Anomalies in Exact Renomalization Group

— 場の理論と弦理論 at YITP, Jul. 6, 2009 —

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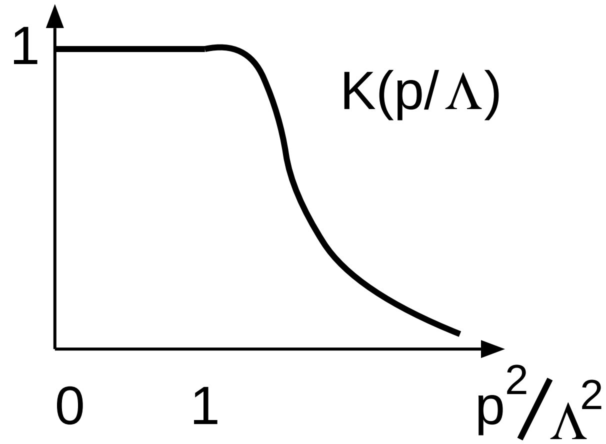
The contents

$$\mathcal{Z}_\phi[J] = \int \mathcal{D}\phi \exp\left(-\mathcal{S}[\phi] - J\cdot\phi\right) \ .$$

- Gradulal integration of higher momentum modes defines a RG flow
- When a symmetry exits, we find an expression, $\Sigma_\Lambda = 0$.
  - $\Sigma_\Lambda = 0$ is the cutoff dependent Ward-Takahashi identity.
- Σ_Λ changes along the flow as a composite operator.
- If anomalous, we find $\Sigma_\Lambda \sim \mathrm{ghost} \times \mathrm{anomaly}$.
  - We study it in a typical example.

The cutoff function



The UV action with the cutoff Λ_0

$$\mathcal{S}[\phi;\Lambda_0] = \frac{1}{2}\phi \cdot K_0^{-1} D \cdot \phi + \mathcal{S}_I[\phi;\Lambda_0]\,.$$

$$\mathcal{Z}_\phi[J] = \int \mathcal{D}\phi \exp\left(-\mathcal{S}[\phi;\Lambda_0] - K_0^{-1} J \cdot \phi\right) \,.$$

$$K_0(p) \equiv K(p/\Lambda_0)$$

$$\phi \cdot D \cdot \phi = \int \frac{d^4p}{(2\pi)^4} \phi^A(-p) D_{AB}(p) \phi^B(p), \qquad J \cdot \phi = \int \frac{d^4p}{(2\pi)^4} J_A(-p) \phi^A(p)$$

Introduce a cutoff $\Lambda(<\Lambda_0)$ with $K(p/\Lambda)$, and decompose ϕ^A into IR fields Φ^A and UV fields χ^A:

$$K_0 D^{-1} = K D^{-1} + (K - K_0) D^{-1}$$

Integration over the UV fields gives

$$\mathcal{Z}_\phi[J] = N_J Z_\Phi[J],$$

$$Z_\Phi[J] \equiv \int \mathcal{D}\Phi \exp\left(-S[\Phi;\Lambda] - K^{-1} J \cdot \Phi\right)$$

The Wilson action with the cutoff Λ $\quad S[\Phi;\Lambda] \equiv \frac{1}{2}\Phi \cdot K^{-1} D \cdot \Phi + S_I[\Phi;\Lambda]$

its interaction part $S_I[\Phi;\Lambda]$

$$\exp\left(-S_I[\Phi;\Lambda]\right) \equiv \int \mathcal{D}\chi \exp\left[-\frac{1}{2}\chi \cdot (K_0 - K)^{-1} D \cdot \chi - \mathcal{S}_I[\Phi + \chi;\Lambda_0]\right] .$$

The normalization factor N_J is given by

$$\ln N_J \quad = \quad -\frac{(-)^{\epsilon_A}}{2} J_A K_0^{-1} K^{-1} (K_0 - K) \left(D^{-1}\right)^{AB} J_B \ .$$

The gradual integration gives a RG flow, or the Polchinski equation

$$\Lambda\frac{\partial}{\partial\Lambda}S = -\int_p \big(K^{-1}\dot{K}\big)(p)\left[\Phi^A(p)\frac{\partial^l S}{\partial\Phi^A(p)}\right]$$
$$+\frac{1}{2}\int_p (-)^{\epsilon_A}\big(\dot{K}D^{-1}(p)\big)^{AB}\left[\frac{\partial^l S}{\partial\Phi^B(-p)}\frac{\partial^r S}{\partial\Phi^A(p)} - \frac{\partial^l\partial^r S}{\partial\Phi^B(-p)\partial\Phi^A(p)}\right]$$

with the initial condition

$$S[\Phi;\Lambda=\Lambda_0] = \mathcal{S}[\Phi;\Lambda_0]$$

The functional integration and solving the Polchinski equation are equivallent; two different languages to describe the same.

The composite operator is a useful notion

Equivallent definitions for the composite operator $\mathcal{O}[\Phi;\Lambda]$

1. Via the linearized Polchinski equation, with an initial condition at Λ_0

$$\Lambda\frac{\partial}{\partial\Lambda}\mathcal{O}[\Phi;\Lambda] = -\mathcal{D}\ \mathcal{O}[\Phi;\Lambda]$$

$$\mathcal{D} \equiv \int_p \left[\left(K^{-1}\dot{K}\right)\Phi^A\frac{\partial^l}{\partial\Phi^A} + (-)^{\epsilon_A}\left(\dot{K}D^{-1}\right)^{AB}\left(\frac{\partial^l S}{\partial\Phi^B}\frac{\partial^r}{\partial\Phi^A} - \frac{1}{2}\frac{\partial^l\partial^r}{\partial\Phi^B\partial\Phi^A}\right)\right]\mathcal{O}$$

2. Given an operator $\mathcal{O}[\phi;\Lambda_0]$ at the UV scale Λ_0, the corresponding IR composite operator $\mathcal{O}[\Phi;\Lambda]$ may be constructed as

$$\mathcal{O}[\Phi;\Lambda]\ e^{-S_I[\Phi;\Lambda]} \equiv \int \mathcal{D}\chi\,\mathcal{O}[\Phi+\chi;\Lambda_0]\ e^{-\frac{1}{2}\chi\cdot(K_0-K)^{-1}D\cdot\chi-\mathcal{S}_I[\Phi+\chi,\phi^*;\Lambda_0]}$$

3. The expectation values in the presence of arbitrary sources satisfy

$$\langle\mathcal{O}[\Phi;\Lambda]\rangle_{\Phi,K^{-1}J} = N_J^{-1}\,\langle\mathcal{O}[\phi;\Lambda_0]\rangle_{\phi,K_0^{-1}J}$$

Consider <u>some transformation</u>

$$\phi^A \to \phi'^A = \phi^A + \delta_\lambda \phi^A \ , \qquad \delta_\lambda \phi^A = \delta\phi^A \lambda = K_0 \mathcal{R}^A[\phi; \Lambda_0] \ \lambda \ .$$

$$\int \mathcal{D}\phi \ \left(K_0^{-1} J \cdot \delta\phi + \Sigma[\phi; \Lambda_0] \right) \ \exp\left(-\mathcal{S}[\phi; \Lambda_0] - K_0^{-1} J \cdot \phi \right) = 0$$

where the quantity $\Sigma[\phi; \Lambda_0]$ is given as

$$\Sigma[\phi; \Lambda_0] \equiv \frac{\partial^r \mathcal{S}}{\partial \phi^A} \delta\phi^A - \frac{\partial^r}{\partial \phi^A} \delta\phi^A \ .$$

$\Sigma[\phi, \Lambda_0]$ is the sum of the change of the original gauge fixed action $\mathcal{S}[\phi; \Lambda_0]$

$$\delta_\lambda \mathcal{S} = \frac{\partial^r \mathcal{S}}{\partial \phi^A} \delta_\lambda \phi^A \, ,$$

and that of the functional measure $\mathcal{D}\phi$

$$\delta_\lambda \ln \mathcal{D}\phi = (-)^{\epsilon_A} \frac{\partial^r}{\partial \phi^A} \delta_\lambda \phi^A = \frac{\partial^r}{\partial \phi^A} \delta\phi^A \lambda \, .$$

- $\Sigma[\phi;\Lambda_0] = 0$ implies that the UV action $\mathcal{S}[\phi;\Lambda_0]$ is invariant under $\delta\phi$

- Appropriate to call $\Sigma[\phi;\Lambda_0]$ as the WT operator

Let us see how the transformation and the WT operator changes as the scale changes.

We have the relation

$$\begin{aligned}
\langle \Sigma[\phi;\Lambda_0] \rangle_{\phi,\ K_0^{-1}J} &= -K_0^{-1}J \cdot \langle \delta\phi \rangle_{\phi,\ K_0^{-1}J} \\
&= -J \cdot \langle \mathcal{R}[\phi;\Lambda_0] \rangle_{\phi,\ K_0^{-1}J} \\
&= -J \cdot \mathcal{R}[K_0 \partial_J^l;\Lambda_0]\ \mathcal{Z}_\phi[J]
\end{aligned}$$

Using the relation $\mathcal{Z}_\phi[J] = N_J Z_\Phi[J]$, we may rewrite the above for $Z_\Phi[J]$.
Then, we expect to find a similar relation for the lower scale Λ.

- Important to note that ∂_J acts on N_J

To find $\delta\Phi$ and Σ at the scale Λ, use (cf. the definition of composite operator)

$$\begin{aligned}\langle K^{-1}\delta\Phi^A\rangle_{\Phi,K^{-1}J} &= N_J^{-1}\langle K_0^{-1}\delta\phi^A\rangle_{\phi,K_0^{-1}J} \\ \langle\Sigma[\Phi;\Lambda]\rangle_{\Phi,K^{-1}J} &= N_J^{-1}\langle\Sigma[\phi;\Lambda_0]\rangle_{\phi,K_0^{-1}J}\end{aligned}$$

As for the transformation $K_0^{-1}\delta\phi = \mathcal{R}[\phi;\Lambda_0]$

$$N_J^{-1}\langle K_0^{-1}\delta\phi^A\rangle_{\phi,K_0^{-1}J} = N_J^{-1}\mathcal{R}^A[K_0\partial_J^l;\Lambda_0]\mathcal{Z}_\phi[J] = N_J^{-1}\mathcal{R}^A[K_0\partial_J^l;\Lambda_0]N_J Z_\Phi[J]$$

Note here ∂_J^l acts on N_j, that produces the scale change of the transformation.

Equating the above expression with the following

$$\langle K^{-1}\delta\Phi^A\rangle_{\Phi,K^{-1}J} = N_J^{-1}\langle K_0^{-1}\delta\phi^A\rangle_{\phi,K_0^{-1}J} = R^A[K\partial_J^l;\Lambda]Z_\Phi[J]$$

we find the transformation of the IR fields

$$\delta\Phi^A[\Phi;\Lambda] = KR^A[\Phi;\Lambda]$$

For the WT operator, we find

$$\langle \Sigma[\Phi;\Lambda] \rangle_{\Phi, K^{-1}J} = -J \cdot R[K\partial_J^l;\Lambda] Z_\Phi[J]$$

Therefore

$$\Sigma[\Phi;\Lambda] = \frac{\partial^r S[\Phi;\Lambda]}{\partial \Phi^A} \delta\Phi^A - \frac{\partial^r}{\partial \Phi^A} \delta\Phi^A$$

The relation

$$\langle \Sigma[\Phi;\Lambda] \rangle_{\Phi, K^{-1}J} = N_J^{-1} \langle \Sigma[\phi;\Lambda_0] \rangle_{\phi, K_0^{-1}J}$$

implies that if the WT operator vanishes at the scale Λ_0, it does at any lower scale.

A different way to observe the same property: the WT operator flows as a composite operator

$$\Lambda \frac{\partial}{\partial \Lambda} \Sigma[\Phi;\Lambda] = -\mathcal{D}\ \Sigma[\Phi;\Lambda]$$

Where to find an anomaly?

- The vanishing of the WT operator implies symmetry: $\Sigma \neq 0$ for an anomalous theory.
- The WT operator Σ evolves as a composite operator.
  Σ may contain the information of anomaly, that also follows the flow equation.
- The well-known form of anomay may be observed in the limit of $\Lambda \to \infty$.

We will see in an example

$$\Sigma[\Phi; \Lambda] \to \mathrm{ghost} \times \mathrm{anomaly}$$

as $\Lambda \to \infty$

An example

$U(1)_V \times U(1)_A$ gauge theory: WT identities and BRS transformations

two sets of gauge sector: $(A_\mu, h_V, c_V, \bar{c}_V)$ and $(B_\mu, h_A, c_A, \bar{c}_A)$

$$\mathcal{S}[\phi;\Lambda_0] = \frac{1}{2}\phi K_0^{-1} \cdot D \cdot \phi + \mathcal{S}_I[\phi;\Lambda_0]$$

$$\begin{aligned}
\frac{1}{2}\phi K_0^{-1} \cdot D \cdot \phi &= \int_p K_0^{-1}\Big[\bar{\psi}(-p)\not{p}\psi(p) \\
&+ \frac{1}{2}A_\mu(p^2\delta_{\mu\nu} - p_\mu p_\nu)A_\nu - h_V\Big(ip\cdot A + \frac{\xi_V}{2}h_V\Big) + \bar{c}_V ip^2 c_V \\
&+ \frac{1}{2}B_\mu(p^2\delta_{\mu\nu} - p_\mu p_\nu)B_\nu - h_A\Big(ip\cdot B + \frac{\xi_A}{2}h_A\Big) + \bar{c}_A ip^2 c_A\Big]
\end{aligned}$$

The BRS transformation for axial gauge symmetry

$$\begin{aligned}
\delta_A B_\mu(p) &= -iK_0(p)p_\mu c_A(p), \quad \delta_A \bar{c}_A = iK_0(p)h_A(p), \quad \delta_A c_A(p) = \delta_A h_A(p) = 0 \\
\delta_A \psi(p) &= -ie_A K_0(p) \int_k \gamma_5 \psi(p-k) c_V(k), \\
\delta_A \bar{\psi}(-p) &= -ie_A K_0(p) \int_k \bar{\psi}(-p-k) c_A(k) \gamma_5 \,, \\
\delta_A A_\mu(p) &= \delta_A h_V(p) = \delta_A c_V(p) = \delta_A \bar{c}_V(p) = 0
\end{aligned}$$

Similar transformation for vector gauge symmetry.

With the following part of the interaction aciton

$$\mathcal{S}_I[\phi; \Lambda_0] \sim -\int_{p,k} \bar{\psi}(-p-k)\Big(e_V \not{A}(k) + e_A \not{B}(k)\gamma_5\Big)\psi(p)$$

we find the contribution of order e^2 relevant for anomalies

$$
\begin{aligned}
S_I[\Phi;\Lambda] \quad = \quad & -\frac{e_V^2}{2}\int_{l,k,q}\ \bar{\psi}(-l-k)\Bigg[\not{A}(k)\frac{(K_0(l)-K(l))}{\not{l}}\not{A}(q) \\
& +\not{A}(q)\frac{(K_0(l+k-q)-K(l+k-q))}{\not{l}+\not{k}-\not{q}}\not{A}(k)\Bigg]\psi(l-q) \\
& -\frac{e_A^2}{2}\int_{l,k,q}\ \bar{\psi}(-l-k)\Bigg[\not{B}(k)\frac{(K_0(l)-K(l))}{\not{l}}\not{B}(q) \\
& +\not{B}(q)\frac{(K_0(l+k-q)-K(l+k-q))}{\not{l}+\not{k}-\not{q}}\not{B}(k)\Bigg]\psi(l-q) \\
& -\frac{e_V e_A}{2}\int_{l,k,q}\ \bar{\psi}(-l-k)\Bigg[\not{B}(k)\gamma_5\frac{(K_0(l)-K(l))}{\not{l}}\not{A}(q) \\
& +\not{A}(q)\frac{(K_0(l+q-k)-K(l+k-q))}{\not{l}+\not{k}-\not{q}}\not{B}(k)\gamma_5 \\
& +\not{A}(k)\frac{(K_0(l)-K(l))}{\not{l}}\not{B}(q)\gamma_5 \\
& +\not{B}(q)\gamma_5\frac{(K_0(l+q-k)-K(l+k-q))}{\not{l}+\not{k}-\not{q}}\not{A}(k)\Bigg]\psi(l-q)
\end{aligned}
$$

Calculate $\Sigma_A[\Phi;\Lambda]$ in the limit of $\Lambda_0 \to \infty$

$$
\begin{aligned}
\Sigma_A \quad = \quad & \Bigg(-ie_A e_V^2 \int_{p,q,k} c_A(-q-k) \frac{K(p)(1-K(p+q+k))}{(p+q+k)^2} \\
& \times Tr\left[\gamma_5 (\not{p}+\not{q}+\not{k}) \not{A}(k) \frac{(1-K(p+q))(\not{p}+\not{q})}{(p+q)^2} \not{A}(q) \right] \\
& + ie_A e_V^2 \int_{p,q,k} c_A(-q-k) \frac{K(p)(1-K(p-q-k))}{(p-q-k)^2} \\
& \times Tr\left[\gamma_5 (\not{p}-\not{q}-\not{k}) \not{A}(k) \frac{(1-K(p-k))(\not{p}-\not{k})}{(p-k)^2} \not{A}(q) \right] \Bigg) \\
& + \left(e_V \to e_A, \quad \not{A} \to \not{B} \right)
\end{aligned}
$$

Similar expression for $\Sigma_V[\Phi;\Lambda]$

The results in the limits of $\Lambda_0, \Lambda \to \infty$ are summarized as

$$\begin{aligned}
\Sigma_A &= i\frac{e^2}{48\pi^2}\int_x c_A(x)\ \epsilon_{\mu\nu\rho\sigma}\left(F^V_{\mu\nu}(x)F^V_{\rho\sigma}(x) + F^A_{\mu\nu}(x)F^A_{\rho\sigma}(x)\right) \\
\Sigma_V &= -i2 \times \frac{e_A e_V^2}{12\pi^2}\epsilon_{\mu\nu\rho\sigma}\int_{q,k}\ c_V(-q-k)k_\mu B_\nu(k) q_\rho A_\sigma(q)
\end{aligned}$$

Add the following counter term to the Wilson action

$$\begin{aligned}
S_I &\to S_I + S_c \\
S_c &= a\ \epsilon_{\mu\nu\rho\sigma}\int_{q,k}\ B_\mu(k)A_\nu(-k-q)q_\rho A_\sigma(q)
\end{aligned}$$

where a ia a constant to be determined below.

Choose the parameter a as

$$a = -\frac{e_A e_V^2}{6\pi^2}$$

so that the vector gauge symmetry is preserved $\Sigma_V = 0$, while, the anomaly in the axial gauge symmetry is shifted to

$$\begin{aligned}\Sigma_A &= -i\frac{e_A e_V^2}{4\pi^2}\epsilon_{\mu\nu\rho\sigma}\int_{q,k} c_A(-q-k)k_\mu A_\nu(k)q_\rho A_\sigma(q) \\ &\quad -i\frac{e_A^3}{12\pi^2}\epsilon_{\mu\nu\rho\sigma}\int_{q,k} c_A(-q-k)k_\mu B_\nu(k)q_\rho B_\sigma(q)\end{aligned}$$

Summary

$$\mathcal{Z}_\phi[J] = \int \mathcal{D}\phi \exp\left(-\mathcal{S}[\phi] - J \cdot \phi\right) \ .$$

- Gradulal integration of higher momentum modes defines a RG flow
- When a symmetry exits, we find an expression, $\Sigma_\Lambda = 0$.
  - For a gauge symmetry, $\Sigma_\Lambda = 0$ is the Ward-Takahashi identity (cutoff dependent).
- Σ_Λ changes along the flow as a composite operator.
- For anomalous symmetry, we have seen $\Sigma_\Lambda \sim \mathrm{ghost} \times \mathrm{anomaly}$ in the limit of $\Lambda \to \infty$.