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Supergravity and Doubled Geometry

based on arXiv:0806.1783 [hep-th]

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Introduction

Start from low energy effective field theory for ten-dimensional string theory including

$$S = \int \mathrm{d}^{10}x \sqrt{-\mathcal{G}}\, \mathrm{e}^{-2\Phi} \Big\{ \mathcal{R} + 4(\nabla\Phi)^2 - \frac{1}{12} \mathcal{H}_{MNP} \mathcal{H}^{MNP} \Big\}$$

$$\mathcal{H} = \mathrm{d}\mathcal{B}$$

Consider the field theory compactified on (twisted) torus in the presence of B-field.

motivation

flux compactifications

duality relations among flux vacua

N. Kaloper, R.C. Myers hep-th/9901045

Reduced D-dim. action compactified on a flat d-torus $(D = 10 - d)$:

$$S = \int \mathrm{d}^D x \sqrt{-g}\, \mathrm{e}^{-2\phi} \Big\{ R + 4(\nabla\phi)^2 - \frac{1}{12} H_{\mu\nu\rho} H^{\mu\nu\rho} + \frac{1}{8} L_{IJ} \nabla_\mu \mathcal{M}^{JK} L_{KL} \nabla^\mu \mathcal{M}^{LI} - \frac{1}{4} F^I_{\mu\nu} L_{IJ} \mathcal{M}^{JK} L_{KL} F^{L\mu\nu} \Big\}$$

This theory has $U(1)^{2d}$ gauge symmetry and a manifest global $O(d,d)$ symmetry with

$$\mathcal{M}_{IJ} = \begin{pmatrix} g_{ij} - B_{ik}\, g^{kl}\, B_{lj} & B_{ik}\, g^{kj} \\ -g^{ik}\, B_{kj} & g^{ij} \end{pmatrix} \; : \; \text{moduli, taking values in } \frac{O(d,d)}{O(d) \times O(d)}$$

$$F^I = \mathrm{d}A^I \,, \qquad A^I_\mu = \begin{pmatrix} V^i{}_\mu \\ B_{\mu i} \end{pmatrix} , \qquad H_{\mu\nu\rho} = 3\, \partial_{[\mu} B_{\nu\rho]} - \frac{3}{2} A^I_{[\mu}\, L_{|IJ|}\, F^J_{\nu\rho]}$$

$$L^{IJ} \equiv \begin{pmatrix} \mathbf{0}_d & \mathbb{1}_d \\ \mathbb{1}_d & \mathbf{0}_d \end{pmatrix} \; : \; \begin{array}{c} O(d,d) \text{ invariant metric s.t.} \\ \forall M \in O(d,d),\; M L M^T = L \end{array}$$

Non-abelian gauge symmetry from a $2d$-dimensional subgroup G of $O(d,d)$:

Fundamental repr. of $O(d,d)$ becomes adjoint repr. of G under embedding

$$[T_I, T_J] \; = \; t_{IJ}{}^K T_K \, , \qquad T_I \; = \; \frac{1}{2} \Theta_I{}^{JK} \, \mathfrak{m}_{JK}$$

$$\begin{cases} T_I : & \text{generators of } G \text{ with structure constant } t_{IJ}{}^K \\ \mathfrak{m}_{JK} : & \text{generators of } O(d,d) \\ \Theta_I{}^{JK} : & \text{embedding tensor} \end{cases}$$

T_I are (non-)abelian generators for gauge fields $A^I_\mu = \big(V^i{}_\mu, B_{\mu i}\big)^T$:

$$T_I \ni \begin{cases} Z_i : & \text{generators for } V^i{}_\mu \\ X^i : & \text{generators for } B_{\mu i} \end{cases} \dashrightarrow \quad \begin{aligned} [Z_i, Z_j] &= f_{ij}{}^k Z_k + h_{ijk} X^k \\ [X^i, X^j] &= 0 \\ [X^i, Z_j] &= f^i{}_{jk} X^k \end{aligned}$$

$f_{ij}{}^k$:	structure constant of twisted torus
h_{ijk} :	(minus) VEV of three-form H_{ijk}

- Twisted torus is introduced by vielbein $\mathrm{d}y^i \to \mathsf{e}^m = \mathsf{e}^m{}_i(y)\,\mathrm{d}y^i$:

$$g_{ij}(x) \quad \to \quad \mathsf{G}_{ij}(x,y) = g_{mn}(x)\, \mathsf{e}_i{}^m(y)\, \mathsf{e}^n{}_j(y)$$

$$g_{ij}(x)\big(\mathrm{d}y^i + V^i{}_\mu \mathrm{d}x^\mu\big)\big(\mathrm{d}y^j + V^j{}_\nu \mathrm{d}x^\nu\big) \quad \to \quad g_{mn}(x)\big(\mathsf{e}^m(y) + V^m{}_\mu \mathrm{d}x^\mu\big)\big(\mathsf{e}^n(y) + V^n{}_\nu \mathrm{d}x^\nu\big)$$

We often switch off 4-dim. fluctuations: $g_{mn}(x) \to \delta_{mn}$, $B_{mn}(x) \to 0$, $\mathsf{G}_{ij}(x,y) \to \mathsf{G}_{ij}(y)$.

$$
\begin{aligned}
[Z_m, Z_n] &= f_{mn}{}^p \, Z_p + h_{mnp} \, X^p \\
[X^m, X^n] &= 0 \\
[X^m, Z_n] &= f^m{}_{np} \, X^p
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$$\Downarrow$$

$$
\begin{aligned}
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[X^m, X^n] &= Q^{mn}{}_p \, X^p + R^{mnp} \, Z_p \\
[X^m, Z_n] &= f^m{}_{np} \, X^p - Q^{mp}{}_n \, Z_p
\end{aligned}
$$

Why should we study additional structure constants $Q^{mn}{}_p$ and R^{mnp}?

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Why should we study additional structure constants $Q^{mn}{}_p$ and R^{mnp}?

$$\downarrow$$

Because they are related via T-duality transformations

J. Shelton, W. Taylor, B. Wecht hep-th/0508133   A. Dabholkar, C.M. Hull hep-th/0512005

h_{mnp}

$\downarrow$ T-duality transformation

$f^m{}_{np}$

$\downarrow$ T-duality transformation

$Q^{mn}{}_p$

$\downarrow$ T-duality transformation

R^{mnp}

h_{mnp}	Flat torus with three-form flux
$\downarrow$	T-duality transformation
$f^m{}_{np}$	
$\downarrow$	T-duality transformation
$Q^{mn}{}_p$	
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R^{mnp}	

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${f^m}_{np}$	Twisted torus with non-trivial isometry group
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$Q^{mn}{}_{p}$	T-fold globally nongeometric, locally geometric: *stringy*
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C.M. Hull

In order to include the above information,

we double the *compactified geometry* $\mathcal{M}_d$ *to* $\mathcal{M}_{2d} = \mathcal{M}_d \times \widetilde{\mathcal{M}}_d$

and study an extended sigma model on it. ⇢ Doubled Formalism

C.M. Hull hep-th/0406102 hep-th/0605149 C.M. Hull, R.A. Reid-Edwards hep-th/0503114 arXiv:0711.4818

Doubled formalism

Start with a sigma model on a space $\mathcal{M}_d$ with metric $G_{ij}(y)$ and B-field $B_{ij}(y)$:

$$S_c = \frac{1}{2} \int_{\Sigma} \Big(G_{ij} \, \mathrm{d}Y^i \wedge * \mathrm{d}Y^j + B_{ij} \, \mathrm{d}Y^i \wedge \mathrm{d}Y^j \Big)$$

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Extend to the action with the Wess-Zumino term on a doubled space $\mathcal{M}_{2d}\,(= G/\Gamma)$

$$S = \frac{1}{4} \int_\Sigma \mathcal{M}_{MN}\, \mathcal{P}^M \wedge *\mathcal{P}^N + \frac{1}{12} \int_V t_{MNP}\, \mathcal{P}^M \wedge \mathcal{P}^N \wedge \mathcal{P}^P$$

- Σ: string worldsheet (without boundary)
- V: an extension of Σ s.t. $\partial V = \Sigma$
- G: $2d$-dim. (non-)compact Lie group w/ $[T_M, T_N] = t_{MN}{}^P T_P$
- Γ: a discrete subgroup of G chosen s.t. $\mathcal{M}_{2d}$ is compact

for doubled sigma model with BOUNDARY: C.Albertsson's POSTER

Constituents of the action $S = \frac{1}{4}\int_{\Sigma} \mathcal{M}_{MN}\, \mathcal{P}^M \wedge *\mathcal{P}^N + \frac{1}{12}\int_V t_{MNP}\, \mathcal{P}^M \wedge \mathcal{P}^N \wedge \mathcal{P}^P$

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✓ Scalar fields of doubled coordinates and doubled vielbeins:

$$\mathbb{Y}^I\, ; \quad \mathcal{P} \;=\; \mathfrak{g}^{-1}\mathrm{d}\mathfrak{g} \;=\; \mathcal{P}^M{}_I \left(r\, T_M\right) \mathrm{d}\mathbb{Y}^I \;, \quad \text{w/ } \mathfrak{g} \in G \subset O(d,d)$$

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✓ Bianchi identity (Maurer-Cartan eq.):

$$\mathrm{d}\mathcal{P}^M = -\frac{r}{2}\, t_{NP}{}^M\, \mathcal{P}^N \wedge \mathcal{P}^P$$

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✓ Doubled metric from doubled vielbeins:

$$\mathcal{M}_{MN} = \mathcal{P}_M{}^I\,\mathcal{M}_{IJ}\,\mathcal{P}^J{}_N, \quad \mathcal{M}_{IJ} \text{ takes values in a coset } \frac{O(d,d)}{O(d)\times O(d)}$$

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✓ Self-duality constraint (to go back to conventional system):

$$\mathcal{P}^M = L^{MN}\mathcal{M}_{NP} * \mathcal{P}^P$$

Generators of the Lie algebra T_M given by $2d \times 2d$ matrix projectors $\Pi^M{}_N$, $\widetilde{\Pi}^M{}_N$:

$$\Pi^M{}_N \Pi^N{}_P \;=\; \Pi^M{}_P \,, \quad \Pi^M{}_N \widetilde{\Pi}^N{}_P \;=\; 0 \,, \quad \Pi^M{}_N + \widetilde{\Pi}^M{}_N \;=\; \delta^M{}_N$$

$$\Pi^M{}_N \;\equiv\; \begin{pmatrix} \Pi^m{}_N \\ 0 \end{pmatrix} , \quad \widetilde{\Pi}^M{}_N \;\equiv\; \begin{pmatrix} 0 \\ \widetilde{\Pi}_{mN} \end{pmatrix}$$

$$X^m \;=\; \Pi^m{}_M \, L^{MN} T_N \,, \quad Z_m \;=\; \widetilde{\Pi}_{mM} \, L^{MN} T_N$$

Then doubled coordinates, vielbeins and metric are polarized as

$$Y^I \;\equiv\; \Pi^I{}_J \mathbb{Y}^J \;=\; \begin{pmatrix} Y^i \\ 0 \end{pmatrix} , \quad \widetilde{Y}^I \;\equiv\; \widetilde{\Pi}^I{}_J \mathbb{Y}^J \;=\; \begin{pmatrix} 0 \\ \widetilde{Y}_i \end{pmatrix}$$

$$\mathcal{M}_{IJ} \;=\; \begin{pmatrix} G_{ij} - B_{ik} \, G^{kl} \, B_{lj} & B_{ik} \, G^{kj} \\ -G^{ik} \, B_{kj} & G^{ij} \end{pmatrix}$$

$$\mathcal{P}^M{}_I \;=\; \begin{pmatrix} e^m{}_i & 0 \\ -e_m{}^j B_{ji} & e_m{}^i \end{pmatrix}$$

$$\rho \in O(d,d); \quad \mathbb{Y}^I \ \to \ \mathbb{Y}'^I = \rho^I{}_J \mathbb{Y}^J, \quad \mathcal{P}^M{}_I(\mathbb{Y}) \ \to \ \mathcal{P}'^M{}_I(\mathbb{Y}') = \rho^P{}_Q \, \mathcal{P}^Q{}_J(\mathbb{Y}') \, \rho^J{}_I$$

$$\rho \in O(d,d); \quad \mathbb{Y}^I \to \mathbb{Y}'^I = \rho^I{}_J \mathbb{Y}^J, \quad \mathcal{P}^M{}_I(\mathbb{Y}) \to \mathcal{P}'^M{}_I(\mathbb{Y}') = \rho^P{}_Q \mathcal{P}^Q{}_J(\mathbb{Y}') \, \rho^J{}_I$$

A realization of fractional transformation of $M_{ij} = G_{ij} + B_{ij}$

$$\rho = \begin{pmatrix} A & \beta \\ \Theta & D \end{pmatrix} : \qquad M \to (DM + \Theta)(\beta M + A)^{-1}$$

$$\begin{cases} \Theta : & \text{gauge transformation of B-field } B \to B + \Theta \\ D, A : & \text{diffeomorphism} \\ \beta : & \text{duality transformation with mixing } Y^i \text{ and } \widetilde{Y}_i \end{cases}$$

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$$\rho = \begin{pmatrix} A & \beta \\ \Theta & D \end{pmatrix} : \qquad M \to (DM + \Theta)\,(\beta M + A)^{-1}$$

- Θ : gauge transformation of B-field $B \to B + \Theta$
- D, A : diffeomorphism
- β : duality transformation with mixing Y^i and $\widetilde{Y}_i$

T-duality transformation (ex. $d = 3$ case):

$$\rho_i = \begin{pmatrix} \mathbb{1}_3 - \mathrm{T}_i & \mathrm{T}_i \\ \mathrm{T}_i & \mathbb{1}_3 - \mathrm{T}_i \end{pmatrix} \in O(3,3;\mathbb{Z})$$

$$\mathrm{T}_1 = \begin{pmatrix} 1 & & \\ & 0 & \\ & & 0 \end{pmatrix}, \quad \mathrm{T}_2 = \begin{pmatrix} 0 & & \\ & 1 & \\ & & 0 \end{pmatrix}, \quad \mathrm{T}_3 = \begin{pmatrix} 0 & & \\ & 0 & \\ & & 1 \end{pmatrix}$$

This action exchanges physical coordinates Y^i with dual coordinates $\widetilde{Y}_i$

Example

Start from a flat three-torus T^3 with a three-form flux H given by the following forms:

$$\mathrm{d}s^2 = (\mathrm{d}x)^2 + (\mathrm{d}y)^2 + (\mathrm{d}z)^2 \,, \qquad H = \mathrm{d}B = m \, \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z$$

with a symmetric gauge $B = k\Big(x \, \mathrm{d}y \wedge \mathrm{d}z + y \, \mathrm{d}z \wedge \mathrm{d}x + z \, \mathrm{d}x \wedge \mathrm{d}y \Big) \,, \qquad k = \dfrac{m}{3}$

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Doubled vielbein $\mathcal{P}^M{}_I$

$$\mathcal{P}^M{}_I = \begin{pmatrix} e^m{}_i & \mathbf{0} \\ -e_m{}^j B_{ji} & e_m{}^i \end{pmatrix} = \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 0 & -kz & ky & 1 & 0 & 0 \\ kz & 0 & -kx & 0 & 1 & 0 \\ -ky & kx & 0 & 0 & 0 & 1 \end{array}\right)$$

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Doubled vielbein $\mathcal{P}^M{}_I$ and doubled metric $\mathcal{M}_{IJ} = \mathcal{P}_I{}^M \delta_{MN} \mathcal{P}^N{}_J$ are given as

$$\mathcal{P}^M{}_I = \begin{pmatrix} e^m{}_i & \mathbf{0} \\ -e_m{}^j B_{ji} & e_m{}^i \end{pmatrix} = \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 0 & -kz & ky & 1 & 0 & 0 \\ kz & 0 & -kx & 0 & 1 & 0 \\ -ky & kx & 0 & 0 & 0 & 1 \end{array}\right)$$

$$\mathcal{M}_{IJ} = \left(\begin{array}{ccc|ccc} 1 + k^2 y^2 + k^2 z^2 & -k^2 xy & -k^2 zx & 0 & kz & -ky \\ -k^2 xy & 1 + k^2 x^2 + k^2 z^2 & -k^2 yz & -kz & 0 & kx \\ -k^2 zx & -k^2 yz & 1 + k^2 x^2 + k^2 y^2 & ky & -kx & 0 \\ \hline 0 & -kz & ky & 1 & 0 & 0 \\ kz & 0 & -kx & 0 & 1 & 0 \\ -ky & kx & 0 & 0 & 0 & 1 \end{array}\right)$$

▶ Bianchi identity of doubled vielbein $\mathcal{P}^M = \begin{pmatrix} \mathcal{P}^m \\ \widetilde{\mathcal{P}}_m \end{pmatrix}$ gives a structure constant $t_{MN}{}^P$:

$$\mathrm{d}\widetilde{\mathcal{P}}_1 \;=\; \frac{2m}{3}\,\mathcal{P}^2 \wedge \mathcal{P}^3\,, \qquad \mathrm{d}\widetilde{\mathcal{P}}_2 \;=\; \frac{2m}{3}\,\mathcal{P}^3 \wedge \mathcal{P}^1\,, \qquad \mathrm{d}\widetilde{\mathcal{P}}_3 \;=\; \frac{2m}{3}\,\mathcal{P}^2 \wedge \mathcal{P}^3$$

$$\therefore \qquad \mathrm{d}\mathcal{P}^m \;=\; 0\,, \qquad \mathrm{d}\widetilde{\mathcal{P}}_m \;=\; -\frac{r}{2}\,t_{mnp}\,\mathcal{P}^n \wedge \mathcal{P}^p$$

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$$\therefore \quad \mathrm{d}\mathcal{P}^m = 0 \,, \quad \mathrm{d}\widetilde{\mathcal{P}}_m = -\frac{r}{2} t_{mnp} \mathcal{P}^n \wedge \mathcal{P}^p$$

Then we can read the structure constant $t_{mnp} \equiv h_{mnp}$ of the Lie algebra as

$$[Z_m, Z_n] = h_{mnp} X^p \,, \qquad h_{123} = -\frac{2m}{3r} \equiv -H_{123}$$

We can also fix the scaling factor in the Bianchi identity:

$$r = \frac{2}{3} \,, \qquad \mathrm{d}\mathcal{P}^M = -\frac{1}{3} t_{NP}{}^M \mathcal{P}^N \wedge \mathcal{P}^P$$

- Periodicity of physical coordinates Y^i and dual coordinates $\widetilde{Y}_i$:

$$(x, \widetilde{y}, \widetilde{z}) \sim (x+1, \widetilde{y} + kz, \widetilde{z} - ky) \qquad \widetilde{x} \sim \widetilde{x} + 1$$

$$(y, \widetilde{z}, \widetilde{x}) \sim (y+1, \widetilde{z} + kx, \widetilde{x} - kz) \qquad \widetilde{y} \sim \widetilde{y} + 1$$

$$(z, \widetilde{x}, \widetilde{y}) \sim (z+1, \widetilde{x} + ky, \widetilde{y} - kx) \qquad \widetilde{z} \sim \widetilde{z} + 1$$

This does not change the metric G_{ij} and the B-field B_{ij}.

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$$
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(z, \widetilde{x}, \widetilde{y}) &\sim (z+1, \widetilde{x} + ky, \widetilde{y} - kx) & \widetilde{z} &\sim \widetilde{z} + 1
\end{aligned}
$$

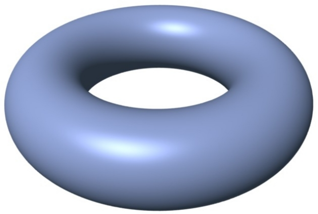
This does not change the metric G_{ij} and the B-field B_{ij}.

- Self-duality constraint: $Y^i = (x, y, z)$, $\widetilde{Y}_i = (\widetilde{x}, \widetilde{y}, \widetilde{z})$ and $\sigma^{\pm} \equiv \sigma^0 \pm \sigma^1$

$$
\partial_{\pm} \widetilde{Y}_i = \big(B_{ij}(Y) \mp \delta_{ij} \big) \, \partial_{\pm} Y^j
$$

We can completely take the projection onto the physical space

$\dashrightarrow$ geometric background



- Doubled vielbein by T-duality along z-direction:

$$(\mathcal{P}_f)^M{}_I = (\rho_z)^M{}_N \mathcal{P}^N{}_J (\rho_z)^J{}_I = \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ -ky & kx & 1 & 0 & 0 & 0 \\ \hline 0 & -k\tilde{z} & 0 & 1 & 0 & ky \\ k\tilde{z} & 0 & 0 & 0 & 1 & -kx \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right)$$

$$(\mathcal{M}_f)_{IJ} = (\rho_z)_I{}^K \mathcal{M}_{KL} (\rho_z)^L{}_J \equiv \begin{pmatrix} G_f - B_f G_f^{-1} B_f & B_f G_f^{-1} \\ -G_f^{-1} B_f & G_f^{-1} \end{pmatrix}$$

"Metric" G_f and "B-field" B_f can be read from the doubled metric as

$$(G_f)_{ij} = \begin{pmatrix} 1 + k^2 y^2 & -k^2 xy & -ky \\ -k^2 xy & 1 + k^2 x^2 & kx \\ -ky & kx & 1 \end{pmatrix}, \quad (B_f)_{ij} = \begin{pmatrix} 0 & k\,\tilde{z} & 0 \\ -k\,\tilde{z} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

▶ Bianchi identity of doubled vielbein $\mathcal{P}^M = \begin{pmatrix} \mathcal{P}^m \\ \widetilde{\mathcal{P}}_m \end{pmatrix}$ gives a structure constant $t_{MN}{}^P$:

$$\mathrm{d}\widetilde{\mathcal{P}}_1 = \frac{2m}{3} \mathcal{P}^2 \wedge \widetilde{\mathcal{P}}_3 \,, \qquad \mathrm{d}\widetilde{\mathcal{P}}_2 = \frac{2m}{3} \widetilde{\mathcal{P}}_3 \wedge \mathcal{P}^1 \,, \qquad \mathrm{d}\mathcal{P}^3 = \frac{2m}{3} \mathcal{P}^1 \wedge \mathcal{P}^2$$

$$\therefore \qquad \mathrm{d}\mathcal{P}^m = -\frac{1}{3} t^m{}_{np} \, \mathcal{P}^n \wedge \mathcal{P}^p \,, \qquad \mathrm{d}\widetilde{\mathcal{P}}_m = -\frac{1}{3} t_{mn}{}^p \, \mathcal{P}^n \wedge \widetilde{\mathcal{P}}_p$$

Then we can read the structure constant $t_{mn}{}^p \equiv f_{mn}{}^p$ as

$$[Z_m, Z_n] = f_{mn}{}^p \, Z_p \,, \qquad [X^m, Z_n] = f^m{}_{np} \, X^p \,, \qquad f^1{}_{23} = -m$$

- Periodicity of physical coordinates Y^i and dual coordinates $\widetilde{Y}_i$:

$$(x, \widetilde{y}, z) \sim (x + 1, \widetilde{y} + kz, z - ky) \qquad \widetilde{x} \sim \widetilde{x} + 1$$

$$(y, z, \widetilde{x}) \sim (y + 1, z + kx, \widetilde{x} - kz) \qquad \widetilde{y} \sim \widetilde{y} + 1$$

$$z \sim z + 1 \qquad (\widetilde{z}, \widetilde{x}, \widetilde{y}) \sim (\widetilde{z} + 1, \widetilde{x} + k\widetilde{y}, \widetilde{y} - k\widetilde{x})$$

$$\mathrm{d}s^2 = \big(\mathrm{d}x\big)^2 + \big(\mathrm{d}y\big)^2 + \big(\mathrm{d}z - ky\,\mathrm{d}x + kx\,\mathrm{d}y\big)^2, \qquad B = k\widetilde{z}\,\mathrm{d}x \wedge \mathrm{d}y$$

The metric is invariant and the B-field is shifted via this periodic shift.

- Periodicity of physical coordinates Y^i and dual coordinates $\widetilde{Y}_i$:

$$(x, \widetilde{y}, z) \sim (x+1, \widetilde{y} + kz, z - ky) \qquad \widetilde{x} \sim \widetilde{x} + 1$$

$$(y, z, \widetilde{x}) \sim (y+1, z + kx, \widetilde{x} - kz) \qquad \widetilde{y} \sim \widetilde{y} + 1$$

$$z \sim z + 1 \qquad (\widetilde{z}, \widetilde{x}, \widetilde{y}) \sim (\widetilde{z} + 1, \widetilde{x} + k\widetilde{y}, \widetilde{y} - k\widetilde{x})$$

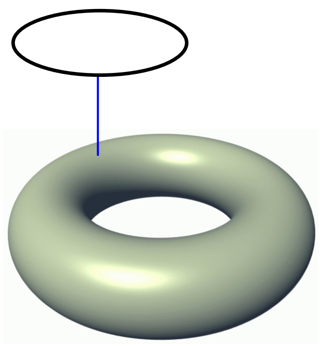
$$\mathrm{d}s^2 = \big(\mathrm{d}x\big)^2 + \big(\mathrm{d}y\big)^2 + \big(\mathrm{d}z - ky\,\mathrm{d}x + kx\,\mathrm{d}y\big)^2 , \qquad B = k\widetilde{z}\,\mathrm{d}x \wedge \mathrm{d}y$$

The metric is invariant and the B-field is shifted via this periodic shift.

- Self-duality constraint is $\partial_{\pm}\widetilde{Y}_i = \big(B_{ij}(\widetilde{Y}) \mp G_{ij}(Y)\big)\,\partial_{\pm}Y^j$

We can completely take the projection on the physical space

$\dashrightarrow$ geometric background



- Doubled vielbein by T-duality along (y, z)-directions:

$$(\mathcal{P}_Q)^M{}_I = (\rho_y \rho_z)^M{}_N \, \mathcal{P}^N{}_J (\rho_z \rho_y)^J{}_I = \left(\begin{array}{ccc:ccc} 1 & 0 & 0 & 0 & 0 & 0 \\ k\widetilde{z} & 1 & 0 & 0 & 0 & -kx \\ -k\widetilde{y} & 0 & 1 & 0 & kx & 0 \\ \hdashline 0 & 0 & 0 & 1 & -k\widetilde{z} & k\widetilde{y} \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right)$$

$$(\mathcal{M}_Q)_{IJ} = (\rho_y \rho_z)_I{}^K \mathcal{M}_{KL} (\rho_z \rho_y)^L{}_J \equiv \begin{pmatrix} G_Q - B_Q G_Q^{-1} B_Q & B_Q G_Q^{-1} \\ -G_Q^{-1} B_Q & G_Q^{-1} \end{pmatrix}$$

The "metric" G_Q and "B-field" B_Q are

$$(G_Q)_{ij} = \frac{1}{1 + k^2 x^2} \begin{pmatrix} 1 + k^2(x^2 + \widetilde{y}^2 + \widetilde{z}^2) & k\widetilde{z} & -k\widetilde{y} \\ k\widetilde{z} & 1 & 0 \\ -k\widetilde{y} & 0 & 1 \end{pmatrix}$$

$$(B_Q)_{ij} = \frac{1}{1 + k^2 x^2} \begin{pmatrix} 0 & -k^2 x\widetilde{y} & -k^2 x\widetilde{z} \\ k^2 x\widetilde{y} & 0 & -kx \\ k^2 x\widetilde{z} & kx & 0 \end{pmatrix}$$

▶ Bianchi identity of doubled vielbein $\mathcal{P}^M = \begin{pmatrix} \mathcal{P}^m \\ \widetilde{\mathcal{P}}_m \end{pmatrix}$ gives a structure constant $t_{MN}{}^P$:

$$\mathrm{d}\widetilde{\mathcal{P}}_1 = \frac{2m}{3} \widetilde{\mathcal{P}}_2 \wedge \widetilde{\mathcal{P}}_3 \,, \qquad \mathrm{d}\mathcal{P}^2 = \frac{2m}{3} \widetilde{\mathcal{P}}_3 \wedge \mathcal{P}^1 \,, \qquad \mathrm{d}\mathcal{P}^3 = \frac{2m}{3} \mathcal{P}^1 \wedge \widetilde{\mathcal{P}}_2$$

$$\therefore \qquad \mathrm{d}\mathcal{P}^m = -\frac{1}{3} t^{mn}{}_p \, \widetilde{\mathcal{P}}_n \wedge \mathcal{P}^p \,, \qquad \mathrm{d}\widetilde{\mathcal{P}}_m = -\frac{1}{3} t_m{}^{np} \, \widetilde{\mathcal{P}}_n \wedge \widetilde{\mathcal{P}}_p$$

Then we can read the structure constant $t^{mn}{}_p \equiv Q^{mn}{}_p$ as

$$[X^m, X^n] = Q^{mn}{}_p X^p \,, \qquad [Z_m, X^n] = Q_m{}^{np} Z_p \,, \qquad Q^{12}{}_3 = -m$$

- Periodicity of physical coordinates Y^i and dual ones $\widetilde{Y}_i$:

$$(x, y, z) \sim (x+1, y + k\widetilde{z}, z - k\widetilde{y}) \qquad \widetilde{x} \sim \widetilde{x} + 1$$

$$y \sim y + 1 \qquad (\widetilde{y}, z, \widetilde{x}) \sim (\widetilde{y}+1, z + kx, \widetilde{x} - k\widetilde{z})$$

$$z \sim z + 1 \qquad (\widetilde{z}, \widetilde{x}, y) \sim (\widetilde{z}+1, \widetilde{x} + k\widetilde{y}, y - kx)$$

$$\mathrm{d}s^2 = (\mathrm{d}x)^2 + \frac{1}{1+k^2x^2}\Big[(\mathrm{d}y + k\widetilde{z}\,\mathrm{d}x)^2 + (\mathrm{d}z - k\widetilde{y}\,\mathrm{d}x)^2\Big]$$

This periodic shift yields a β-trsf: duality trsf. --→ globally nongeometric

- Periodicity of physical coordinates Y^i and dual ones $\widetilde{Y}_i$:

$$(x, y, z) \sim (x+1, y+k\widetilde{z}, z-k\widetilde{y}) \qquad \widetilde{x} \sim \widetilde{x}+1$$

$$y \sim y+1 \qquad (\widetilde{y}, z, \widetilde{x}) \sim (\widetilde{y}+1, z+kx, \widetilde{x}-k\widetilde{z})$$

$$z \sim z+1 \qquad (\widetilde{z}, \widetilde{x}, y) \sim (\widetilde{z}+1, \widetilde{x}+k\widetilde{y}, y-kx)$$

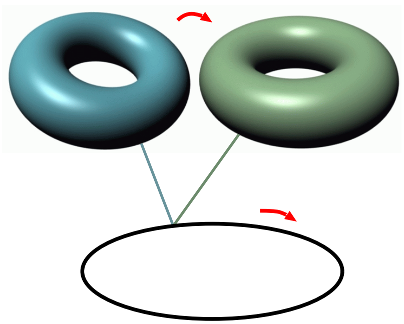
$$\mathrm{d}s^2 = (\mathrm{d}x)^2 + \frac{1}{1+k^2x^2}\Big[(\mathrm{d}y + k\widetilde{z}\,\mathrm{d}x)^2 + (\mathrm{d}z - k\widetilde{y}\,\mathrm{d}x)^2\Big]$$

This periodic shift yields a β-trsf: duality trsf. --→ globally nongeometric

However, imposing the self-duality constraint,

we see that this duality transformation is interpreted as T-duality on fibred T^2

in terms of only the physical coordinate objects --→ locally geometric



- Doubled vielbein by T-duality along (x,y,z)-directions:

$$(\mathcal{P}_R)^M{}_I = (\rho_x\rho_y\rho_z)^M{}_N \mathcal{P}^N{}_J (\rho_z\rho_y\rho_x)^J{}_I = \left(\begin{array}{ccc:ccc} 1 & 0 & 0 & 0 & -k\widetilde{z} & k\widetilde{y} \\ 0 & 1 & 0 & k\widetilde{z} & 0 & -k\widetilde{x} \\ 0 & 0 & 1 & -k\widetilde{y} & k\widetilde{x} & 0 \\ \hdashline 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right)$$

$$(\mathcal{M}_R)_{IJ} = (\rho_x\rho_y\rho_z)_I{}^K \mathcal{M}_{KL} (\rho_z\rho_y\rho_x)^L{}_J \equiv \begin{pmatrix} G_R - B_R G_R^{-1} B_R & B_R G_R^{-1} \\ -G_R^{-1} B_R & G_R^{-1} \end{pmatrix}$$

The "metric" G_R and "B-field" B_R can be read from the doubled metric as

$$(G_R)_{ij} = \frac{1}{1 + k^2\widetilde{x}^2 + k^2\widetilde{y}^2 + k^2\widetilde{z}^2} \begin{pmatrix} 1 + k^2\widetilde{x}^2 & k^2\widetilde{x}\widetilde{y} & k^2\widetilde{z}\widetilde{x} \\ k^2\widetilde{x}\widetilde{y} & 1 + k^2\widetilde{y}^2 & k^2\widetilde{y}\widetilde{z} \\ k^2\widetilde{z}\widetilde{x} & k^2\widetilde{y}\widetilde{z} & 1 + k^2\widetilde{z}^2 \end{pmatrix}$$

$$(B_R)_{ij} = \frac{1}{1 + k^2\widetilde{x}^2 + k^2\widetilde{y}^2 + k^2\widetilde{z}^2} \begin{pmatrix} 0 & -k\widetilde{z} & k\widetilde{y} \\ k\widetilde{z} & 0 & -k\widetilde{x} \\ -k\widetilde{y} & k\widetilde{x} & 0 \end{pmatrix}$$

- Bianchi identity of doubled vielbein $\mathcal{P}^M = \begin{pmatrix} \mathcal{P}^m \\ \widetilde{\mathcal{P}}_m \end{pmatrix}$ gives a structure constant $t_{MN}{}^P$:

$$\mathrm{d}\mathcal{P}^1 \;=\; \frac{2m}{3}\,\widetilde{\mathcal{P}}_2 \wedge \widetilde{\mathcal{P}}_3\,, \qquad \mathrm{d}\mathcal{P}^2 \;=\; \frac{2m}{3}\,\widetilde{\mathcal{P}}_3 \wedge \widetilde{\mathcal{P}}_1\,, \qquad \mathrm{d}\mathcal{P}^3 \;=\; \frac{2m}{3}\,\widetilde{\mathcal{P}}_1 \wedge \widetilde{\mathcal{P}}_2$$

$$\therefore \quad \mathrm{d}\mathcal{P}^m \;=\; -\frac{m}{3}\,t^{mnp}\,\widetilde{\mathcal{P}}_n \wedge \widetilde{\mathcal{P}}_p\,, \qquad \mathrm{d}\widetilde{\mathcal{P}}_m \;=\; 0$$

Then we can read the structure constant $t^{mnp} \equiv R^{mnp}$ as

$$[X^m, X^n] \;=\; R^{mnp}\,Z_p\,, \qquad R^{123} \;=\; -m$$

- Periodicity of physical coordinates Y^i and dual coordinates $\widetilde{Y}_i$:

$$x \sim x + 1 \qquad (\widetilde{x}, y, z) \sim (\widetilde{x} + 1, y + k\widetilde{z}, z - k\widetilde{y})$$

$$y \sim y + 1 \qquad (\widetilde{y}, z, x) \sim (\widetilde{y} + 1, z + k\widetilde{x}, x - k\widetilde{z})$$

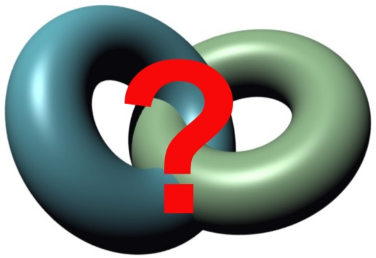
$$z \sim z + 1 \qquad (\widetilde{z}, x, y) \sim (\widetilde{z} + 1, x + k\widetilde{y}, y - k\widetilde{x})$$

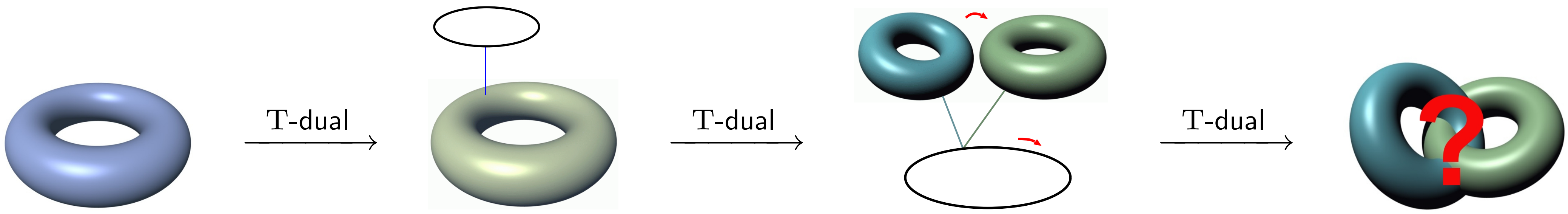
$$\mathrm{d}s^2 = \frac{1}{1 + k^2\widetilde{x}^2 + k^2\widetilde{y}^2 + k^2\widetilde{z}^2}\Big[(\mathrm{d}x)^2 + (\mathrm{d}y)^2 + (\mathrm{d}z)^2 + k^2(\widetilde{x}\,\mathrm{d}x + \widetilde{y}\,\mathrm{d}y + \widetilde{z}\,\mathrm{d}z)^2\Big]$$

This periodic shift yields a β-trsf: duality trsf. --→ globally nongeometric

The self-duality constraint does not yield a well-defined projection

--→ locally nongeometric





h_{mnp}	$f^{m}{}_{np}$	$Q^{mn}{}_{p}$	R^{mnp}
flat-torus	nilmanifold	T-fold	nongeometric
		stringy	stringy

T-duality in the presence of B-field generates geometric/nongeometric backgrounds.

They also have to be investigated as low energy stringy geometries.

Extended formalism can proceed the analysis.

Generalized geometry would also know the existence of Q- and R-fluxes.

M. Graña, J. Louis, D. Waldram hep-th/0612237

Discussions

- Start from scalar moduli matrix in supergravity on $\mathcal{M}_d$
- Introduce doubled space $\mathcal{M}_{2d}$ induced by B-field
- Perform $\mathbb{T}$-duality transformations
- Evaluate Lie algebra and geometries

Introduction of D-branes in the doubled geometry

C. Albertsson's POSTER

Extend to $\mathbb{U}$-fold endowed with $\mathbb{U}$-duality transformation (hidden symmetry)

? Supersymmetry on doubled geometry ?

? Investigate quantum aspects of the doubled sigma model ?

Appendix

Decomposition of fields by Kaluza-Klein compactification on a flat d-torus

$$
\begin{aligned}
\mathrm{d}s^2 &= \mathcal{G}_{\mu\nu}(x,y)\mathrm{d}x^\mu \otimes \mathrm{d}x^\nu + \mathcal{G}_{ij}(x,y)\big(\mathrm{d}y^i + \mathcal{V}^i{}_\mu(x,y)\mathrm{d}x^\mu\big) \otimes \big(\mathrm{d}y^j + \mathcal{V}^j{}_\nu(x,y)\mathrm{d}x^\nu\big) \\
\mathcal{B} &= \frac{1}{2}\mathcal{B}_{\mu\nu}(x,y)\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu + \mathcal{B}_{\mu i}(x,y)\mathrm{d}x^\mu \wedge \mathrm{d}y^i + \frac{1}{2}\mathcal{B}_{ij}(x,y)\mathrm{d}y^i \wedge \mathrm{d}y^j
\end{aligned}
$$

with Ansatz (truncation of massive Kaluza-Klein modes)

$$
\begin{aligned}
\mathcal{G}_{\mu\nu}(x,y) &= g_{\mu\nu}(x)\,, \qquad \mathcal{G}_{ij}(x,y) = g_{ij}(x)\,, \qquad \mathcal{V}^i{}_\mu(x,y) = V^i{}_\mu(x) \\
\mathcal{B}_{\mu\nu}(x,y) &= \mathcal{B}_{\mu\nu}(x)\,, \qquad \mathcal{B}_{\mu i}(x,y) = \mathcal{B}_{\mu i}(x)\,, \qquad \mathcal{B}_{ij}(x,y) = B_{ij}(x)
\end{aligned}
$$

$$
\Phi(x,y) = \phi(x) + \frac{1}{4}\log\big|\det g_{ij}(x)\big|
$$

Reduced degrees of freedom to demonstrate manifest gauge invariance:

$$
\begin{aligned}
B_{\mu i} &= \mathcal{B}_{\mu i} + B_{ij}V^j{}_\mu \\
B_{\mu\nu} &= \mathcal{B}_{\mu\nu} + V^i{}_{[\mu}B_{\nu]i} - B_{ij}V^i{}_\mu V^j{}_\nu
\end{aligned}
$$

$$\mathcal{M}_{IJ} \text{ takes values in a coset } \frac{O(d,d)}{O(d) \times O(d)}$$

This sigma model on the doubled space $\mathcal{M}_{2d}$ has

- $O(d,d)$ global symmetry by $\rho \in O(d,d)$ w/ $\rho^M{}_P L^{PQ} \rho_Q{}^N = L^{MN}$:

$$\begin{aligned}
\mathbb{Y}^I &\to \mathbb{Y}'^I = \rho^I{}_J \mathbb{Y}^J \\
\mathcal{P}^M{}_I(\mathbb{Y}) &\to \mathcal{P}'^M{}_I(\mathbb{Y}') = \rho^P{}_Q \, \mathcal{P}^Q{}_J(\mathbb{Y}') \, \rho^J{}_I \\
\mathcal{M}_{IJ}(\mathbb{Y}) &\to \mathcal{M}'_{IJ}(\mathbb{Y}') = \rho_I{}^K \, \mathcal{M}_{KL}(\mathbb{Y}') \rho^L{}_J
\end{aligned}$$

- $O(d) \times O(d)$ local symmetry: $\mathcal{P}^M{}_I(\mathbb{Y}) \to \mathcal{P}'^M{}_I(\mathbb{Y}) = h^M{}_N(\mathbb{Y}) \, \mathcal{P}^N{}_I(\mathbb{Y})$

Using the worldsheet coordinates σ^a, we see the self-duality constraint as

$$\mathcal{P}^M \;=\; L^{MN}\mathcal{M}_{NP} * \mathcal{P}^P \qquad \longleftrightarrow \qquad \mathrm{d}\mathbb{Y}^I \;=\; L^{IJ}\mathcal{M}_{JK} * \mathrm{d}\mathbb{Y}^K$$

$$\therefore \qquad \Big(\partial_a \mathbb{Y}^I - \sqrt{-\eta}\,\varepsilon_a{}^b\, L^{IJ}\mathcal{M}_{JK}\,\partial_b \mathbb{Y}^K\Big)\,\mathrm{d}\sigma^a \;=\; 0 \qquad \text{w/} \quad \begin{cases} \eta_{ab} = \mathrm{diag.}(+,-) \\ \varepsilon_{01} = 1 = \varepsilon^{10} \end{cases}$$

Taking the polarization, we obtain a set of non-trivial equations:

$$\boxed{(\partial_0 \pm \partial_1)\widetilde{Y}_i \;=\; \Big(B_{ij}(Y,\widetilde{Y}) \mp G_{ij}(Y,\widetilde{Y})\Big)\,(\partial_0 \pm \partial_1)Y^j}$$

Then the dual coordinates $\widetilde{Y}_i$ are related to the physical coordinates Y^i.

Reduction of (co)tangent bundle of doubled space $\mathcal{M}_{2d}$

$$L^{MN} = \langle \mathcal{P}^M, \mathcal{P}^N \rangle = \begin{pmatrix} \mathbf{0}_d & \mathbb{1}_d \\ \mathbb{1}_d & \mathbf{0}_d \end{pmatrix}, \quad L^{IJ} \equiv \langle \mathrm{d}\mathbb{Y}^I, \mathrm{d}\mathbb{Y}^J \rangle = \mathcal{P}^I{}_M \, L^{MN} \, \mathcal{P}_N{}^J$$

This implies $T^*\mathcal{M}_{2d} = T\mathcal{M}_d \oplus T^*\mathcal{M}_d$ s.t.

$$\langle \mathrm{d}Y^i, \mathrm{d}\widetilde{Y}_j \rangle = \delta^i_j \quad \to \quad \mathrm{d}\widetilde{Y}_i = \frac{\partial}{\partial Y^i}$$

$$\mathcal{P}^M = \mathcal{P}^M{}_I \, \mathrm{d}\mathbb{Y}^I = \begin{pmatrix} e^m{}_i \, \mathrm{d}Y^i \\ e_m{}^i \big(\mathrm{d}\widetilde{Y}_i - B_{ij} \, \mathrm{d}Y^j \big) \end{pmatrix} = \begin{pmatrix} e^m{}_i \, \mathrm{d}Y^i \\ e_m{}^i \big(\frac{\partial}{\partial Y^i} - B_{ij} \, \mathrm{d}Y^j \big) \end{pmatrix}$$

a connection to Generalized Geometry

✓ a flat three-torus: *introduction*

$$H = \mathrm{d}B \equiv m\,\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z$$

✓ a nilmanifold given by T-duality along z-direction of the three-torus

$$H = k\,\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}\widetilde{z}\,, \qquad k = \frac{m}{3}$$

✓ a T-fold given by T-duality along (y, z)-directions of the three-torus

$$H = \frac{k(-1 + k^2 x^2)}{(1 + k^2 x^2)^2} \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z - \frac{k^2 x}{1 + k^2 x^2} \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}\widetilde{y} + \frac{k^2 x}{1 + k^2 x^2} \mathrm{d}z \wedge \mathrm{d}x \wedge \mathrm{d}\widetilde{z}$$

✓ a nongeometric space given by T-duality along (x, y, z)-directions of the three-torus

$$\begin{aligned} H = \chi^2 \Big\{ & 2k^2 \widetilde{x}\widetilde{z}\,\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}\widetilde{x} - \big(\chi^{-1} - 2k^2 \widetilde{x}^2\big) \mathrm{d}y \wedge \mathrm{d}z \wedge \mathrm{d}\widetilde{x} + 2k^2 \widetilde{x}\widetilde{y}\,\mathrm{d}z \wedge \mathrm{d}x \wedge \mathrm{d}\widetilde{x} \\ & + 2k^2 \widetilde{y}\widetilde{z}\,\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}\widetilde{y} + 2k^2 \widetilde{x}\widetilde{y}\,\mathrm{d}y \wedge \mathrm{d}z \wedge \mathrm{d}\widetilde{y} - \big(\chi^{-1} - 2k^2 \widetilde{y}^2\big) \mathrm{d}z \wedge \mathrm{d}x \wedge \mathrm{d}\widetilde{y} \\ & - \big(\chi^{-1} - 2k^2 \widetilde{z}^2\big) \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}\widetilde{z} + 2k^2 \widetilde{x}\widetilde{z}\,\mathrm{d}y \wedge \mathrm{d}z \wedge \mathrm{d}\widetilde{z} + 2k^2 \widetilde{y}\widetilde{z}\,\mathrm{d}z \wedge \mathrm{d}x \wedge \mathrm{d}\widetilde{z} \Big\} \end{aligned}$$

$$\chi^{-1} = 1 + k^2 \big(\widetilde{x}^2 + \widetilde{y}^2 + \widetilde{z}^2\big)$$