area law!

- For a fixed $n \gg 1$, for asymptotically large Λ, noting $L_\mu^{(n)} \equiv L_\mu / b^n$

$$\langle W(C) \rangle \leq \exp\left(-\frac{\widehat{\rho}(n)}{b^{2n}} A_c \right)$$

- Since $\langle W(C) \rangle$ cannot decay faster than the area law (Seiler '78),

$$\langle W(C) \rangle \sim \exp\left(-\widehat{\sigma} A_c\right) \qquad \widehat{\sigma} \geq \frac{\widehat{\rho}(n)}{b^{2n}}$$

This completes the proof of the area law for any fixed β!

Is this correct?

Area law!

- For a fixed $n \gg 1$, for asymptotically large Λ, noting $L_\mu^{(n)} \equiv L_\mu / b^n$

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This completes the proof of the area law for any fixed β!

Is this correct?

Question arose by Ito-Seiler

- The proof could be applied even to 4-dimensional (compact) U(1) lattice gauge theory!
- In particular, the convergence of the MK transformation cannot distinguish U(1) from SU(2)!
- Once after going toward the strong coupling region, U(1) is almost the same as SU(2) (cluster expansion etc.)...
- The weak coupling region of the U(1) theory must be in the Coulomb phase! (Guth '80, Fröhlich-Spencer '82, Seiler '82)

Something is wrong???

After all, what is wrong?

A serious leap of logic! (Kanazawa)

- I said that, in Step 3, $t^* = t(1)$ will be obtained by iteratively solving

$$t(\lambda) = t_n - \int_0^\lambda d\lambda' \, \frac{\partial\Psi/\partial\lambda}{\partial\Psi/\partial t}(\lambda', t(\lambda'))$$

if

$$\frac{\partial\Psi(\lambda, t)}{\partial t} \neq 0 \qquad \text{for } 0 \leq \lambda \leq 1$$

But this is wrong!

- $\partial\Psi(\lambda, t)/\partial t \neq 0$ is not a sufficient condition for the existence of $t(\lambda)$!

After all, what is wrong? (cont'd)

- For example, for

$$\Psi(\lambda, t) = e^{-t} - 1 + 2\lambda$$

we have

$$\frac{\partial \Psi(\lambda, t)}{\partial t} = -e^{-t} \neq 0 \quad \text{for } 0 \leq \lambda \leq 1$$

- However, the solution to $\Psi(\lambda, t) = 0$ is

$$t(\lambda) = -\log(1 - 2\lambda)$$

and this cannot be continued beyond $\lambda = 1/2$ and $t(1)$ is not real

- The reason is

$$\frac{\partial \Psi(\lambda, t(\lambda))}{\partial t} = 0 \quad \text{for } \lambda = 1/2 \text{ at which } t = \infty$$

After all, what is wrong? (cont'd)

- The right condition is

$$\frac{\partial \Psi(\lambda, t(\lambda))}{\partial t} \neq 0 \qquad \text{for } 0 \leq \lambda \leq 1$$

and not

$$\frac{\partial \Psi(\lambda, t)}{\partial t} \neq 0 \qquad \text{for } 0 \leq \lambda \leq 1 \text{ and } |t| < \infty$$

- This difference can be crucial in the real problem, because

$$\left|\frac{\partial \Psi(\lambda, t)}{\partial t}\right| \leq \left|\frac{\partial h(\alpha_n(t), t)}{\partial t}\right| \xrightarrow{t\to\infty} 0 \quad \text{for typical } h(\alpha, t)$$

Conclusion

The existence of $t^* = t(\lambda = 1)$ is not yet guaranteed and the proof is incomplete...

t^* really exists? (H.S., unpublished)

- If the interpolation function $h(\alpha, t)$ satisfies

$$h(\alpha, t) \leq C\alpha \qquad C > 0$$

and $b \geq 3$, then t^* does not exist...

- Examples of such $h(\alpha, t)$ ($t > 0$)

$$h(\alpha, t) = \exp\left(-t\frac{1-\alpha}{\alpha}\right), \qquad h(\alpha, t) = \tanh\left(\frac{\alpha}{t(1-\alpha)}\right)$$

- The inequality

$$k_L \exp(-6\hat{\sigma} b^{2n}) \lesssim \alpha_{n+1}(t^*) \lesssim k_U \exp(-b^2 \hat{\sigma} b^{2n})$$

clashes for $n \to \infty$ if $b^2 > 6$

- Such restriction on $h(\alpha, t)$ and on b is not obvious in the proof...

Simplification and generalization (Kanazawa)

- For $G = \mathrm{U(N)}$ or $G = \mathrm{SU(N)}$, there exists $\overline{\alpha}_\Lambda = \overline{\alpha}_\Lambda(\{c_r\}, \{\alpha c'_r\})$ such that

$$|\alpha - \overline{\alpha}_\Lambda| \le O\left(\frac{1}{|\Lambda|}\right)$$

and cf.

$$\frac{\mathcal{Z}_\Lambda(\{c_r\})}{Z_\Lambda(\{c_r\})} = \frac{\mathcal{Z}_{\Lambda^{(n)}}(\{\alpha c'_r\})}{Z_{\Lambda^{(n)}}(\{\overline{\alpha}_\Lambda c'_r\})} \qquad \mathcal{Z}_\Lambda(\{c_r\}) \equiv \sum_{g \in H} A^g Z^g_\Lambda(\{c_r\})$$

where $A^g > 0$ and H is a discrete subgroup of the center of U(N) or SU(N)

- This statement can be proved without using the MK!
- After proving $Z^g_\Lambda \le Z_\Lambda$ ($\simeq$ RP), just 1 page is enough to prove this!

Simplification and generalization (Kanazawa)

- Generalization of the Tomboulis-Yaffe inequality to SU(N)

$$|\langle W_r(C)\rangle| \leq 2\left\{1 - \frac{1}{N}\sum_{g\in Z_N}\frac{Z^g_\Lambda(\{c_r\})}{Z_\Lambda(\{c_r\})}\right\}^{A_c/L_1L_2}$$

where

$$W_r(C) \equiv \frac{1}{d_r}\chi_r(\mathcal{P}\prod_{b\in C} U_b)$$

and N-ality of the representation r is non-zero

- Then, if $\{\alpha c'_r\}$ can be chosen such that $\alpha_0 \equiv \overline{\alpha}_\Lambda = \alpha$, then the area law will follow

$$|\langle W_r(C)\rangle| \leq e^{-\widehat{\rho}A_C/b^{2n}}$$

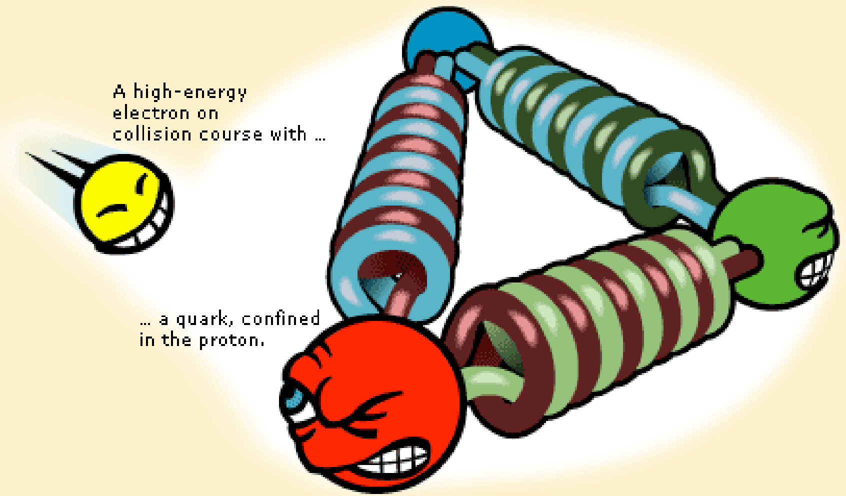
Conclusion

- Suggestive, but incomplete...
- It appears that the representation

$$\frac{\mathcal{Z}_\Lambda(\{c_r\})}{Z_\Lambda(\{c_r\})} = \frac{\mathcal{Z}_{\Lambda^{(n)}}(\{\alpha c'_r\})}{Z_{\Lambda^{(n)}}(\{\overline{\alpha}_\Lambda c'_r\})} \qquad \mathcal{Z}_\Lambda(\{c_r\}) \equiv \sum_{g \in H} A^g Z^g_\Lambda(\{c_r\})$$

  does not possess (much) dynamical information
  $\because$ Kanazawa's derivation does not use the MK at all
- Quark confinement is still far from having been proved
- Possible remedy for the argument for t^*? (Tomboulis, private communication)

THIS IS THE QUARK CONFINEMENT!



from http://nobelprize.org/nobel_prizes/physics/laureates/2004/illpres/

Clay Millennium Problems (Quantum Yang-Mills theory)

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Yang-Mills and Mass Gap

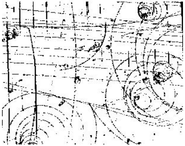
The laws of quantum physics stand to the world of elementary particles in the way that Newton's laws of classical mechanics stand to the macroscopic world. Almost half a century ago, Yang and Mills introduced a remarkable new framework to describe elementary particles using structures that also occur in geometry. Quantum Yang-Mills theory is now the foundation of most of elementary particle theory, and its predictions have been tested at many experimental laboratories, but its mathematical foundation is still unclear. The successful use of Yang-Mills theory to describe the strong interactions of elementary particles depends on a subtle quantum mechanical property called the "mass gap:" the quantum particles have positive masses, even though the classical waves travel at the speed of light. This property has been discovered by physicists from experiment and confirmed by computer simulations, but it still has not been understood from a theoretical point of view. Progress in establishing the existence of the Yang-Mills theory and a mass gap and will require the introduction of fundamental new ideas both in physics and in mathematics.

The Millennium Problems

Official Problem Description — Arthur Jaffe and Edward Witten

Report on the Status of the Yang-Mill Millenium Prize Problem by Michael Douglas (April 2004).

Lecture by Lorenzo Sadun at University of Texas (video)



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Character expansion

- SU(2) character: $U = \begin{pmatrix} e^{i\theta/2} & 0 \\ 0 & e^{-i\theta/2} \end{pmatrix} \in \mathrm{SU}(2)$

$$\chi_j(U) = \mathrm{Tr}_j\, U = \frac{\sin\left(\frac{1}{2}(2j+1)\theta\right)}{\sin\left(\frac{1}{2}\theta\right)} \qquad j = 0, 1/2, 1, 3/2, \ldots$$

- Decomposition law (DL) return

$$\chi_i(U)\chi_j(U) = \sum_{k=|i-j|}^{i+j} 1\,\chi_k(U)$$

- Orthonormality ($\int dU\, \chi_0(U) = 1$)

$$\int dU\, \chi_i(U)\chi_j(U) = \delta_{ij} \qquad \text{where } \int dU \times = \int_0^{4\pi} \frac{d\theta}{2\pi} \sin^2\left(\frac{1}{2}\theta\right) \times$$

(This follows from the orthogonality of irreducible representations $\int dU\, R_j(U)_{ab} R_k(U^\dagger)_{cd} = \frac{1}{d_j}\delta_{jk}\delta_{ad}\delta_{bc}$)

Character expansion (cont'd)

- Character expansion of the plaquette action (U: plaquette)

$$\exp\left(\frac{\beta}{2}\mathrm{Tr}_{1/2}\,U\right) = \sum_j d_j F_j \chi_j(U) \qquad d_j \equiv 2j+1$$

$$\equiv F_0\left[1 + \sum_{j\neq 0} d_j c_j \chi_j(U)\right] \equiv F_0 f_p(U)$$

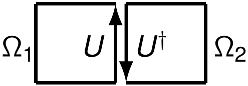
- Coefficients

$$F_j = \frac{1}{d_j}\int dU \exp\left(\frac{\beta}{2}\mathrm{Tr}_{1/2}\,U\right)\chi_j(U) = \frac{2}{\beta} I_{2j+1}(\beta)$$

$$c_j = \frac{I_{2j+1}(\beta)}{I_1(\beta)} = \frac{\beta^{2j}}{(2j+1)!} + O(\beta^{2j+2}) \qquad \text{for } \beta \ll 1$$

Power of the character expansion

- For 2 adjacent plaquettes



- Integration over U yields

$$\int dU \sum_j d_j F_j \chi_j(\Omega_1 U) \sum_k d_k F_k \chi_j(U^\dagger \Omega_2) = \sum_j d_j F_j^2 \chi_j(\Omega_1 \Omega_2)$$

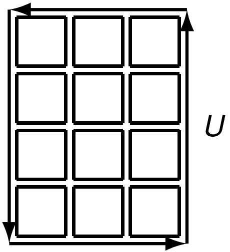
- because

$$\int dU \, \chi_j(\Omega_1 U) \chi_k(U^\dagger \Omega_2) = \frac{1}{d_j} \delta_{j,k} \chi_j(\Omega_1 \Omega_2)$$

(This also follows from the orthogonality of irreducible representations)

Power of the character expansion (cont'd)

- For a 2-dimensional collection of plaquettes



- Integrations over internal U_b's yield

$$\sum_j d_j (F_j)^{\text{number of plaquettes}} \chi_j(U)$$

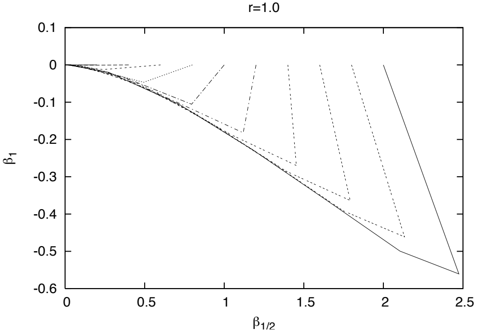
return

Flow of the MK transformation

- Coupling constants

$$A_p(U,n) = \left(\frac{\beta_{1/2}(n)}{2} \chi_{1/2}(U) + \frac{\beta_1(n)}{3} \chi_1(U) + \cdots \right)$$

- RG flow in the $(\beta_{1/2}, \beta_1)$-plane

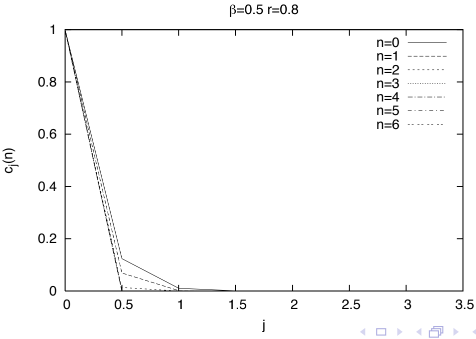


Flow of the MK transformation (cont'd)

A rough approximation that neglects β_j with $j \geq 1$ yields...

- Sufficiently strong couplings always flow to the strong coupling limit, as long as $b^2 r > 1$

$$\beta_{1/2}(n) \simeq 4\left(\frac{b^{d-2}}{4}\beta_{1/2}(n-1)\right)^{b^2 r} \to 0 \qquad \text{as } n \to \infty$$



Flow of the MK transformation (cont'd)

- In the weak coupling region,

$$\beta_{1/2}(n) \simeq \frac{b^{d-4}}{r}\beta_{1/2}(n-1) - 8\widetilde{b}_0 \ln b \qquad \widetilde{b}_0 \equiv \frac{1 - b^{-2}/r}{24 \ln b}$$

- When $r = 1$, for $d = 4$, $\beta_{1/2}$ is certainly marginal and asymptotically free:

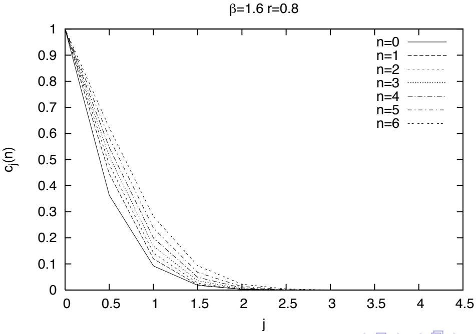
$$\widetilde{b}_0 = 0.04508\cdots \quad \text{for } b = 2$$

is 97.08% of the right answer, $b_0 = 11/24\pi^2 = 0.04643\cdots$

Flow of the MK transformation (cont'd)

- When $r < 1$, the parameter r effectively increases the dimensionality as $d > 4$ (Ito-Seiler) and this makes $\beta_{1/2}$ "irrelevant": $\beta_{1/2}(n) \to \infty$ as $n \to \infty$
- Weak couplings flow toward weak region! ◂ return

$$c_j(n) \to 1 \qquad \text{as } n \to \infty$$



t and t^+

- Suppose that we first set $t_1 = t_1^+$. Then

$$\frac{Z_\Lambda^+(\beta)}{Z_\Lambda(\beta)} = \frac{F_0(1)^{h(\alpha_1^+(t_1^+),t_1^+)|\Lambda^{(1)}|} Z_1^+(\{\alpha_1^+(t_1^+)c_j(1)\})}{F_0(1)^{h(\alpha_1(t_1^+),t_1^+)|\Lambda^{(1)}|} Z_1(\{\alpha_1(t_1^+)c_j(1)\})}$$

$$= \frac{F_0(1)^{h(\alpha_1^+(t_1^+),t_1^+)|\Lambda^{(1)}|}}{F_0(1)^{h(\alpha_1(t_1^+),t_1^+)|\Lambda^{(1)}|} } \frac{Z_1^+(\{\alpha_1^+(t_1^+)c_j(1)\})}{Z_1^+(\{\alpha_1(t_1^+)c_j(1)\})} \frac{Z_1^+(\{\alpha_1(t_1^+)c_j(1)\})}{Z_1(\{\alpha_1(t_1^+)c_j(1)\})}$$

- Since $0 < Z^{(-)} < Z$ (see Tomboulis, Appendix A §5)

$$\frac{1}{2} < \frac{Z_\Lambda^+(\beta)}{Z_\Lambda(\beta)} < 1 \qquad \frac{1}{2} < \frac{Z_1^+(\{\alpha_1(t_1^+)c_j(1)\})}{Z_1(\{\alpha_1(t_1^+)c_j(1)\})} < 1$$

and thus

$$\frac{1}{2} < \frac{F_0(1)^{h(\alpha_1^+(t_1^+),t_1^+)|\Lambda^{(1)}|}}{F_0(1)^{h(\alpha_1(t_1^+),t_1^+)|\Lambda^{(1)}|} } \frac{Z_1^+(\{\alpha_1^+(t_1^+)c_j(1)\})}{Z_1^+(\{\alpha_1(t_1^+)c_j(1)\})} < 2$$

t and t^+ (cont'd)

- Taking the logarithm

$$\Bigg| h(\alpha_1^+(t_1^+), t_1^+) - h(\alpha_1(t_1^+), t_1^+) + \frac{1}{|\Lambda^{(1)}| \log F_0(1)} \log \frac{Z_1^+(\{\alpha_1^+(t_1^+) c_j(1)\})}{Z_1^+(\{\alpha_1(t_1^+) c_j(1)\})} \Bigg| < \frac{\log 2}{\log F_0(1)} \frac{1}{|\Lambda^{(1)}|}$$

- Since $h(\alpha, t)$ and $Z_1^+(\{\alpha c_j(1)\})$ are monotonically increasing functions of α,

$$\left| h(\alpha_1^+(t_1^+), t_1^+) - h(\alpha_1(t_1^+), t_1^+) \right| + \frac{1}{|\Lambda^{(1)}| \log F_0(1)} \left| \log \frac{Z_1^+(\{\alpha_1^+(t_1^+) c_j(1)\})}{Z_1^+(\{\alpha_1(t_1^+) c_j(1)\})} \right| < \frac{\log 2}{\log F_0(1)} \frac{1}{|\Lambda^{(1)}|}$$

t and t^+ (cont'd)

- This shows

$$\left|h(\alpha_1^+(t_1^+), t_1^+) - h(\alpha_1(t_1^+), t_1^+)\right| \leq O\left(\frac{1}{|\Lambda^{(1)}|}\right)$$

- Thus, if

$$\frac{dh(\alpha_1(t), t)}{dt} \neq 0 \qquad \text{as } |\Lambda| \to \infty$$

as shown in Tomboulis, Appendix B (the parameter $r < 1$ is introduced to guarantees this), there always exists, for asymptotically large $|\Lambda|$, $t_1 = t_1^+ + O(1/|\Lambda^{(1)}|)$ such that

$$h(\alpha_1^+(t_1^+), t_1^+) = h(\alpha_1(t_1), t_1)$$

- Repeat this procedure from $m = 1$ to $m = n$

Various properties of the interpolation

- Using

$$Z_{n-1} = F_0(n)^{h(\alpha_n(t),t)|\Lambda^{(n)}|} Z_n(\{\alpha_n(t) c_j(n)\})$$

- We have

$$\frac{d\alpha_n(t)}{dt} = -\frac{|\Lambda^{(n)}| \log F_0(n)}{\frac{\partial h}{\partial \alpha}|\Lambda^{(n)}| \log F_0(n) + \frac{\partial}{\partial \alpha} \log Z_n} \frac{\partial h}{\partial t}\bigg|_{\alpha=\alpha_n(t)} > 0$$

and, equivalently,

$$\frac{dh(\alpha_n(t),t)}{dt} = -\frac{d\alpha_n(t)}{dt} \frac{1}{|\Lambda^{(n)}| \log F_0(n)} \frac{\partial}{\partial \alpha} \log Z_n\bigg|_{\alpha=\alpha_n(t)} < 0$$

where the argument of the partition functions is $\{\alpha c_j(n)\}$ and we have used $\delta' < \alpha_n(t) < 1 - \delta$ (Tomboulis, Appendix B)

Various properties of the interpolation (cont'd)

- It is also straightforward to see that

$$\frac{\partial \Psi(\lambda,t)}{\partial t} = \left[-\frac{\frac{\partial h}{\partial \alpha}|\Lambda^{(n)}| \log F_0(n) + \lambda \frac{\partial}{\partial \alpha} \log Z_n^+}{\frac{\partial h}{\partial \alpha}|\Lambda^{(n)}| \log F_0(n) + \frac{\partial}{\partial \alpha} \log Z_n} + 1 \right] \frac{\partial h}{\partial t} \bigg|_{\alpha=\alpha_n(t)}$$

where the argument of all the partition functions is $\{\alpha c_j(n)\}$ and

$$\frac{\partial \Psi(\lambda,t)}{\partial \lambda} = \frac{1}{|\Lambda^{(n)}| \log F_0(n)} \times \left[\log Z_n^+(\{\alpha_n(t) c_j(n)\}) - \log Z_n^+(\{\alpha_n^+(t_n^+) c_j(n)\}) \right]$$

Various properties of the interpolation (cont'd)

- For $n \gg 1$, using the result of the cluster expansion,

$$\frac{\partial}{\partial \alpha}\log(Z_n) - \frac{\partial}{\partial \alpha}\log(Z_n^+)$$
$$= \frac{1}{\alpha} L_1^{(n)} L_2^{(n)} L_3^{(n)} L_4^{(n)} \exp(-\widehat{\rho}(n) L_1^{(n)} L_2^{(n)}) \frac{Z_n^{(-)}}{Z_n^+}\Bigg|_{\alpha=\alpha_n(t)} > 0$$

therefore (since $\partial h / \partial t < 0$)

$$\frac{\partial \Psi(\lambda, t)}{\partial t} < 0 \quad \text{for } 0 \le \lambda \le 1$$

return

Various properties of the interpolation (cont'd)

- Now, as a possible case, assume

$$\alpha_n(t_n^+) > \alpha^+(t_n^+)$$

then since $\partial h/\partial\alpha > 0$

$$h(\alpha_n(t_n^+), t_n^+) > h(\alpha_n^+(t_n^+), t_n^+)$$

This, combined with

$$h(\alpha_n^+(t_n^+), t_n^+) = h(\alpha_n(t_n), t_n)$$

and $dh(\alpha_n(t), t)/dt < 0$, implies

$$t_n > t_n^+$$

Finally, since $d\alpha_n(t)/dt > 0$,

$$\alpha_n(t) > \alpha_n(t_n^+) \qquad \text{for } t \geq t_n > t_n^+$$

Various properties of the interpolation (cont'd)

- For this case, therefore

$$\frac{\partial \Psi(\lambda, t)}{\partial \lambda} > 0 \qquad \text{for } t \geq t_n > t_n^+$$

and

$$t(\lambda) = t_n - \int_0^\lambda d\lambda' \, \frac{\partial \Psi / \partial \lambda}{\partial \Psi / \partial t}(\lambda', t(\lambda'))$$

will give $t(\lambda) > t_n$ for $\lambda > 0$

Various properties of the interpolation (cont'd)

- As a second case, assume

$$\alpha_n(t_n^+) < \alpha^+(t_n^+)$$

then since $\partial h/\partial\alpha > 0$

$$h(\alpha_n(t_n^+), t_n^+) < h(\alpha_n^+(t_n^+), t_n^+)$$

This, combined with

$$h(\alpha_n^+(t_n^+), t_n^+) = h(\alpha_n(t_n), t_n)$$

and $dh(\alpha_n(t), t)/dt < 0$, implies

$$t_n < t_n^+$$

Finally, since $d\alpha_n(t)/dt > 0$,

$$\alpha_n(t) < \alpha_n(t_n^+) \qquad \text{for } t \leq t_n < t_n^+$$

Various properties of the interpolation (cont'd)

- For this case, therefore

$$\frac{\partial \Psi(\lambda, t)}{\partial \lambda} < 0 \qquad \text{for } t \leq t_n < t_n^+$$

and

$$t(\lambda) = t_n - \int_0^\lambda d\lambda' \, \frac{\partial \Psi / \partial \lambda}{\partial \Psi / \partial t}(\lambda', t(\lambda'))$$

will give $t(\lambda) < t_n$ for $\lambda > 0$