トポロジカル絶縁体・超伝導体

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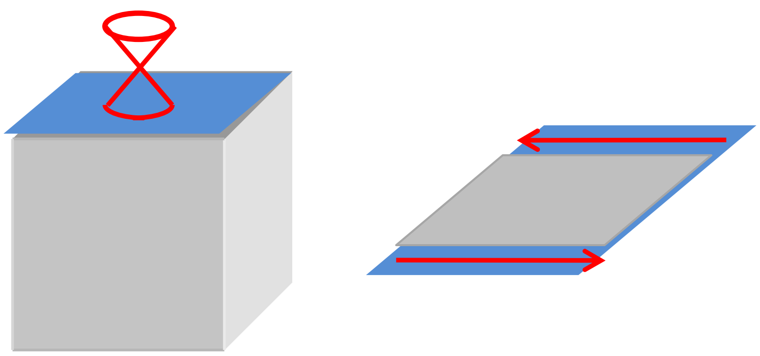


July 9, 2009

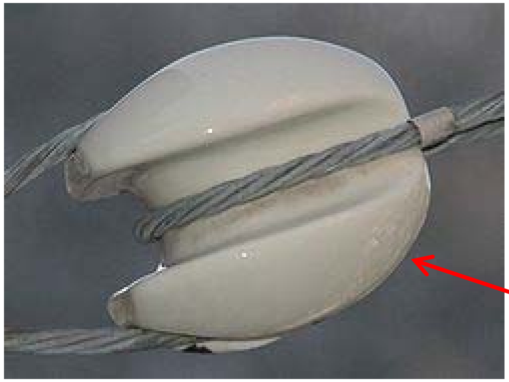
topological insulators (3d and 2d)

Outline

- イントロダクション
  バンド絶縁体, トポロジカル絶縁体･超伝導体
- トポロジカル絶縁体･超伝導体の例:
  整数量子ホール系
  p+ip 超伝導体
  Z_2 トポロジカル絶縁体: 2d & 3d
- トポロジカル絶縁体･超伝導体の分類

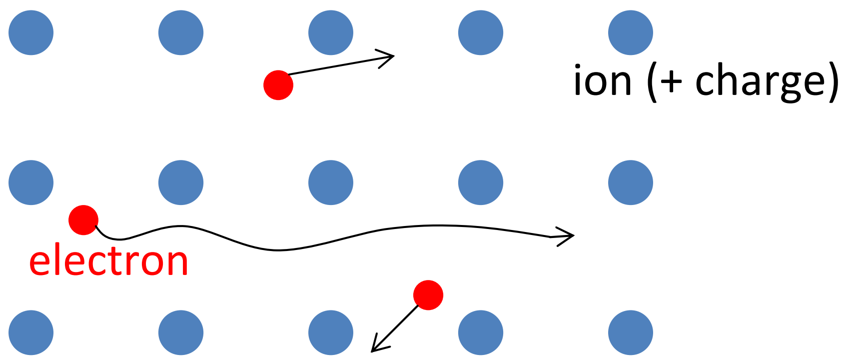
絶縁体　Insulator

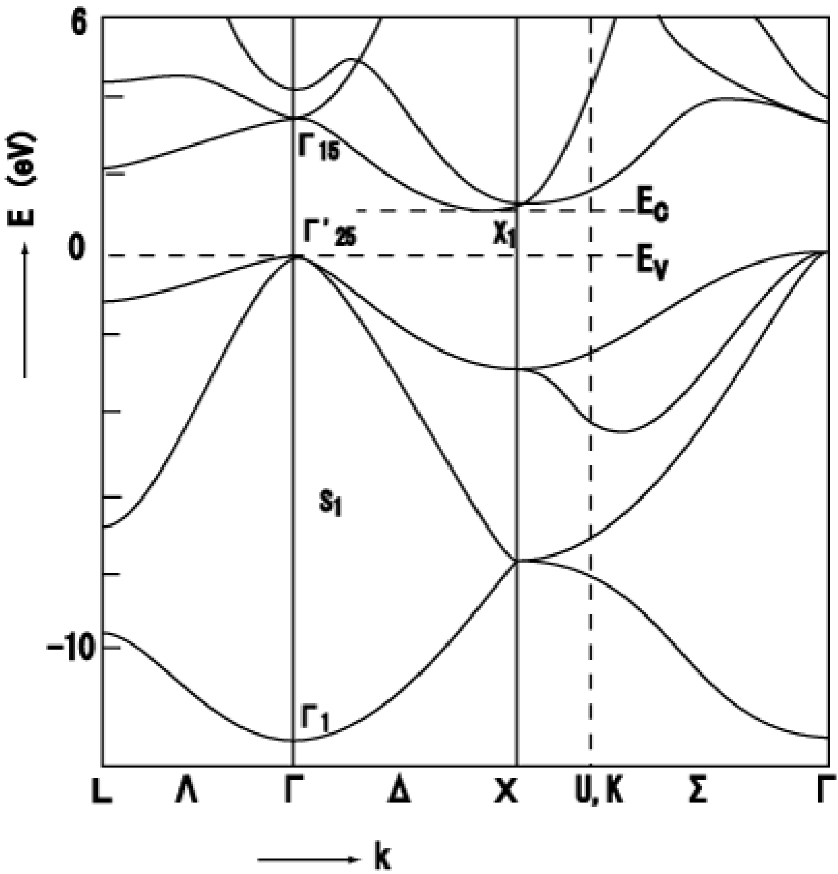
- 電気を流さない物質



絶縁体

Band theory of electrons in solids





- Schroedinger equation

$$\left[-\frac{\hbar^2}{2m}\nabla^2 + V(r)\right]\psi(r) = E\psi(r)$$

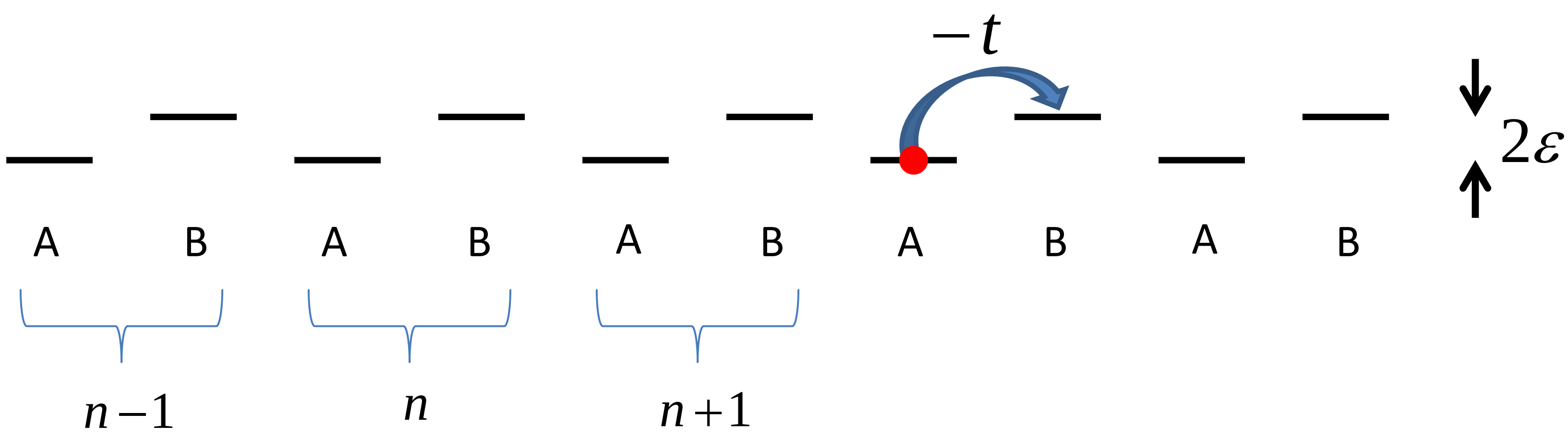
Bloch's theorem:

$$\psi(r) = e^{ikr}u_{n,k}(r), \quad -\frac{\pi}{a} < k < \frac{\pi}{a}, \quad u_{n,k}(r+a) = u_{n,k}(r)$$

$E_n(k)$ Energy band dispersion

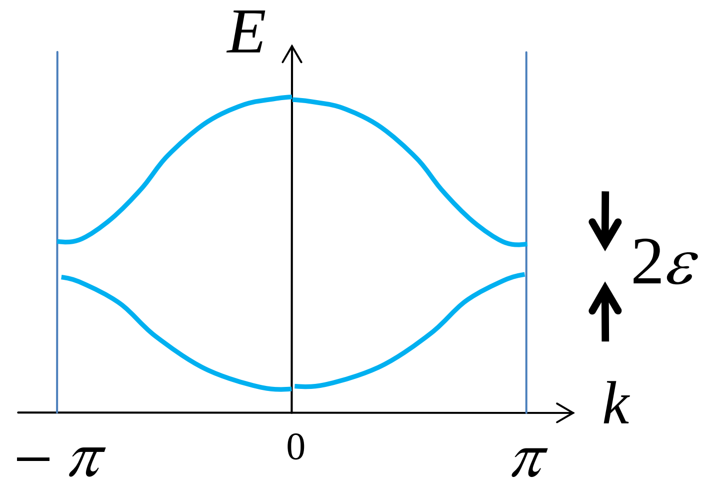
$n:$ band index

例：一次元格子模型

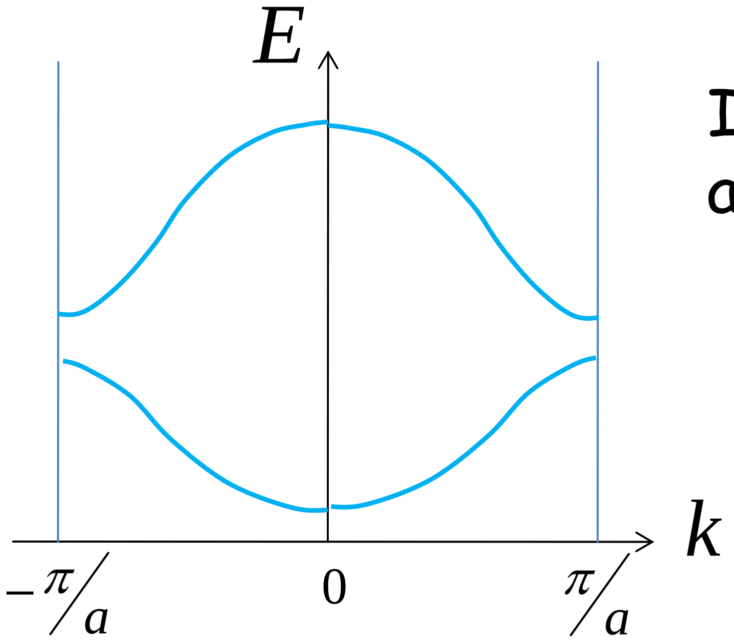


$$\psi(n) = e^{ikn}\begin{pmatrix} u_A \\ u_B \end{pmatrix} \qquad \begin{pmatrix} -\varepsilon & -2t\cos(k/2) \\ -2t\cos(k/2) & +\varepsilon \end{pmatrix}\begin{pmatrix} u_A \\ u_B \end{pmatrix} = E\begin{pmatrix} u_A \\ u_B \end{pmatrix}$$

$$E = \pm\sqrt{4t^2\cos^2(k/2) + \varepsilon^2}$$



Metal and insulator in the band theory

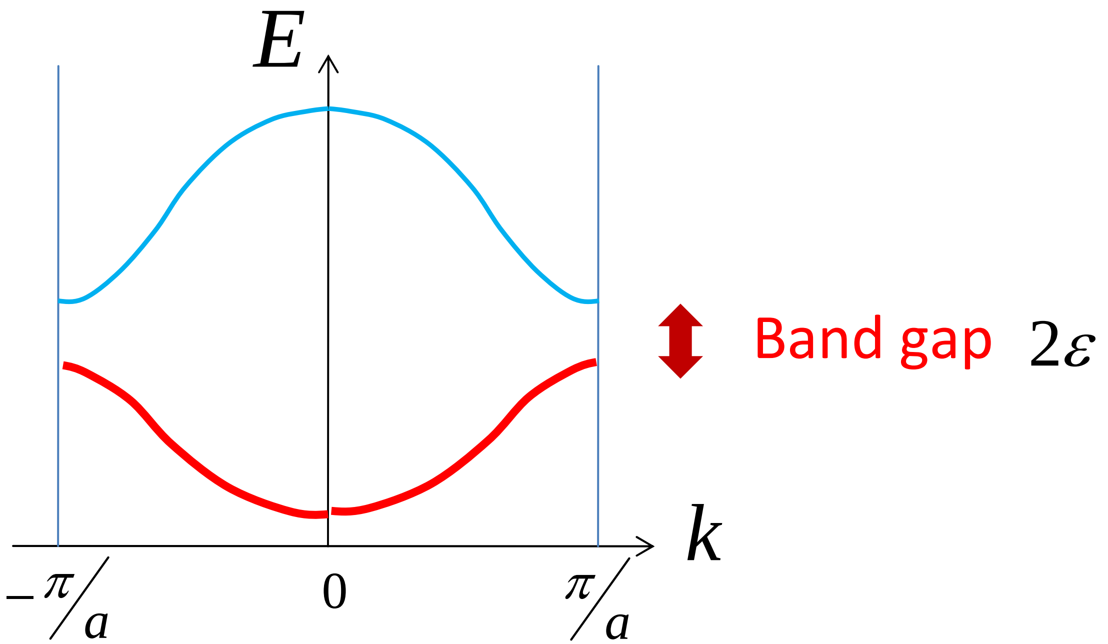


Interactions between electrons are ignored. (free fermion)

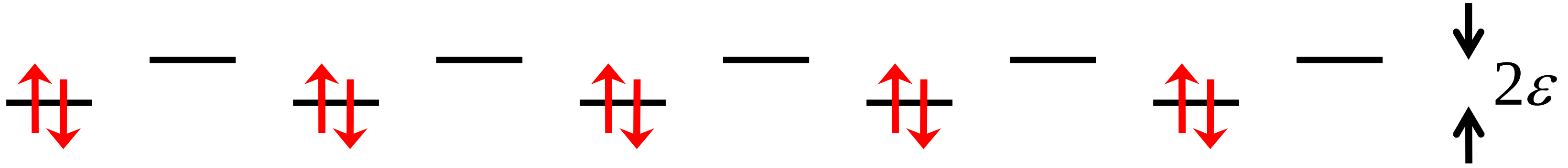
Each state (n,k) can accommodate up to two electrons (up, down spins). ↑↓

Pauli principle

Band Insulator



All the states in the lower band are completely filled.
(2 electrons per unit cell)



Electric current does not flow under (weak) electric field.

Topological (band) insulators

- バンド絶縁体
- トポロジカル数をもつ
- 端にgapless励起(Dirac fermion)をもつ

stable

Condensed-matter realization of domain wall fermions

Examples: integer quantum Hall effect,
quantum spin Hall effect, Z_2 topological insulator, ….

Topological superconductors

- BCS超伝導体
- トポロジカル数をもつ
- 端にgapless励起 (Dirac or Majorana) をもつ

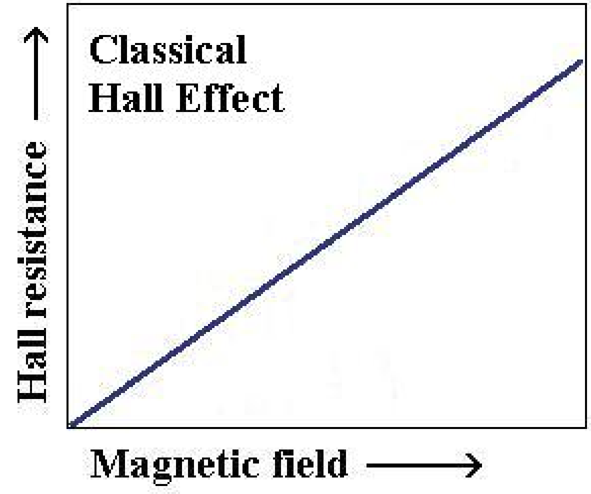
stable

Condensed-matter realization of domain wall fermions

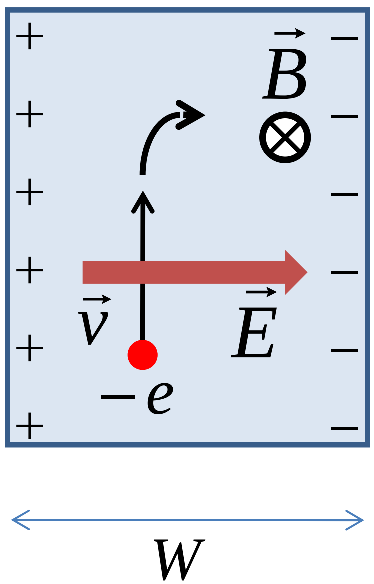
Examples: integer quantum Hall effect, p+ip superconductor, ^{3}He
quantum spin Hall effect, Z_2 topological insulator, ...

Example 1: Integer QHE

Prominent example: quantum



- Classical Hall effect



n : electron density

Electric current $I = -nevW$

Electric field $E = \frac{v}{c} B$

Hall voltage $V_H = EW = \frac{B}{-ne} I$

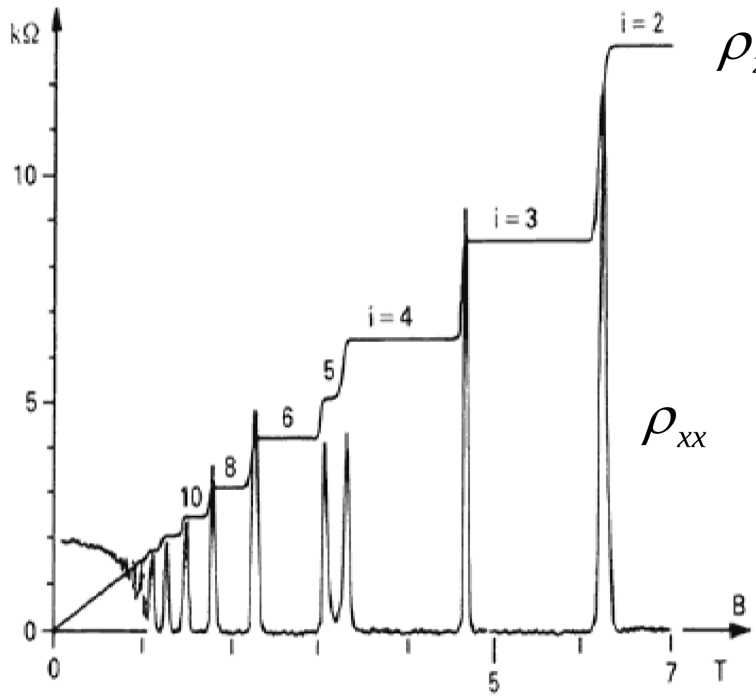
Hall resistance $R_H = \frac{B}{-ne}$

Hall conductance $\sigma_{xy} = \frac{1}{R_H}$

Lorentz force

$\vec{F} = -e\vec{v} \times \vec{B}$

Integer quantum Hall effect (von Klitzing 1980)



$\rho_{xy} = \rho_H$

ρ_{xx}

$$\frac{h}{e^2} = 25812.807\Omega$$

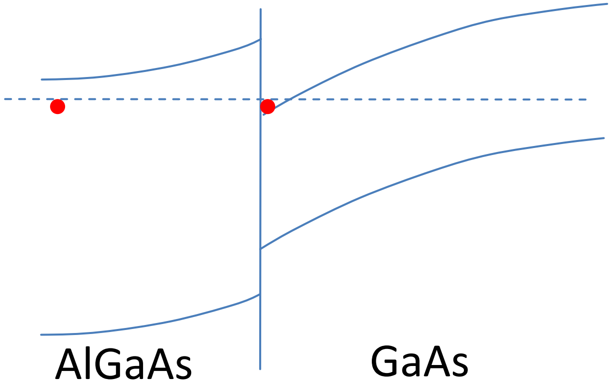
Quantization of Hall conductance

$$\sigma_{xy} = i\frac{e^2}{h}$$

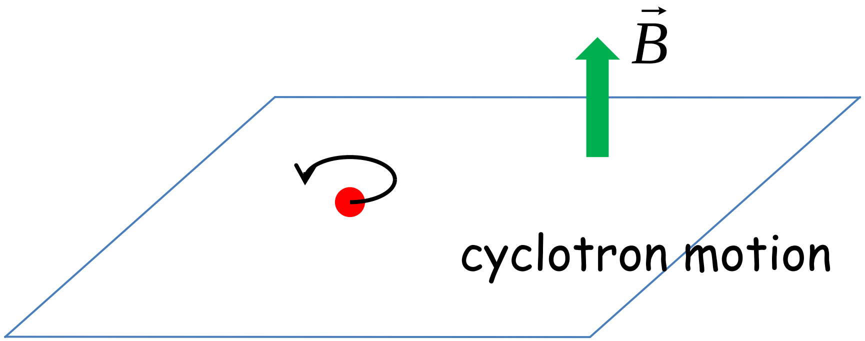
exact, robust against disorder etc.

Integer quantum Hall effect

- Electrons are confined in a two-dimensional plane. (ex. AlGaAs/GaAs interface)

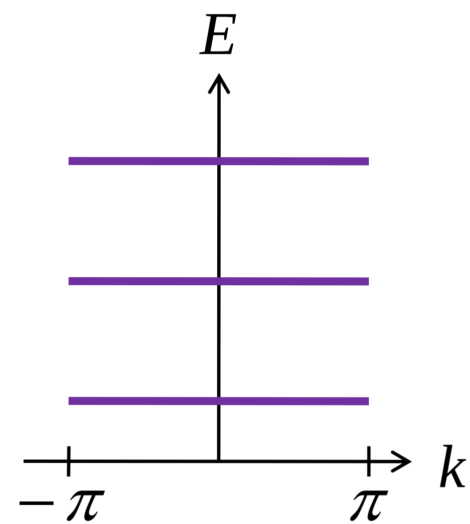


- Strong magnetic field is applied (perpendicular to the plane)



Landau levels:

$$E_n = \hbar\omega_c\left(n+\tfrac{1}{2}\right), \quad \omega_c = \frac{eB}{mc}, \quad n = 0,1,2,\ldots$$



TKNN number (Thouless-Kohmoto-Nightingale-den Nijs)

TKNN (1982); Kohmoto (1985)

$$\sigma_{xy} = -\frac{e^2}{h} C$$

Chern number (topological invariant)

$$\psi = e^{i\vec{k}\cdot\vec{r}} u_{\vec{k}}(\vec{r})$$

$$C = \frac{1}{2\pi i} \int_{\text{filled band}} d^2k \int d^2r \left(\frac{\partial u^*}{\partial k_y} \frac{\partial u}{\partial k_x} - \frac{\partial u^*}{\partial k_x} \frac{\partial u}{\partial k_y} \right)$$

integer valued

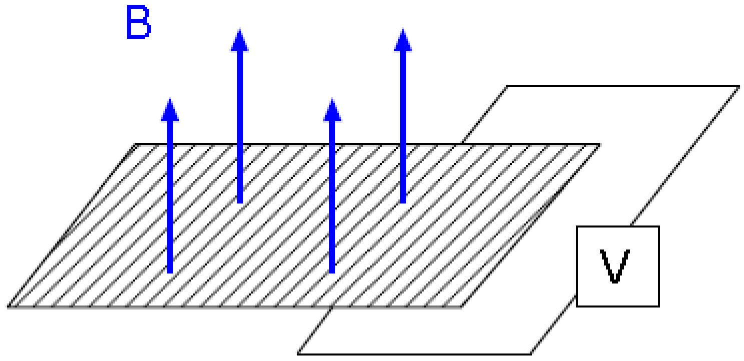
$$= \frac{1}{2\pi i} \int d^2k \, \vec{\nabla}_k \times \vec{A}(k_x, k_y)$$

$$\vec{A}(k_x, k_y) = \left\langle u_{\vec{k}} \middle| \vec{\nabla}_k \middle| u_{\vec{k}} \right\rangle$$

Topological insulators:
band insulators characterized by a topological number

Best known example:

IQHE in 2DEG

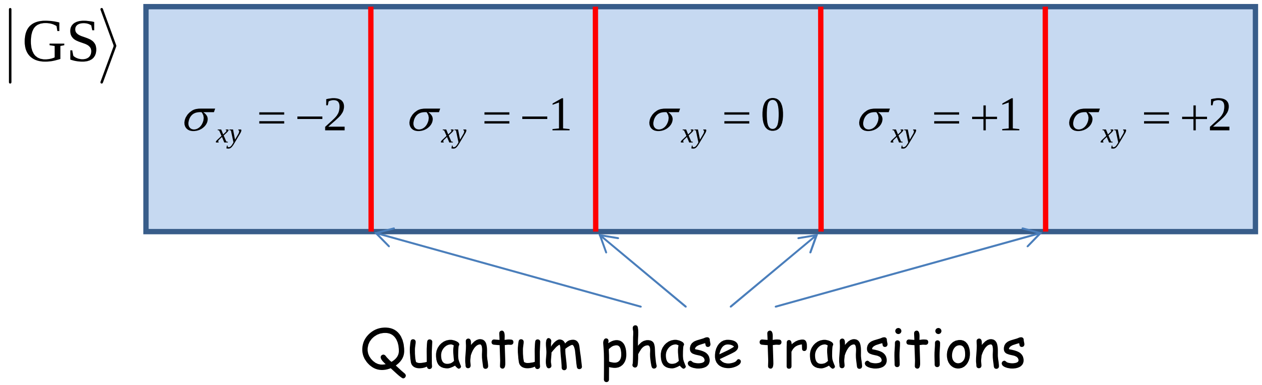


Quantized Hall conductance

$$\sigma_{xy} \in Z \times \frac{e^2}{h}$$ TKNN

Edge states $\# \in Z$

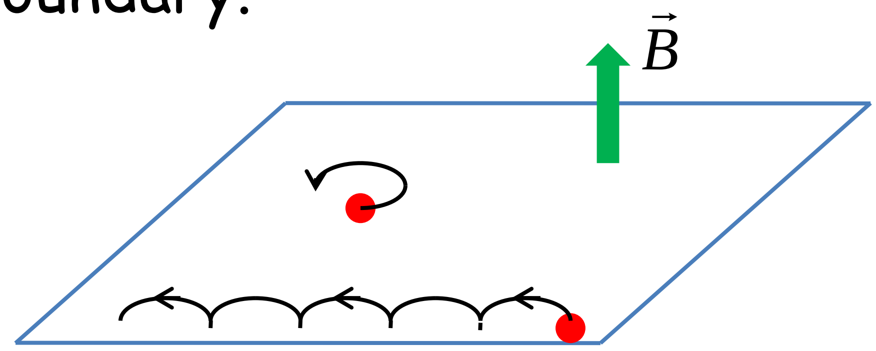
stable against disorder

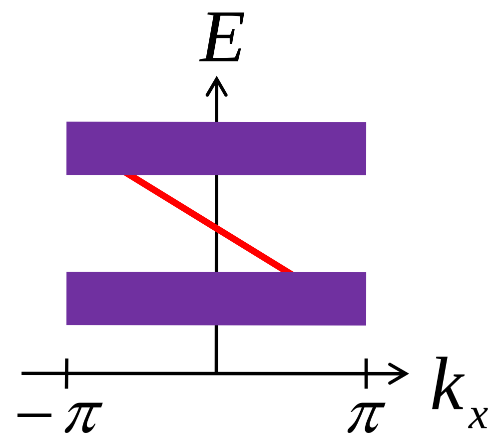


Space of gapped ground states is partitioned into topological sectors.

Edge states

- There is a gapless chiral edge mode along the sample boundary.





Number of edge modes $= \dfrac{-\sigma_{xy}}{e^2/h} = C$

Robust against disorder (chiral fermions cannot be backscattered)

Bulk: (2+1)d Chern-Simons theory

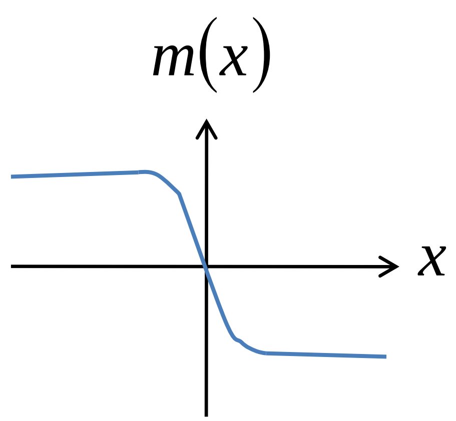
Edge: (1+1)d CFT

Effective field theory

$$H = -iv(\sigma_x \partial_x + \sigma_y \partial_y) + m\sigma_z$$

parity anomaly → $\sigma_{xy} = \frac{1}{2}\mathrm{sgn}(m)$

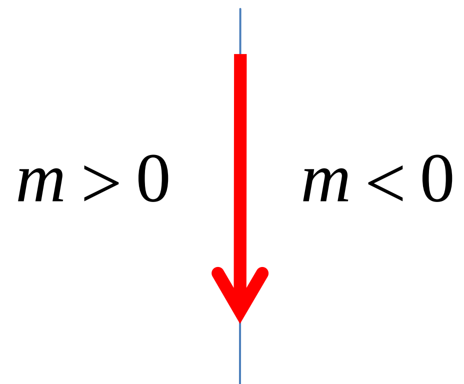
Domain wall fermion



$$H = -iv(\sigma_x \partial_x + \sigma_y \partial_y) + m(x)\sigma_z$$

$$\psi(x,y) = \exp\left[iky - \frac{1}{v}\sigma_y \int_0^x m(x')dx'\right]\begin{pmatrix} 1 \\ -i \end{pmatrix}$$

$$E = -vk$$



Example 2:
chiral p-wave superconductor

BCS理論 （平均場理論）

Integrating out ψ_σ

⇒ Ginzburg-Landau

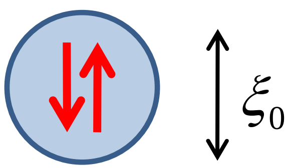
$$H=\int d^dr\sum_{\sigma=\uparrow,\downarrow}\psi_\sigma^+(r)\left[-\frac{\hbar^2}{2m}\left(\vec{\nabla}-ie\vec{A}\right)^2-\mu\right]\psi_\sigma(r)$$

$$+\frac{1}{2}\int d^drd^dr'\sum_{\alpha,\beta=\uparrow,\downarrow}\left[\Delta_{\alpha,\beta}(r,r')\psi_\alpha^+(r)\psi_\beta^+(r')+\Delta_{\alpha,\beta}^*(r,r')\psi_\alpha(r)\psi_\beta(r')\right]$$

$$\Delta_{\alpha,\beta}(r,r')=V(r-r')\langle\psi_\alpha(r)\psi_\beta(r')\rangle$$

超伝導秩序変数

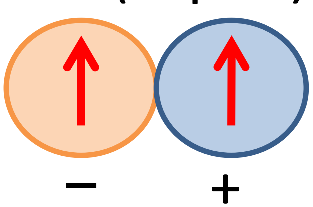
S-wave (singlet)



$$\Delta_{\alpha,\beta}(r,r')=\delta(r-r')\frac{1}{\sqrt{2}}\left(|\uparrow\rangle|\downarrow\rangle-|\downarrow\rangle|\uparrow\rangle\right)\Psi(r)$$

高温超伝導体 $d_{x^2-y^2}$ $\Delta(\vec{k})\propto\langle\psi_k\psi_{-k}\rangle\propto k_x^2-k_y^2$

P-wave (triplet)



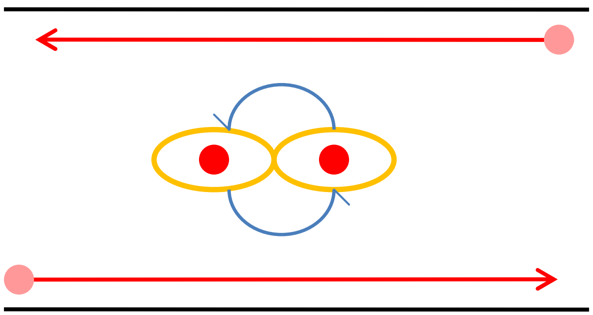
$$\Delta_{\alpha,\beta}(r,r')=\frac{\partial}{\partial r'}\delta(r-r')|\uparrow\rangle|\uparrow\rangle\Psi(r)$$

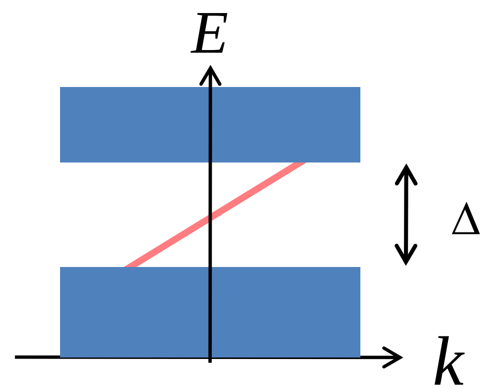
$$\Delta(\vec{k})\propto\langle\psi_k\psi_{-k}\rangle\propto\Delta_0k_x$$

Spinless p_x+ip_y superconductor in 2 dim.

- Order parameter $\Delta(\vec{k}) \propto \langle \psi_k \psi_{-k} \rangle \propto \Delta_0 (k_x + ik_y)$ $L_z = 1$

- Chiral (Majorana) edge state





$$\Psi = \begin{pmatrix} \psi_k \\ \psi^+_{-k} \end{pmatrix}$$

Hamiltonian density

$$H = \frac{1}{2}\Psi^+ \begin{pmatrix} (k^2 - k_F^2)/2m & \Delta(k_x + ik_y)/k_F \\ \Delta(k_x - ik_y)/k_F & -(k^2 - k_F^2)/2m \end{pmatrix} \Psi = \frac{1}{2}\Psi^+ \vec{h}_k \cdot \vec{\sigma}\, \Psi$$

$$\vec{h}_k = \begin{pmatrix} \dfrac{\Delta k_x}{k_F} & -\dfrac{\Delta k_y}{k_F} & \dfrac{k_x^2 + k_y^2 - k_F^2}{2m} \end{pmatrix} \qquad \frac{\vec{h}_k}{|\vec{h}_k|} : S^2 \to S^2$$

winding (wrapping) number=1

Hamiltonian density

$$H = \psi^{+}\left(-\frac{\hbar^2}{2m}\nabla^2 - \mu\right)\psi - i\frac{\Delta}{2k_F}\left[\psi^{+}\left(\partial_x + i\partial_y\right)\psi^{+} + \psi\left(\partial_x - i\partial_y\right)\psi\right]$$

Bogoliubov-de Gennes equation

$$i\frac{\partial\psi}{\partial t} = [\psi, H]$$

$$\begin{pmatrix} h_0 & -i\Delta\left(\partial_x + i\partial_y\right) \\ -i\Delta\left(\partial_x - i\partial_y\right) & -h_0 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = E\begin{pmatrix} u \\ v \end{pmatrix}$$

$$\begin{pmatrix} \psi(r,t) \\ \psi^{+}(r,t) \end{pmatrix} = e^{-iEt/\hbar}\begin{pmatrix} u(r) \\ v(r) \end{pmatrix}$$

$$E \to -E, \begin{pmatrix} u \\ v \end{pmatrix} \to \begin{pmatrix} v^{*} \\ u^{*} \end{pmatrix}$$

Particle-hole symmetry (charge conjugation)

$$\begin{pmatrix} \psi(\boldsymbol{r},t) \\ \psi^{\dagger}(\boldsymbol{r},t) \end{pmatrix} = \sum_{E_n>0}\left[e^{-iE_n t/\hbar}\gamma_n \begin{pmatrix} u_n(\boldsymbol{r}) \\ v_n(\boldsymbol{r}) \end{pmatrix} + e^{iE_n t/\hbar}\gamma_n^{\dagger}\begin{pmatrix} v_n^{*}(\boldsymbol{r}) \\ u_n^{*}(\boldsymbol{r}) \end{pmatrix}\right] + \gamma_0\begin{pmatrix} u_0(\boldsymbol{r}) \\ v_0(\boldsymbol{r}) \end{pmatrix}$$

zeromode: Majorana fermion $\gamma_0 = \gamma_0^{\dagger}$ $u_0(\boldsymbol{r}) = v_0^{*}(\boldsymbol{r})$

Majorana edge state $|E|<\Delta$

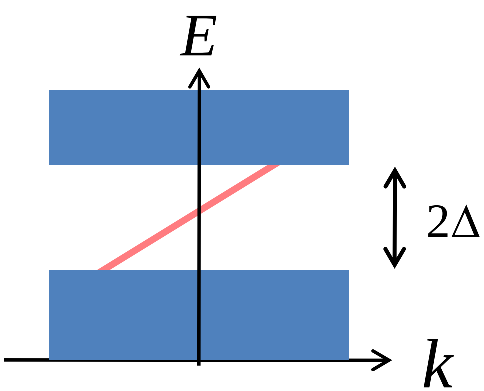
p_x+ip_y superconductor: $y>0$ vacuum: $y<0$

$$\frac{\partial}{\partial y}\begin{pmatrix} u \\ v \end{pmatrix}=0$$

$$\begin{pmatrix} -\frac{1}{2m}\left(\partial_x^2+\partial_y^2+k_F^2\right) & -i\frac{\Delta}{k_F}\left(\partial_x+i\partial_y\right) \\ -i\frac{\Delta}{k_F}\left(\partial_x-i\partial_y\right) & \frac{1}{2m}\left(\partial_x^2+\partial_y^2+k_F^2\right) \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}=E\begin{pmatrix} u \\ v \end{pmatrix}$$

$$E=\Delta\frac{k}{k_F}$$

$$\begin{pmatrix} u_k \\ v_k \end{pmatrix}=\exp\left(ikx+\frac{m\Delta}{k_F}y\right)\cos\left(\sqrt{k_F^2-k^2}\,y\right)\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

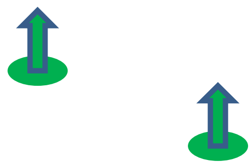


$$\psi(x,t)=\int_0^{k_F}\frac{dk}{\sqrt{4\pi}}\left(e^{ik(x-\Delta t/k_F)}\gamma_k+e^{-ik(x-\Delta t/k_F)}\gamma_k^+\right)$$

$$\psi=\psi^+$$

$$H_{\text{edge}}=\int_0^{k_F}dk\frac{k\Delta}{k_F}\gamma_k^+\gamma_k=-i\frac{\Delta}{k_F}\int dy\,\psi(y)\partial_y\psi(y)$$

- # Majorana bound state in a quantum vortex



vortex $\phi = \dfrac{hc}{e}$

Bogoliubov-de Gennes equation

$$\begin{pmatrix} h_0 & \Delta e^{i\varphi} \\ \Delta e^{-i\varphi} & -h_0^* \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = \varepsilon \begin{pmatrix} u \\ v \end{pmatrix} \qquad h_0 = \frac{1}{2m}\left(\vec{p} + e\vec{A}\right)^2 - E_F \qquad \begin{pmatrix} \Psi \\ \Psi^+ \end{pmatrix} \Leftrightarrow \begin{pmatrix} u \\ v \end{pmatrix}$$

energy spectrum of vortex bound states

$$\varepsilon_n = n\omega_0, \quad \omega_0 \approx \Delta_0^2 / E_F$$

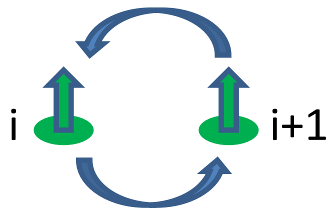
zero mode $\varepsilon_0 = 0$ $\quad \Psi = \Psi^+ \ (= \gamma)$ Majorana (real) fermion!

2N vortices ➡ GS degeneracy = 2^N

phase φ $\quad \begin{pmatrix} u \\ v \end{pmatrix} \Rightarrow \begin{pmatrix} ue^{i\varphi/2} \\ ve^{-i\varphi/2} \end{pmatrix}$

2π phase winding: $\gamma \rightarrow -\gamma$

interchanging vortices ⇒ braid groups, non-Abelian statistics



$$\gamma_i \rightarrow \gamma_{i+1}$$

$$\gamma_{i+1} \rightarrow -\gamma_i$$

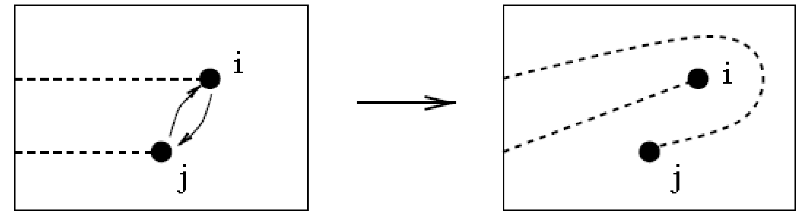
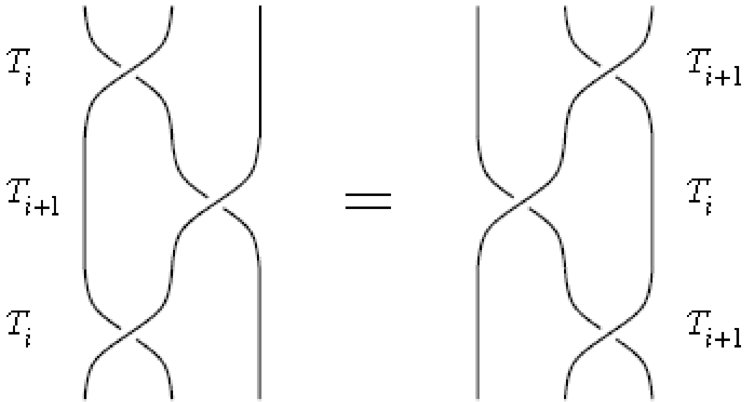


FIG. 3. Elementary braid interchange of two vortices.



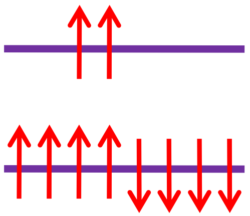
$$T_i T_j = T_j T_i, \qquad |i-j|>1,$$

$$T_i T_j T_i = T_j T_i T_j, \qquad |i-j|=1.$$

$$\tau(T_i) = \exp\left(\frac{\pi}{4}\gamma_{i+1}\gamma_i\right) = \frac{1}{\sqrt{2}}(1+\gamma_{i+1}\gamma_i)$$

D.A. Ivanov, PRL (2001)

Fractional quantum Hall effect at $\nu = \frac{5}{2}$



- 2nd Landau level
- Even denominator (cf. Laughlin states: odd denominator)
- Moore-Read (Pfaffian) state

$$z_j = x_j + iy_j$$

$$\psi_{\mathrm{MR}} = \mathrm{Pf}\left(\frac{1}{z_i - z_j}\right)\prod_{i<j}\left(z_i - z_j\right)^2 e^{-\sum|z_i|^2/4} \qquad \mathrm{Pf}\left(A_{ij}\right) = \sqrt{\det A_{ij}}$$

Pf( ) is equal to the BCS wave function of p_x+ip_y pairing state.

Excitations above the Moore-Read state obey non-Abelian statistics.

Effective field theory: level-2 SU(2) Chern-Simons theory

G. Moore & N. Read (1991); C. Nayak & F. Wilczek (1996)

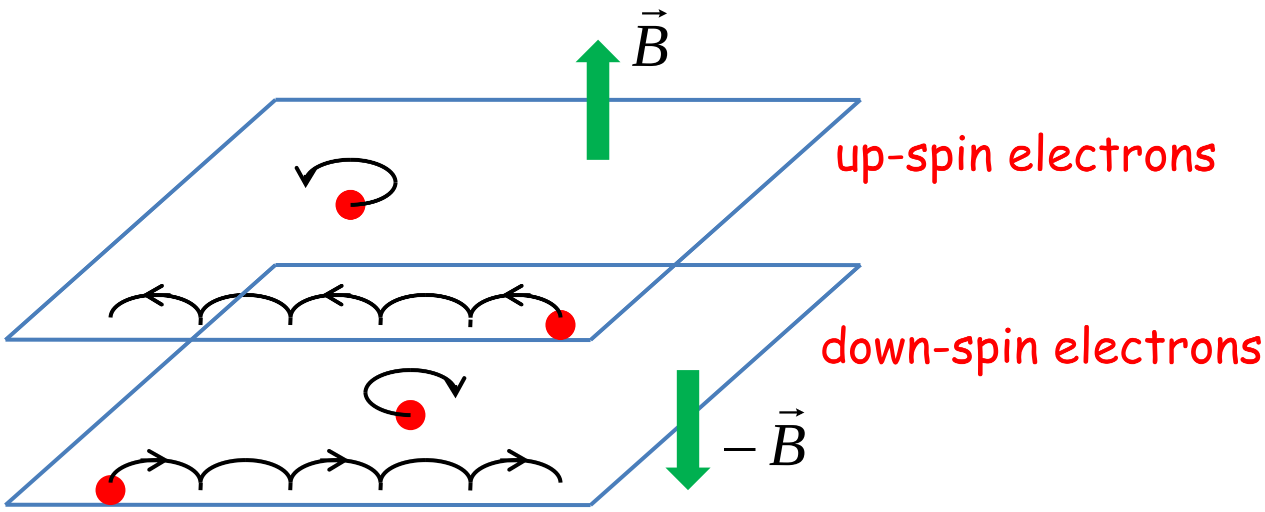
Example 3: Z_2 topological insulator

Quantum spin Hall effect

Quantum spin Hall effect (Z_2 top. Insulator)

Kane & Mele (2005, 2006); Bernevig & Zhang (2006)

- Time-reversal invariant band insulator
- Strong spin-orbit interaction $\lambda \vec{L} \cdot \vec{\sigma}$
- Gapless helical edge mode (Kramers pair)

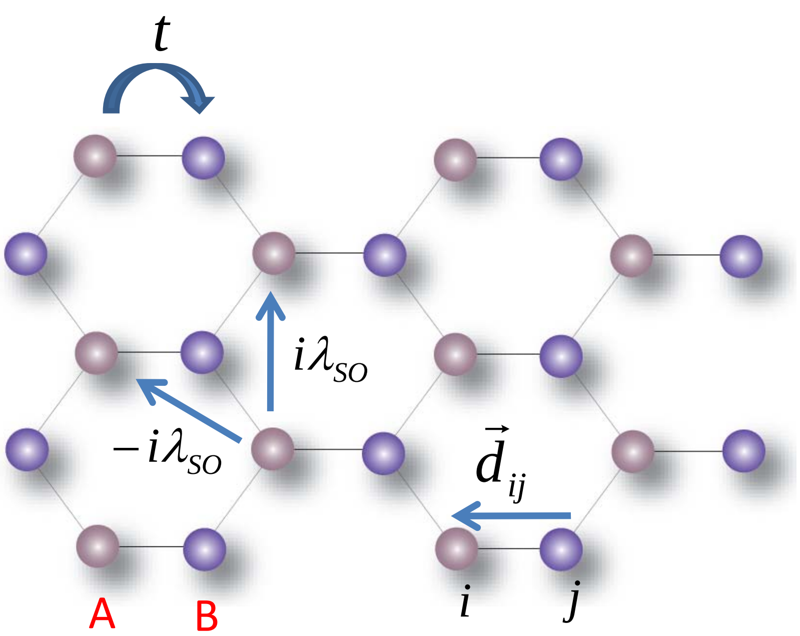


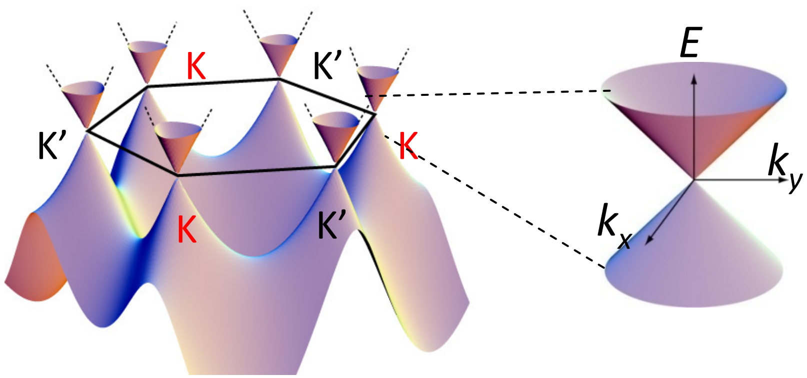
No spin rotation symmetry

Kane-Mele model (PRL, 2005)

$\xi_i = +1\ (\mathrm{A}),\ -1\ (\mathrm{B})$

$$H = t\sum_{\langle ij\rangle} c_i^+ c_j + i\lambda_{SO}\sum_{\langle\langle ij\rangle\rangle} v_{ij} c_i^+ s^z c_j + i\lambda_R \sum_{\langle ij\rangle} c_i^+ \left(\vec{s}\times\vec{d}_{ij}\right)_z c_j + \lambda_v \sum_i \xi_i c_i^+ c_i$$





$$\Psi = e^{i\vec{k}\cdot\vec{r}}\left(\phi_{A,s}\left(\vec{k}\right)\quad \phi_{B,s}\left(\vec{k}\right)\right) \qquad s = \uparrow,\downarrow$$

$$K:\ H_K = iv\left(\sigma^x\partial_x + \sigma^y\partial_y\right) - 3\sqrt{3}\lambda_{SO}\sigma^z s^z + \frac{3}{2}\lambda_R\left(\sigma^y s^x - \sigma^x s^y\right) + \lambda_v\sigma^z$$

$$K':\ H_{K'} = iv\left(-\sigma^x\partial_x + \sigma^y\partial_y\right) + 3\sqrt{3}\lambda_{SO}\sigma^z s^z + \frac{3}{2}\lambda_R\left(\sigma^y s^x + \sigma^x s^y\right) + \lambda_v\sigma^z$$

$$-is^y H_K^* is^y = H_{K'}$$

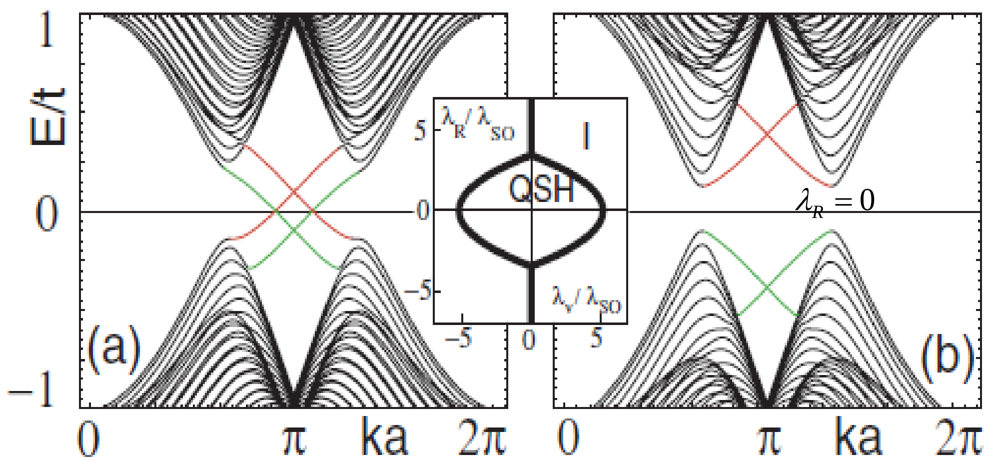
time reversal symmetry

- Quantum spin Hall insulator is characterized by Z_2 topological index ν

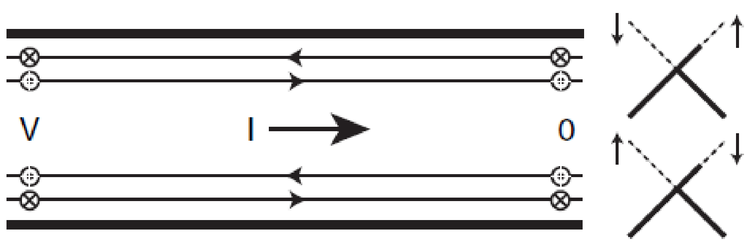
$\nu = 1$ an odd number of helical edge modes; Z_2 topological insulator

$\nu = 0$ an even (0) number of helical edge modes

1 0



Kane-Mele model
graphene + SOI
[PRL 95, 146802 (2005)]



Quantum spin Hall effect $\sigma_{xy}^{s} = \dfrac{e}{2\pi}$

(if S^z is conserved)

Edge states stable against disorder (and interactions)

Z_2: stability of gapless edge states

(1) A single Kramers doublet

$$H = iv s^z \partial_x + V_0 + V_x s^x + V_y s_y + V_z s_z \qquad -is^y H^* is^y = H$$

remain to be gapless

(2) Two Kramers doublets

$$H = iv\left(I \otimes s^z\right)\partial_x + V_0 + \tau^y \otimes \left(V_x s^x + V_y s_y + V_z s_z\right)$$

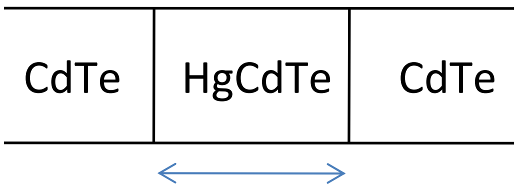
opens a gap

Odd number of Kramers doublet ➡ (1)

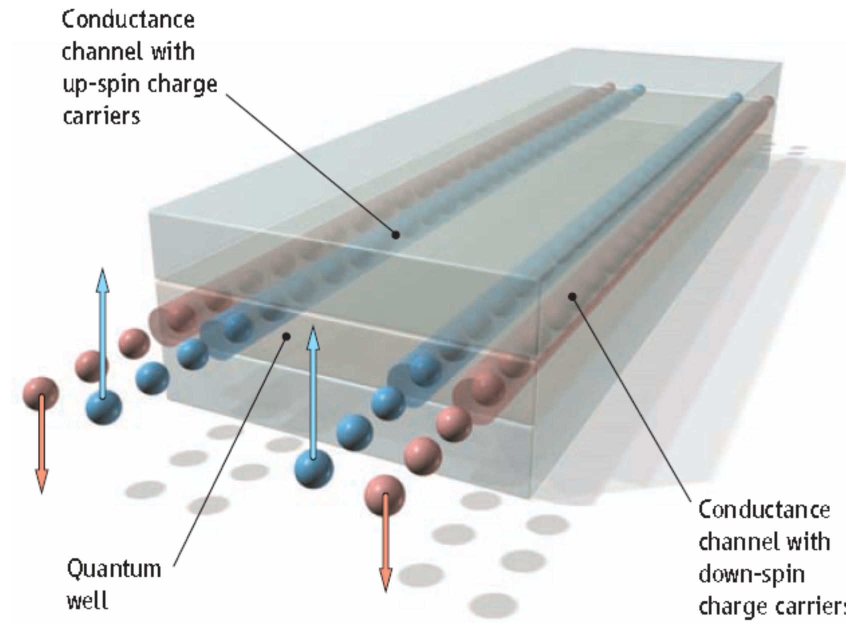
Even number of Kramers doublet ➡ (2)

Experiment

HgTe/(Hg,Cd)Te quantum wells



Konig et al. [Science 318, 766 (2007)]



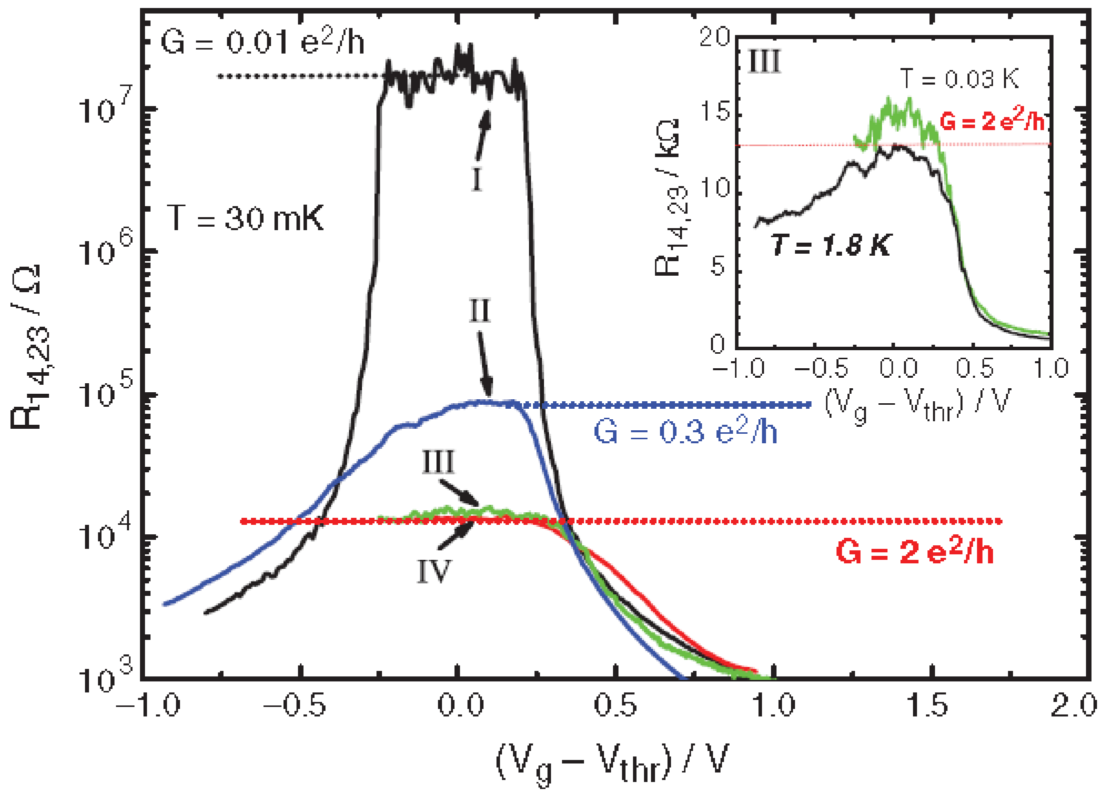


Fig. 4. The longitudinal four-terminal resistance, $R_{14,23}$, of various normal (d = 5.5 nm) (I) and inverted (d = 7.3 nm) (II, III, and IV) QW structures as a function of the gate voltage measured for B = 0 T at T = 30 mK. The device sizes are (20.0 × 13.3) μm^2 for devices I and II, (1.0 × 1.0) μm^2 for device III, and (1.0 × 0.5) μm^2 for device IV. The inset shows $R_{14,23}(V_g)$ of two samples from the same wafer, having the same device size (III) at 30 mK (green) and 1.8 K (black) on a linear scale.

Example 4: 3-dimensional Z_2 topological insulator

3-dimensional Z_2 topological insulator

Moore & Balents; Roy; Fu, Kane & Mele (2006, 2007)

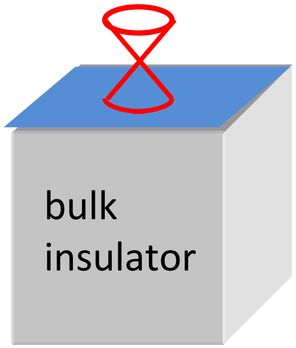
(strong) topological insulator

bulk: band insulator

surface: an odd number of surface Dirac modes

characterized by Z_2 topological numbers

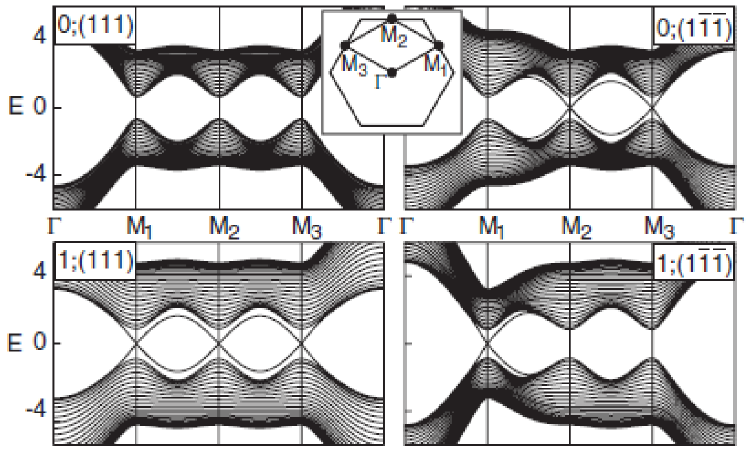
surface Dirac fermion



Ex: tight-binding model with SO int. on the diamond lattice
[Fu, Kane, & Mele; PRL 98, 106803 (2007)]

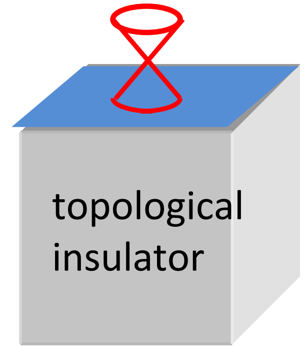
trivial insulator

Z_2 topological insulator

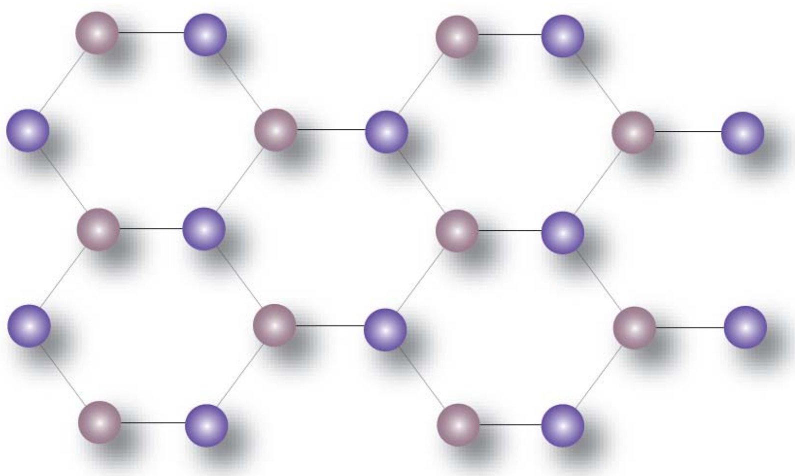


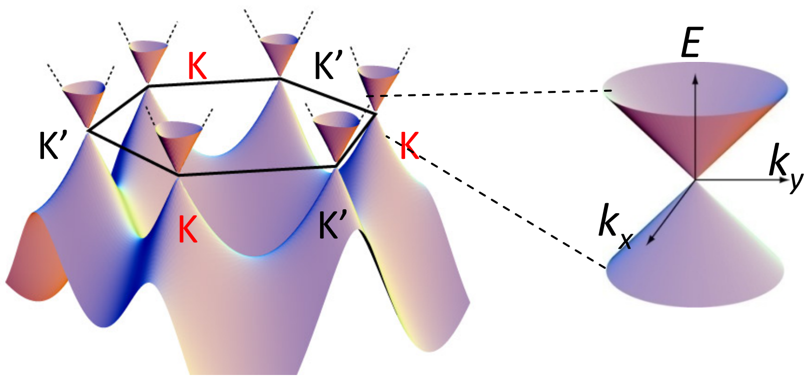
trivial band insulator:
0 or an even number of surface Dirac modes

Surface Dirac fermions



- A "half" of graphene



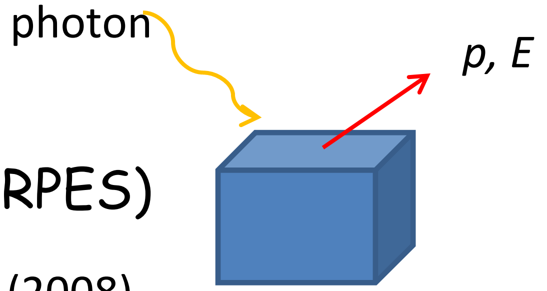


- An odd number of Dirac fermions in 2 dimensions

cf. Nielsen-Ninomiya's no-go theorem

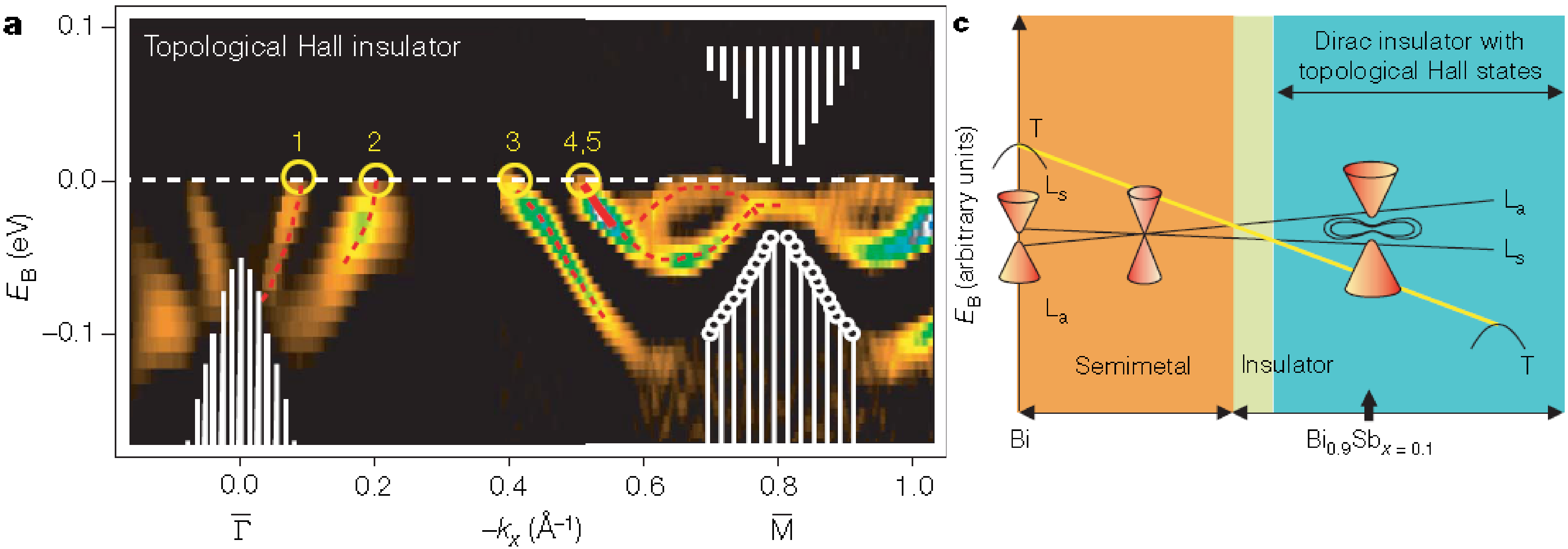
Experiments

- *Angle-resolved photoemission spectroscopy (ARPES)*



$Bi_{1-x}Sb_x$

Hsieh et al., Nature 452, 970 (2008)



An odd (5) number of surface Dirac modes were observed.

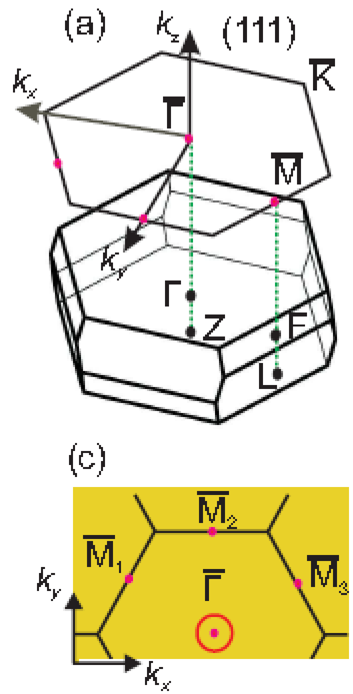
Experiments II

Bi_2Se_3

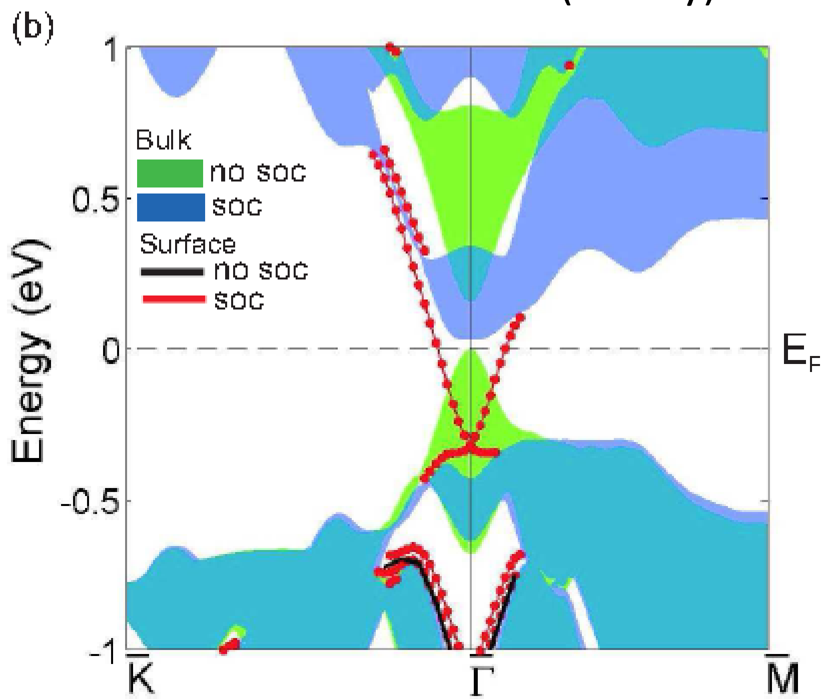
“hydrogen atom” of top. ins.

a single Dirac cone

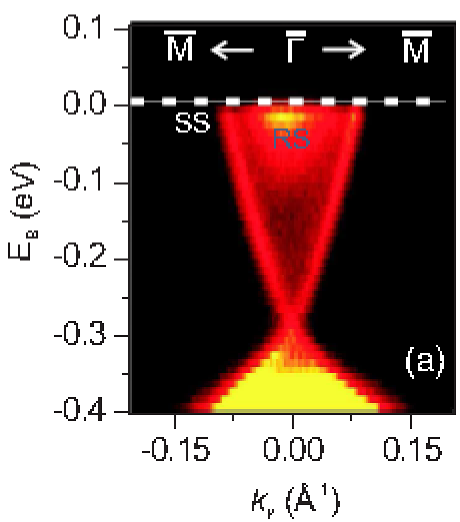
Xia et al.,
Nature Physics 5, 398 (2009)



Band calculations (theory)



ARPES experiment



トポロジカル絶縁体･超伝導体の分類

Schnyder, Ryu, AF, and Ludwig, PRB 78, 195125 (2008)

arXiv:0905.2029 (Landau100)

Classification of topological insulators/superconductors

AZ\d	0	1	2	3	4	5	6	7	8	9
A	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	$\cdots$
AIII	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\cdots$
AI	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	$\cdots$
BDI	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\cdots$
D	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\cdots$
DIII	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\cdots$
AII	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	$\cdots$
CII	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\cdots$
C	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	$\cdots$
CI	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	$\cdots$

Schnyder, Ryu, AF, and Ludwig, PRB (2008)

Kitaev, arXiv:0901.2686

spatial dimensions

AZ\d	0	1	2	3	4
A	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
AIII	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0
AI	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$
BDI	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0
D	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
DIII	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0
AII	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
CII	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
C	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$
CI	0	0	0	$\mathbb{Z}$	0

symmetry classes of
quadraic fermionic Hamiltonians

presence/absence of
topological band structure

Altland-Zirnbauer (97)

$\mathbb{Z}$: integer classification

$\mathbb{Z}_2$: Z2 classification

0 : no top. insulator/SC

Zoo of topological insulators/superconductors

AZ\d	0	1	2	3	4
A	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
AIII	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0
AI	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$
BDI	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0
D	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
DIII	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0
AII	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
CII	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
C	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$
CI	0	0	0	$\mathbb{Z}$	0

IQHE

p+ip wave SC

polyacetylene

3He B

TMTSF

Z2 topological insulator

QSHE

d+id wave SC

Classification of topological insulators/SCs

Topological insulators are stable against (weak) perturbations.

↓ Random deformation of Hamiltonian

Natural framework: random matrix theory
(Wigner, Dyson, Altland & Zirnbauer)

Assume only basic discrete symmetries:

(1) time-reversal symmetry

$$TH^{*}T^{-1} = H$$

TRS =
- 0 no TRS
- +1 TRS with $T^{\mathrm{T}} = +T$ (integer spin)
- -1 TRS with $T^{\mathrm{T}} = -T$ (half-odd integer spin)

$$T = is^{y}$$

(2) particle-hole symmetry

$$CH^{\mathrm{T}}C^{-1} = -H$$

PHS =
- 0 no PHS
- +1 PHS with $C^{\mathrm{T}} = +C$ (odd parity: p-wave)
- -1 PHS with $C^{\mathrm{T}} = -C$ (even parity: s-wave)

(3) TRS $\times$ PHS $=$ chiral symmetry [sublattice symmetry (SLS)]

$$TCH(TC)^{-1} = -H$$

$$3\times 3+1=10$$

(2) particle-hole symmetry   Bogoliubov-de Gennes

p_x+ip_y

$$H=\frac{1}{2}\begin{pmatrix}c_k^+ & c_{-k}\end{pmatrix}h_k\begin{pmatrix}c_k\\ c_{-k}^+\end{pmatrix}\qquad h_k=\Delta\left(k_x\sigma_x+k_y\sigma_y\right)+\varepsilon_k\sigma_z$$

$$\sigma_x h_{-k}^*\sigma_x=-h_k\qquad C=\sigma_x=C^T$$

$d_{x2-y2}+id_{xy}$

$$H=\frac{1}{2}\begin{pmatrix}c_{k\uparrow}^+ & c_{-k\downarrow}\end{pmatrix}h_k\begin{pmatrix}c_{k\uparrow}\\ c_{-k\downarrow}^+\end{pmatrix}\qquad h_k=\Delta\left[\left(k_x^2-k_y^2\right)\sigma_x+k_yk_y\sigma_y\right]+\varepsilon_k\sigma_z$$

$$\sigma_y h_{-k}^*\sigma_y=-h_k\qquad C=i\sigma_y=C^T$$

10 random matrix ensembles

		TRS	PHS	SLS	description
Wigner-Dyson (standard)	A	0	0	0	unitary IQHE
	AI	+1	0	0	orthogonal
	AII	-1	0	0	symplectic (spin-orbit) Z_2 TPI
chiral (sublattice)	AIII	0	0	1	chiral unitary
	BDI	+1	+1	1	chiral orthogonal
	CII	-1	-1	1	chiral symplectic
BdG	D	0	+1	0	singlet/triplet SC p_x+ip_y
	C	0	-1	0	singlet SC $d_{x2-y2}+id_{xy}$
	DIII	-1	+1	1	singlet/triplet SC with TRS
	CI	+1	-1	1	singlet SC with TRS

Examples of topological insulators in 2 spatial dimensions

Integer quantum Hall Effect

Z_2 topological insulator (quantum spin Hall effect) also in 3D

Moore-Read Pfaffian state (spinless p+ip superconductor)

Complex case:

n	classifying space BG	$\pi_0(B)$	AZ class
0	$\mathrm{U}(N+M)/\mathrm{U}(N)\times\mathrm{U}(M)\times\mathbb{Z}$	$\mathbb{Z}$	A
1	$\mathrm{U}(N)$	0	AIII

Real case:

n	classifying space BG	$\pi_0(B)$	AZ class
0	$\mathrm{O}(N+M)/\mathrm{O}(N)\times\mathrm{O}(M)\times\mathbb{Z}$	$\mathbb{Z}$	AI
1	$\mathrm{O}(N)$	$\mathbb{Z}_2$	BDI
2	$\mathrm{O}(2N)/\mathrm{U}(N)$	$\mathbb{Z}_2$	D
3	$\mathrm{U}(N)/\mathrm{Sp}(N)$	0	DIII
4	$\mathrm{Sp}(N+M)/\mathrm{Sp}(N)\times\mathrm{Sp}(M)\times\mathbb{Z}$	$\mathbb{Z}$	AII
5	$\mathrm{Sp}(N)$	0	CII
6	$\mathrm{Sp}(2N)/\mathrm{U}(N)$	0	C
7	$\mathrm{U}(N)/\mathrm{O}(N)$	0	CI

Table of topological insulators in 1, 2, 3 dim.

Schnyder, Ryu, Furusaki & Ludwig, PRB (2008)

random matrix ensemble		TRS	PHS	chS		d=1	d=2	d=3
Wigner-Dyson (standard)	A	0	0	0	unitary	0	Z [(a)]	0
	AI	+1	0	0	orthogonal	0	0	0
	AII	−1	0	0	symplectic	0	Z_2 [(b)]	Z_2 [(c)]
Chiral (sublattice)	AIII	0	0	1	chiral unitary	Z	0	Z
	BDI	+1	+1	1	chiral orthogonal	Z	0	0
	CII	−1	−1	1	chiral symplectic	Z	0	Z_2
BdG	D	0	+1	0	(triplet) SC	Z_2	Z [(d)]	0
	C	0	−1	0	singlet SC	0	Z [(e)]	0
	DIII	−1	+1	1	triplet SC	Z_2	Z_2 [(f)]	Z [(g)]
	CI	+1	−1	1	singlet SC	0	0	Z

Examples:
(a)Integer Quantum Hall Insulator, (b) Quantum Spin Hall Insulator,
(c) 3d Z_2 Topological Insulator, (d) Spinless chiral p-wave (p+ip) superconductor (Moore-Read),
(e)Chiral d-wave $(d_{x^2-y^2} + i d_{xy})$superconductor, (f) $(p_x + i p_y)_\uparrow \otimes (p_x - i p_y)_\downarrow$ superconductor,
(g) ^{3}He B phase.

Classification of 3d topological insulators/SCs

strategy (bulk ⟺ boundary)

- Bulk topological invariants

integer topological numbers : 3 random matrix ensembles (AIII, CI, DIII)

$$\nu = \int_{\mathrm{Bz}} \frac{d^3k}{24\pi^2} \epsilon^{\mu\nu\rho} \mathrm{tr}\left[(q^{-1}\partial_\mu q)(q^{-1}\partial_\nu q)(q^{-1}\partial_\rho q)\right]$$

$$q : \mathrm{BZ} \longrightarrow U(m)$$

spectral projector

BZ: Brillouin zone

- Classification of 2d Dirac fermions Bernard & LeClair ('02)

13 classes (13=10+3) AIII, CI, DIII AII, CII

$$\mathscr{H} = \begin{pmatrix} \mathbf{V}_+ + \mathbf{V}_- & -i\frac{\partial}{\partial z}\mathbf{1} + \mathbf{A}_+ \\ +i\frac{\partial}{\partial \bar{z}}\mathbf{1} + \mathbf{A}_- & \mathbf{V}_+ - \mathbf{V}_- \end{pmatrix}$$

- Anderson delocalization in 2d

nonlinear sigma models Z_2 topological term (2) or WZW term (3)

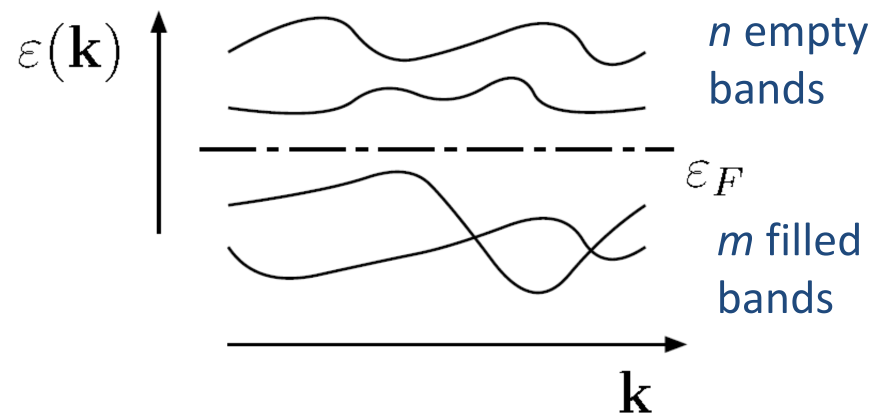
Topological distinction of ground states

deformed "Hamiltonian"

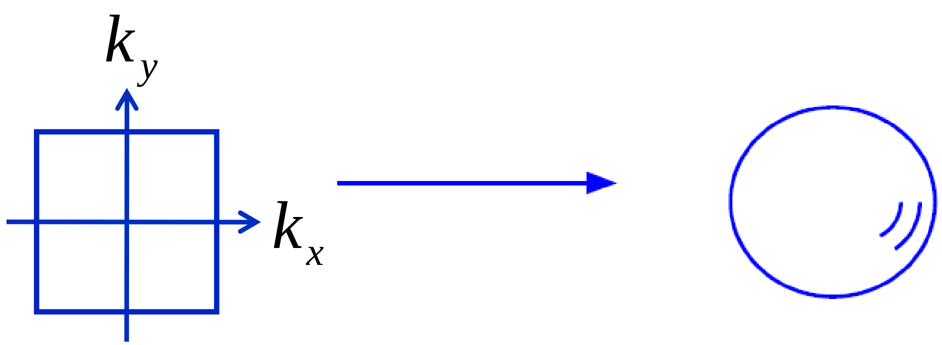
$$Q(k) = 2 \sum_{a \in \text{filled}} |u_a(k)\rangle\langle u_a(k)| - 1$$

$$Q^2 = 1,\ \ Q^\dagger = Q,\ \ \mathrm{tr}\, Q = m - n$$

filled (→ m) empty (→ n)



Q : BZ $\longrightarrow$ $U(m+n)/U(m)\times U(n)$



map from BZ to Grassmannian

$$\pi_2\left[U(m+n)/U(m)\times U(n)\right] = Z \Longrightarrow \text{IQHE (2 dim.)}$$

homotopy class

$$\pi_3[U(m+n)/U(m)\times U(n)] = 0$$

In classes AIII, BDI, CII, CI, DIII, Hamiltonian can be made off-diagonal.

$$\mathcal{H} = \begin{pmatrix} 0 & D \\ D^\dagger & 0 \end{pmatrix}$$

Projection operator is also off-diagonal.

$$Q(k) = \begin{pmatrix} 0 & q(k) \\ q^\dagger(k) & 0 \end{pmatrix}$$

q : BZ $\longrightarrow U(m)$

$q^T(-k) = -q(k)$ DIII

$q^T(-k) = q(k)$ CI

$q^*(-k) = q(k)$ BDI

$i\sigma_y q^*(-k)(-i\sigma_y) = -q(k)$ CII

$\pi_3[U(n)] = Z$ ⟶ topological insulators labeled by an integer

$$\nu[q] = \int \frac{d^3k}{24\pi^2} \varepsilon^{\lambda\mu\nu} \mathrm{tr}\left[\left(q^{-1}\partial_\lambda q\right)\left(q^{-1}\partial_\mu q\right)\left(q^{-1}\partial_\nu q\right)\right]$$

Discrete symmetries limit possible values of $\nu[q]$

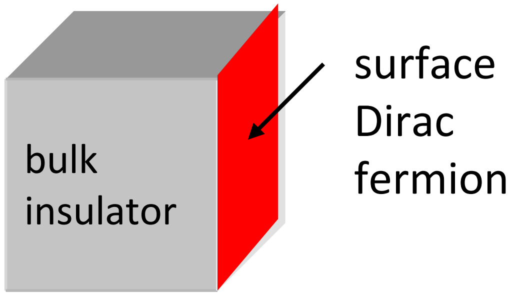
AIII & DIII $\nu \in \mathbf{Z}$

CI $\nu \in 2\mathbf{Z}$

CII & BDI $\nu = 0$

Z_2 insulators in CII (chiral symplectic)

The integer number $\nu(q)$ ⟺ # of surface Dirac (Majorana) fermions



(3+1)D 4-component Dirac Hamiltonian

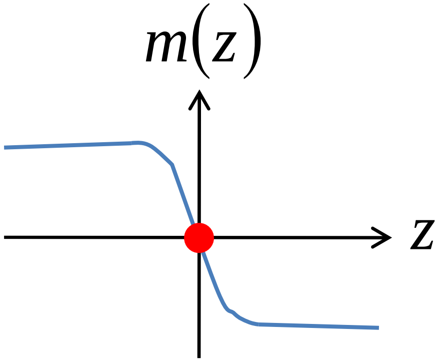
$$H(k) = k_\mu \tau_x \otimes \sigma_\mu + m\tau_\mu = \begin{pmatrix} m & \vec{k}\cdot\vec{\sigma} \\ \vec{k}\cdot\vec{\sigma} & -m \end{pmatrix}$$

$$\nu[q] = \frac{1}{2}\mathrm{sgn}(m)$$

AII: $-i\sigma_y H^*(k) i\sigma_y = H(-k)$ TRS

DIII: $\tau_y \otimes \sigma_y H^*(k) \tau_y \otimes \sigma_y = -H(-k)$ PHS

AIII: $\tau_y H(k) \tau_y = -H(k)$ chS



random matrix ensemble		TRS	PHS	chS		d=1	d=2	d=3
Wigner-Dyson (standard)	A	0	0	0	unitary	0	Z [(a)]	0
	AI	+1	0	0	orthogonal	0	0	0
	AII	−1	0	0	symplectic	0	Z_2 [(b)]	Z_2 [(c)]
Chiral (sublattice)	AIII	0	0	1	chiral unitary	Z	0	Z
	BDI	+1	+1	1	chiral orthogonal	Z	0	0
	CII	−1	−1	1	chiral symplectic	Z	0	Z_2
BdG	D	0	+1	0	(triplet) SC	Z_2	Z [(d)]	0
	C	0	−1	0	singlet SC	0	Z [(e)]	0
	DIII	−1	+1	1	triplet SC	Z_2	Z_2 [(f)]	Z [(g)]
	CI	+1	−1	1	singlet SC	0	0	Z

(3+1)D 8-component Dirac Hamiltonian

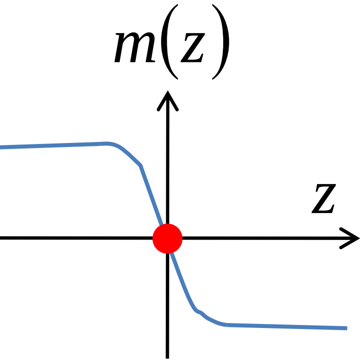
$$H=\begin{pmatrix} 0 & D \\ D^{+} & 0 \end{pmatrix}$$

CI: $D(k)=i\sigma_y\beta(k_\mu\alpha_\mu - i\gamma^5) = -\tau_y \otimes \sigma_y (k_\mu\sigma_\mu - im)$

$$D^T(k)=D(-k) \qquad \nu[q]=\frac{1}{2}\mathrm{sgn}(m)\times 2$$

CII: $D(k)=k_\mu\alpha_\mu + m\beta = D^{+}(k) \qquad -i\sigma_y D^*(k) i\sigma_y = D(-k)$

$$\nu[q]=0$$



random matrix ensemble		TRS	PHS	chS		d=1	d=2	d=3
Wigner-Dyson	A	0	0	0	unitary	0	Z [(a)]	0
(standard)	AI	+1	0	0	orthogonal	0	0	0
	AII	−1	0	0	symplectic	0	Z_2 [(b)]	Z_2 [(c)]
Chiral	AIII	0	0	1	chiral unitary	Z	0	Z
(sublattice)	BDI	+1	+1	1	chiral orthogonal	Z	0	0
	CII	−1	−1	1	chiral symplectic	Z	0	Z_2
BdG	D	0	+1	0	(triplet) SC	Z_2	Z [(d)]	0
	C	0	−1	0	singlet SC	0	Z [(e)]	0
	DIII	−1	+1	1	triplet SC	Z_2	Z_2 [(f)]	Z [(g)]
	CI	+1	−1	1	singlet SC	0	0	Z

Classification of 3d topological insulators

strategy (bulk ⟺ boundary)

- Bulk topological invariants

integer topological numbers: 3 random matrix ensembles (AIII, CI, DIII)

$$\nu = \int_{\mathrm{Bz}} \frac{d^3k}{24\pi^2} \epsilon^{\mu\nu\rho} \mathrm{tr}\left[(q^{-1}\partial_\mu q)(q^{-1}\partial_\nu q)(q^{-1}\partial_\rho q)\right]$$

q : BZ $\longrightarrow$ $U(m)$ spectral projector

- Classification of 2d Dirac fermions Bernard & LeClair ('02)

13 classes (13=10+3) AIII, CI, DIII AII, CII

$$\mathscr{H} = \begin{pmatrix} \mathbf{V}_+ + \mathbf{V}_- & -i\frac{\partial}{\partial z}\mathbf{1} + \mathbf{A}_+ \\ +i\frac{\partial}{\partial \bar{z}}\mathbf{1} + \mathbf{A}_- & \mathbf{V}_+ - \mathbf{V}_- \end{pmatrix}$$

- Anderson delocalization in 2d

nonlinear sigma models Z_2 topological term (2) or WZW term (3)

Nonlinear sigma approach to Anderson localization

- (fermionic) replica

Wegner, Efetov, Larkin, Hikami, ….

- Matrix field Q describing diffusion
- Localization ⟷ massive
- Extended or critical ⟷ massless ⟷ topological Z_2 term or WZW term

AZ class	Fermionic replica NLσM target space	Topological or WZW term
A	$U(2N)/U(N)\times U(N)$	Pruisken
AI	$Sp(2N)/Sp(N)\times Sp(N)$	N/A
AII	$O(2N)/O(N)\times O(N)$	$\mathbb{Z}_2$ ←
AIII	$U(N)\times U(N)/U(N)$	WZW ←
BDI	$U(2N)/Sp(N)$	N/A
CII	$U(2N)/O(2N)$	$\mathbb{Z}_2$ ←
D	$O(2N)/U(N)$	Pruisken
C	$Sp(N)/U(N)$	Pruisken
DIII	$O(2N)\times O(2N)/O(2N)$	WZW ←
CI	$Sp(N)\times Sp(N)/Sp(N)$	WZW ←

$\pi_2(M) = Z_2$

$\pi_3(M) = Z$

Table of topological insulators in 1, 2, 3 dim.

Schnyder, Ryu, Furusaki & Ludwig, PRB (2008)
arXiv:0905.2029

random matrix ensemble		TRS	PHS	chS		d=1	d=2	d=3
Wigner-Dyson (standard)	A	0	0	0	unitary	0	$\mathbb{Z}$ $^{(a)}$	0
	AI	+1	0	0	orthogonal	0	0	0
	AII	−1	0	0	symplectic	0	$\mathbb{Z}_2$ $^{(b)}$	$\mathbb{Z}_2$ $^{(c)}$
Chiral (sublattice)	AIII	0	0	1	chiral unitary	$\mathbb{Z}$	0	$\mathbb{Z}$
	BDI	+1	+1	1	chiral orthogonal	$\mathbb{Z}$	0	0
	CII	−1	−1	1	chiral symplectic	$\mathbb{Z}$	0	$\mathbb{Z}_2$
BdG	D	0	+1	0	(triplet) SC	$\mathbb{Z}_2$	$\mathbb{Z}$ $^{(d)}$	0
	C	0	−1	0	singlet SC	0	$\mathbb{Z}$ $^{(e)}$	0
	DIII	−1	+1	1	triplet SC	$\mathbb{Z}_2$	$\mathbb{Z}_2$ $^{(f)}$	$\mathbb{Z}$ $^{(g)}$
	CI	+1	−1	1	singlet SC	0	0	$\mathbb{Z}$

Examples:

(a)Integer Quantum Hall Insulator, (b) Quantum Spin Hall Insulator,

(c) 3d Z_2 Topological Insulator, (d) Spinless chiral p-wave (p+ip) superconductor (Moore-Read),

(e)Chiral d-wave $(d_{x^2-y^2} + id_{xy})$superconductor, (f) $(p_x + ip_y)_\uparrow \otimes (p_x - ip_y)_\downarrow$ superconductor,

(g) ^{3}He B phase.

Reordered Table

Kitaev, arXiv:0901.2686

	TRS	PHS	chS		d=1	d=2	d=3	
A	0	0	0	unitary	0	Z	0	complex
AIII	0	0	1	chiral unitary	Z	0	Z	K-theory
AI	+1	0	0	orthogonal	0	0	0	
BDI	+1	+1	1	chiral orthogonal	Z	0	0	
D	0	+1	0	(triplet) SC	Z_2	Z	0	
DIII	−1	+1	1	triplet SC	Z_2	Z_2	Z	real
AII	−1	0	0	symplectic	0	Z_2	Z_2	K-theory
CII	−1	−1	1	chiral symplectic	Z	0	Z_2	
C	0	−1	0	singlet SC	0	Z	0	
CI	+1	−1	1	singlet SC	0	0	Z	

Periodic table of topological insulators

Classification in any dimension

Bott periodicity:

$$\tilde{K}_{\mathbb{C}}^{n+2}(X) \cong \tilde{K}_{\mathbb{C}}^{n}(X)$$

$$\tilde{K}_{\mathbb{R}}^{n+8}(X) \cong \tilde{K}_{\mathbb{R}}^{n}(X)$$

Classification of topological insulators/superconductors

AZ\d	0	1	2	3	4	5	6	7	8	9
A	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	$\cdots$
AIII	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\cdots$
AI	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	$\cdots$
BDI	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\cdots$
D	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\cdots$
DIII	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\cdots$
AII	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	$\cdots$
CII	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\cdots$
C	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	$\cdots$
CI	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	$\cdots$

Schnyder, Ryu, AF, and Ludwig, PRB (2008)

Kitaev, arXiv:0901.2686

Summary

- Many topological insulators of non-interacting fermions have been found.

interacting fermions??

- Gapless boundary modes (Dirac or Majorana)

stable against any (weak) perturbation disorder

- Majorana fermions

to be found experimentally in solid-state devices

Z_2 T.I. + s-wave SC Majorana bound state (Fu & Kane)

Z_2 T.I. + test charge Dyon (Qi, Li, Zang, & S.-C. Zhang)

$$S_\theta = \frac{\theta}{2\pi}\frac{e^2}{hc}\int d^3x dt \vec{E}\cdot\vec{B}$$