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Superstring in the plane-wave background with RR-flux as a conformal field theory

Naoto Yokoi

Institute of Physics, University of Tokyo (Komaba)

1. Introduction : PP-wave as a Limit of $\mathbf{AdS_5} \times \mathbf{S^5}$
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Based on the Collaboration with Yoichi Kazama, **JHEP 0803 (2008) 057** (arXiv:0801.1561).

1 Introduction : PP-Wave as a Limit of $\mathrm{AdS}_5 \times \mathrm{S}_5$

- AdS/CFT Correspondence : One of the Most Profound Structures in String Theory

Type IIB Superstring Theory on $\mathbf{AdS_5} \times \mathbf{S^5}$
$\Updownarrow$ Duality
4-Dim. $\mathcal{N}$=4 $SU(N)$ Super Yang-Mills Theory (CFT)

◇ Parameter Correspondence

$$g_{\text{YM}}^2 N = 4\pi g_s N = R^4/\alpha'^2$$

★ Balance Between RR-Flux N and Curvature R $\sim$ Large RR-Flux is Crucial.

- Green-Schwarz Action on $\mathbf{AdS_5} \times \mathbf{S^5}$ with RR-Flux Has Been Constructed.

  Bosonic Part[1] is a Non-Linear Sigma Model on $\dfrac{SO(2,4)}{SO(1,4)} \times \dfrac{SO(6)}{SO(5)}$.

  - Theory is Highly Non-Linear $\Longrightarrow$ Hard to Solve.
  - In General, Theory Becomes “Massive” $\Longrightarrow$ Left and Right Moving Sectors Couple.

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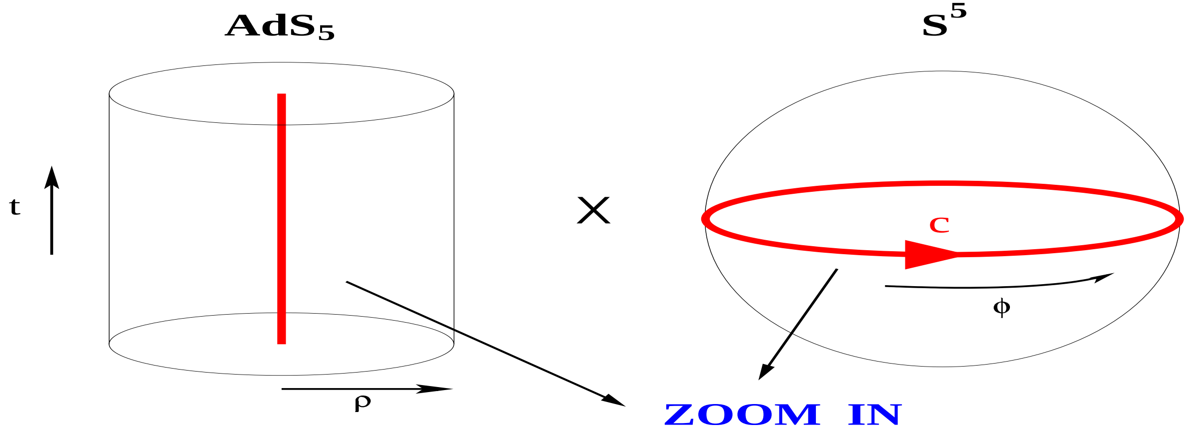
[1] Fermionic part is Much More Complicated.

● PP-wave as a Limit of $\mathbf{AdS_5} \times \mathbf{S^5}$ $\Longleftarrow$ Zoom-In of a Null Geodesics



PP-wave Geometry : $ds^2 = 2\,dx^+dx^- - \mu^2 x_I^2\,dx^{+2} + dx_I^2 \quad (I = 1 \sim 8),$

RR-flux : $F_{+1234} = F_{+5678} = \frac{\mu}{2}$

★ Non-Trivial Curvature and RR-flux is Still There !

The Green-Schwarz Action in Light-Cone Gauge $\Longrightarrow$ Massive Free Field Theory. (Metsaev)

However, ANY String Theory Should Have Massless Conformally Inv. Description.

How Can We Reconcile the “Massive” Picture with Powerful CFT Description ?
$\Longrightarrow$ We Try to Formulate and Quantize the String Theory as EXACT CFT.

2 Classical Analysis of Superstring in the PP-wave

Green-Schwarz Action in a Conformally Inv. Gauge $g_{ij} = \eta_{ij}$:

$$
\begin{aligned}
\mathcal{L}_{\text{GS}} &= \mathcal{L}_{\text{Kin}} + \mathcal{L}_{\text{WZ}}, \\
\mathcal{L}_{\text{Kin}} &= -\frac{T}{2}\eta^{ij}\Big(2\partial_i X^+ \partial_j X^- + \partial_i X^I \partial_j X^I \underbrace{-\mu^2 X^{I2} \partial_i X^+ \partial_j X^+}_{\text{Coupling to Curvature}}\Big) \\
&\quad + i\,T\,\eta^{ij}\Big(\partial_i X^+ \left(\theta^1 \partial_j \theta^1 + \theta^2 \partial_j \theta^2\right) \underbrace{+2\mu \partial_i X^+ \partial_j X^+ \theta^1 \theta^2}_{\text{Coupling to RR-Flux}}\Big), \\
\mathcal{L}_{\text{WZ}} &= -i\sqrt{2}T\epsilon^{ij}\partial_i X^+ \left(\theta^1 \partial_j \theta^1 - \theta^2 \partial_j \theta^2\right),
\end{aligned}
$$

where the Fermionic (Semi) Light-Cone Gauge $\gamma^+ \theta^A = 0 \ (A = 1, 2)$ is Imposed[2].

- Cubic Couplings Exist in the Fermionic Part Even for the Flat Case.
- Non-Trivial Curvature and Background RR-Flux Give Quartic Couplings.

◇ This Becomes FREE MASSIVE Theory in the Full LC Gauge $\partial_0 X^+ \propto p^+$.

[2] To Fix κ-Symmetry. γ^i is $SO(8)$ Gamma Matrices and γ^+ is a Chiral Projection Op. on $SO(8)$ Spinors.

In this Semi-LC Conformal Gauge, Action and Eq. of Motion are Still Non-Linear.

However, We can Exactly Obtain General Solutions for All the Fields.

- For X^+, $\partial_+\partial_- X^+ = 0$: $\mathcal{X}^+(\sigma_+, \sigma_-) = \mathcal{X}^+_L(\sigma_+) + \mathcal{X}^+_R(\sigma_-)$.
- For Transverse X^I, $\partial_+\partial_- X^I + \mu^2 \left(\partial_+ \mathcal{X}^+_L \partial_- \mathcal{X}^+_R\right) X^I = 0$.
  - ◇ General 2π-Periodic Soln. in σ

$$
\begin{aligned}
\mathcal{X}^I(t, \sigma) &= \sum_n (a^I_n u_n + \tilde{a}^I_n \tilde{u}_n)\,, \\
u_n &= e^{-i(\lambda^+_n \mathcal{X}^+_R + \lambda^-_n \mathcal{X}^+_L)}\,, \quad \tilde{u}_n = e^{-i(\lambda^-_n \mathcal{X}^+_R + \lambda^+_n \mathcal{X}^+_L)}\,.
\end{aligned}
$$

Here a^I_n and $\tilde{a}^I_n$ are Constant Coefficients and $\lambda^\pm_n$ are Given by[3]

$$
\lambda^\pm_n = \frac{1}{2\ell_s^2 p^+}(\omega_n \pm n)\,, \quad \omega_n = \frac{n}{|n|}\sqrt{n^2 + M^2} \quad (n \neq 0).
$$

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[3] In Fact, Zero-Mode Parts are Separately Treated, But Similar.

- For Fermionic Fields θ^A, $\partial_+\partial_-\theta^A + \mu^2\left(\partial_+\mathcal{X}_L^+\partial_-\mathcal{X}_R^+\right)\theta^A = 0$.

$$\vartheta^A(t,\sigma) = \sum_n \left(b_n^A u_n + \tilde{b}_n^A \tilde{u}_n\right) \qquad (b_n \text{ and } \tilde{b}_n \text{ are Grassmann Coeff.}).$$

- For X^-, Introducing $(\rho_+,\rho_-) \equiv (\mathcal{X}_L^+(\sigma_+), \mathcal{X}_R^+(\sigma_-))$ and $\tilde{\partial}_\pm \equiv \frac{\partial}{\partial\rho_\pm}$,

$$\tilde{\partial}_+\tilde{\partial}_- X^- = \mu^2\mathcal{X}^I(\tilde{\partial}_+ + \tilde{\partial}_-)\mathcal{X}^I + i\sqrt{2}\mu(\vartheta^1\tilde{\partial}_+\vartheta^2 - \vartheta^2\tilde{\partial}_-\vartheta^1)\,.$$

Since the RHS are Known Fn., This Eq. Can be Solved by the Inverse of the Laplacian.

★ All Soln. of "Physical" Fields are Composed of Mode Fn. u_n and $\tilde{u}_n$.

◇ Mode Fn. u_n and $\tilde{u}_n$ Both Consist of the Product of Left- and Right- Moving Fn.[4]

Note : Completeness Relations for the Mode Fn. u_n and $\tilde{u}_n$ are NOT Known, at Present.

★ The Soln. of Eq. of Motions are Inseparable Functions of σ^+ and σ^-.
How can we Construct PURELY Left (or Right) Moving Virasoro Generator ?

[4]Cf.: For Free Boson, $\mathcal{T}_+ \sim \frac{1}{2}\left(\partial_+\phi_L(\sigma_+)\right)^2$.

● Virasoro Generators $\mathcal{T}_\pm$ in terms of the Classical Solutions.

$$\frac{\mathcal{T}_+}{T} = \frac{1}{2}\partial_+ X^+ \partial_+ X^- + \frac{1}{4}\left(\partial_+ X^I\right)^2 - \frac{i}{\sqrt{2}}\partial_+ X^+\left(\theta^1\partial_+\theta^1 + \theta^2\partial_+\theta^2\right)$$
$$-\frac{1}{4}\left(\partial_+ X^+\right)^2\left(\mu^2 X_I^2 + 4\sqrt{2}\, i\,\mu\,\theta^1\theta^2\right).$$

Using EoM for X^+ and θ^A, with the $\rho_\pm$ variables, $\mathcal{T}_+$ Reduces to

$$\frac{\mathcal{T}_+}{T} = \frac{1}{2}(\partial_+\rho_+)^2\left[\tilde{\partial}_+\mathcal{X}^- + \frac{1}{2}\left((\tilde{\partial}_+\mathcal{X}_I)^2 - \mu^2\mathcal{X}_I^2\right) - i\sqrt{2}(\vartheta^2\tilde{\partial}_+\vartheta^2 - \vartheta^1\tilde{\partial}_+\vartheta^1)\right].$$

Also, the Once-Integrated EoM for X^- Becomes

$$\tilde{\partial}_+\mathcal{X}^- = -\frac{1}{2}\left((\tilde{\partial}_+\mathcal{X}_I)^2 - \mu^2\mathcal{X}_I^2\right) + i\sqrt{2}(\vartheta^2\tilde{\partial}_+\vartheta^2 - \vartheta^1\tilde{\partial}_+\vartheta^1) + f_+(\sigma_+)\,,$$

where $f_+(\sigma_+)$ is an Arbitrary Fn. of σ_+ (“Integration Constant”).

Substituting the Solutions of $\mathcal{X}^-$,

$$\mathcal{T}_+ = \frac{T}{2}(\partial_+\mathcal{X}_L^+)^2 f_+(\sigma_+) \quad \left(\text{Similarly, } \mathcal{T}_- = \frac{T}{2}(\partial_-\mathcal{X}_R^+)^2 f_-(\sigma_-)\right).$$

★ $\mathcal{T}_+$ is a “Chiral” Fn. Only of σ_+ $\Longrightarrow$ Completely Different from Flat ($\mu = 0$) Case.

3 Phase Space Formalism and Quantum Virasoro Algebra

Bosonic Momenta : $P^{+} = T\partial_0 X^{+}, \quad P^{I} = T\partial_0 X^{I},$

$$P^{-} = T\left[\partial_0 X^{-} - \partial_0 X^{+}\left(\mu^2 X_I^2 + 4\sqrt{2}i\mu\theta^1\theta^2\right) - 2\sqrt{2}i\left(\theta^1\partial_+\theta^1 + \theta^2\partial_+\theta^2\right)\right]$$

Fermionic Momenta : $p^1 = i\sqrt{2}T(\partial_0 X^{+} - \partial_1 X^{+})\theta^1 = i\pi^{+1}\theta^1\,,$

$$p^2 = i\sqrt{2}T(\partial_0 X^{+} + \partial_1 X^{+})\theta^2 = i\pi^{+2}\theta^2\,,$$

where $\pi^{+1} \equiv \sqrt{2}(P^{+} - T\partial_1 X^{+})\,, \quad \pi^{+2} \equiv \sqrt{2}(P^{+} + T\partial_1 X^{+})\,.$

● Poisson-Dirac Bracket for Basic Canonical Pairs (at Equal-Time)

$$\{X^I(\sigma),\, P^J(\sigma')\}_D = \delta^{IJ}\delta(\sigma - \sigma'), \quad \{X^{\pm}(\sigma),\, P^{\mp}(\sigma')\}_D = \delta(\sigma - \sigma')$$

$$\{\Theta_a^A(\sigma),\, \Theta_b^B\}_D = i\delta^{AB}\delta_{ab}\delta(\sigma - \sigma'),$$

with the 2nd Class Constraints $d^A \equiv p^A - i\pi^{+A}\theta^A = 0$.

◇ Here, We Have Defined New Canonical Fields $\Theta_a^A \equiv \sqrt{2\pi^{+A}}\,\theta_a^A$ (with $\pi_0^{+A} \neq 0$).

● In Principle, We Can Obtain the Commutators for Modes ($a_n, \tilde{a}_n, b_n, \tilde{b}_n$).

$\Longrightarrow$ We Do NOT Know the Completeness for u_n etc., and Can NOT Obtain the Brackets.

● Phase-Space Fomulation for Canonical Quantization

Usually, Equal-Time Commutator is NOT Sufficient for Solving the Dynamics.

◇ Soln. of EoM $+$ $\underbrace{\text{Brackets for } t\text{-Indep. Modes}}_{\text{Not Obtained Here}}$ $\Longrightarrow$ Correlators at Unequal-Times.

However, String Theory in Conformal Gauge Has a LARGE Symmetry

Including the HAMILTONIAN

$\Downarrow$

★ Representation Theory of the Symmetry Should Also Know the Dynamics !

Physical Spectrum	$\Longleftarrow$	(Gauge) Constraints
Dynamics	$\Longleftarrow$	Construction of Physical Primary Fields

★ In This Strategy, Only the Equal-Time Commutator is Sufficient for the Dynamics.

Introduce Dimensionless Fields $\Phi^\star \equiv \{A, B, S\}$ and a Dimensionless Const. $\hat{\mu}$:

$$X^\star = \frac{1}{\sqrt{2\pi T}} A^\star , \quad P^\star = \sqrt{\frac{T}{2\pi}} B^\star , \quad \widetilde{\Pi}^\star = \frac{1}{\sqrt{2}} (B + \partial_1 A)^\star ,$$

$$\Pi^\star = \frac{1}{\sqrt{2}} (B - \partial_1 A)^\star , \quad \Theta^A = -\frac{i}{\sqrt{2\pi}} S^A , \quad \hat{\mu} = \frac{\mu}{\sqrt{2\pi T}} .$$

Dirac Brackets for the Fourier Modes $\Phi^\star(\sigma) = \sum_n \Phi^\star_n e^{-in\sigma}$ (at $t = 0$)[5] :

$$\{A^\pm_m, B^\mp_n\}_D = \delta_{m+n,0} , \quad \{A^I_m, B^J_n\}_D = \delta^{IJ} \delta_{m+n,0} ,$$

$$\left\{S^A_{a,m}, S^B_{b,n}\right\}_D = -i\, \delta^{AB} \delta_{ab} \delta_{m+n,0} .$$

One Can Obtain the Classical Virasoro Alg. in terms of These Fields :

$$\{\mathcal{T}_\pm(\sigma, t), \mathcal{T}_\pm(\sigma', t)\}_D = \pm 2\mathcal{T}_\pm(\sigma, t)\delta'(\sigma - \sigma') \pm \partial_\sigma \mathcal{T}_\pm(\sigma, t)\delta(\sigma - \sigma'),$$ with

$$\mathcal{T}_+ = \frac{1}{2\pi} \left(\widetilde{\Pi}^+ \widetilde{\Pi}^- + \frac{1}{2} \widetilde{\Pi}^2_I + \frac{i}{2} S^2 \partial_\sigma S^2 + \frac{\hat{\mu}^2}{2} \widetilde{\Pi}^+ \Pi^+ A^2_I - \frac{i\hat{\mu}}{\sqrt{2}} \sqrt{\widetilde{\Pi}^+ \Pi^+} S^1 S^2 \right)$$

[5] Relation between the Fourier Modes and (a_n , b_n) Modes are Quite Complicated.

- Quantization : Replace Poisson-Dirac Brackets with Quantum Commutators at $t=0$.

$$\{A^I_m, B^J_n\}_D \implies [A^I_m, B^J_n] = i\delta^{IJ}\delta_{m+n,0}.$$

Quantum Virasoro Generator $\implies$ Fourier Modes of $\mathcal{T}_+(\sigma) = \sum_n L^+_n e^{-in\sigma}\ (t=0)$.

Quantum Op. L^+_n Requires Ordering $\implies$ ★ Phase-Space Normal Ordering
★ $A^\star_n\ (n\geq 1), B^\star_n\ (n\geq 0), S^A_n\ (n\geq 1)$ as "Annihilation Operators".

Quantum Operator Anomalies Appear through the Calc. (Different from Free CFT)

$$C_B = \frac{1}{(2\pi)^2}\left(\left[\frac{1}{2}\tilde{\Pi}^2_I(\sigma),\ \frac{\hat{\mu}^2}{2}\tilde{\Pi}^+\Pi^+A^2_I(\sigma')\right] - (\sigma\leftrightarrow\sigma')\right)$$

$$C_F = \frac{1}{(2\pi)^2}\left[\frac{i\hat{\mu}}{\sqrt{2}}\sqrt{\tilde{\Pi}^+\Pi^+}S^1S^2(\sigma),\ \frac{i\hat{\mu}}{\sqrt{2}}\sqrt{\tilde{\Pi}^+\Pi^+}S^1S^2(\sigma')\right]$$

$$C_B = -C_F = -\frac{i\hat{\mu}^2}{\pi}\Big(2\tilde{\Pi}^+\Pi^+\delta'(\sigma-\sigma') + \partial_\sigma(\tilde{\Pi}^+\Pi^+)\delta(\sigma-\sigma')\Big).$$

★ These Two Operator Anomalies Exactly Cancel Out ! [6]

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[6] Other (Natural) Orderings Suffer from Operator Anomalies.

4 BRS Quantization and Physical States

BRS Quantization Requires NILPOTENT BRS Charge Q_B

★ Virasoro Generator with Central Charge 26 is Needed.

$\Longrightarrow$ Quantum Correction Term $\Delta\mathcal{T}_+ = -\frac{1}{2\pi}\partial_\sigma^2 \ln\tilde{\Pi}^+$ should be Added to $\mathcal{T}_+$.

From the Virasoro Generator, $Q_B = \sum_n \left(\tilde{c}_{-n}L_n^+ - \frac{1}{2}\sum_m (m-n)\tilde{c}_{-m}\tilde{c}_{-n}\tilde{b}_{m+n}\right)$.

● Physical States as $\tilde{Q}_B$-Cohomology

Decomposition $\tilde{Q}_B = \tilde{Q}_{-1} + \tilde{Q}_0 + \tilde{Q}_{n\geq 1}$ by Light-Cone No. $\tilde{\Pi}_n^\pm \to \pm 1$,

where, $\tilde{Q}_{-1} = -p^+ \sum_n \tilde{c}_n \tilde{\Pi}_{-n}^-$.

One Can Show, in the Same Way as the Free Bosonic String,

Isomorphism : $\tilde{Q}_B$-Cohomology $\simeq$ $\tilde{Q}_{-1}$-Cohomology $\simeq$ $\mathcal{H}_T$ with $L_0^+|\Psi\rangle = 0$.

Here, $\mathcal{H}_T$ is the Transverse Hilbert Space Dropped All the Non-Zero Modes of $(\tilde{b},\ \tilde{c},\ \tilde{\Pi}^\pm)$.

★ Combining Q_B-Cohomology, the Physical States are $|\Psi\rangle \in \mathcal{H}_T$ with the Constraints $H = L_0^+ + L_0^- = 0$ and $P = L_0^+ - L_0^- = 0$.

- Hamiltonian in the Transverse Hilbert Space $\mathcal{H}_T$ w/o Unphysical Non-Zero Modes

$$
\begin{aligned}
H &= H_B + H_F\,, \\
H_B &= \alpha' p^+ p^- + \frac{1}{2}\sum (B^I_{-n}B^I_n + (n^2+M^2)A^I_{-n}A^I_n)\,, \\
H_F &= \frac{1}{2}\sum(-nS^1_{-n}S^1_n + nS^2_{-n}S^2_n - iMS^1_{-n}S^2_n + iMS^2_{-n}S^1_n)\,.
\end{aligned}
$$

◇ This Hamiltonian Describes a Free "Massive" Field Theory.

- Diagonalization of H Leads to the "Physical" Hamiltonian (Re-Normal Ordered) :

$$
\begin{aligned}
H = H^{LC} &= \alpha' p^+ p^- + \alpha_0^{I\,\dagger}\alpha_0^I + \sum_{n\geq 1}(\alpha^I_{-n}\alpha^I_n + \tilde{\alpha}^I_{-n}\tilde{\alpha}^I_n) \\
&\quad + M S_0^\dagger S_0 + \sum_{n\geq 1}\omega_n(S_n^\dagger S_n + \widetilde{S}_n^\dagger \widetilde{S}_n)\,,
\end{aligned}
$$

where

$$
\begin{aligned}
[\tilde{\alpha}_m, \tilde{\alpha}_n] &= [\alpha_m, \alpha_n] = \omega_n \delta_{m+n,0}\,, \quad [\tilde{\alpha}_m, \alpha_n] = 0\,, \\
\left\{\widetilde{S}_{a,m}, \widetilde{S}^\dagger_{b,n}\right\} &= \left\{S_{a,m}, S^\dagger_{b,n}\right\} = \delta_{ab}\delta_{m,n}\,, \quad \left\{S_{a,m}, \widetilde{S}^\dagger_{b,n}\right\} = 0\,.
\end{aligned}
$$

★ This Correctly Reproduces the Light-Cone Hamiltonian Obtained in the LC-Gauge.

Note : $L_0^+ - L_0^- = 0$ Also Gives the Same "Level-Matching" Condition as the LC-Gauge.

5 Summary and Future Problems

- Summary

- We Have Investigated Both the Classical and Quantum Aspects of Superstring Theory in the PP-Wave Background with a Conformally Invariant Gauge as an Exact CFT.
- In Particular, Two Commuting Virasoro Generators are Constructed Quantum Mechanically from the Action with Non-linear Coupling Between Left and Right-Moving Degrees.
- We Have Correctly Reproduced the Light-Cone Gauge Spectrum as the Physical States Defined by the BRS-Cohomology.

- Future Problems

- Analysis of Global Symmetries **:** Realization of the PP-Wave Superalgebra.
- Construction of $(\mathbf{1}, \mathbf{1})$ Primary Fields and Calculation of Correlation Fn.
- Application to the BMN-Correspondence.
- Application to the Phase-Space Formalism to Superstring on $\mathbf{AdS_5} \times \mathbf{S^5}$.
- Modular Invariance, Boundary States and D-Branes, etc.