Supersymmetry restoration in small volume lattices

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- D. Kadoh, H.S., work in progress

Sugino's lattice formulation of 2D $\mathcal{N} = (2,2)$ SYM (we assume $G = SU(k)$)

- Lattice action (fermion: $\Psi^T \equiv (\psi_0, \psi_1, \chi, \eta/2) \Leftrightarrow (Q^{(0)}, Q^{(1)}, \tilde{Q}, Q)$)

$$
\begin{aligned}
&S_{\mathrm{2DSYM}}^{\mathrm{LAT}} \\
&= Q \frac{1}{a^2 g^2} \sum_x \mathrm{tr}\Bigg[-i\chi(x)\hat{\Phi}(x) + \chi(x)H(x) + \frac{1}{4}\eta(x)[\phi(x), \bar{\phi}(x)] \\
&\qquad - i \sum_{\mu=0}^{1} \psi_\mu(x) \left(U_\mu(x)\bar{\phi}(x + a\hat{\mu})U_\mu(x)^{-1} - \bar{\phi}(x) \right) \Bigg],
\end{aligned}
$$

where the lattice field strength $\hat{\Phi}(x)$ ($\Leftrightarrow 2F_{01}$) is given (basically) by the plaquette

$$
\hat{\Phi}(x) \simeq -iU_0(x)U_1(x + a\hat{0})U_0(x + a\hat{1})^{-1}U_1(x)^{-1} + \mathrm{h.c.}
$$

Sugino's lattice formulation

- Lattice Q-transformation

$$
\begin{aligned}
&QU_\mu(x) = i\psi_\mu(x)U_\mu(x),\\
&Q\psi_\mu(x) = i\psi_\mu(x)\psi_\mu(x) - i\left(\phi(x) - U_\mu(x)\phi(x+a\hat{\mu})U_\mu(x)^{-1}\right),\\
&Q\phi(x) = 0,\\
&Q\bar{\phi}(x) = \eta(x), \qquad Q\eta(x) = \left[\phi(x), \bar{\phi}(x)\right],\\
&Q\chi(x) = H(x), \qquad QH(x) = [\phi(x), \chi(x)]
\end{aligned}
$$

is nilpotent on the lattice

$$Q^2 = \delta_\phi \simeq 0$$

- Q is a manifest lattice symmetry, $QS^{\mathrm{LAT}}_{\mathrm{2DSYM}} = 0$

Restoration of full SUSY?

- The above lattice formulation possesses a manifest fermionic symmetry Q
- Other manifest (bosonic) symmetries: $U(1)_A$ symmetry,

$$\Psi(x) \to \exp(\alpha\Gamma_2\Gamma_3)\,\Psi(x),$$
$$\phi(x) \to \exp(2i\alpha)\,\phi(x), \qquad \bar{\phi}(x) \to \exp(-2i\alpha)\,\bar{\phi}(x),$$

- "Reflection" symmetry, $x \to \tilde{x} \equiv (x_1, x_0)$ and

$$U_0(x) \to U_1(\tilde{x}), \qquad U_1(x) \to U_0(\tilde{x}), \qquad \text{etc.}$$

- But how about other $Q^{(0)}$, $Q^{(1)}$, $\tilde{Q}$, full SUSY?
- Full SUSY is restored in the continuum limit $a \to 0$? (perturbative argument on the basis of the effective action: Kaplan et al., Sugino)

What is the best characterization of SUSY restoration?

- Scalar 2-point function? ($\Leftarrow$ not gauge invariant)
- Fermion-boson degeneracy? ($\Leftarrow$ cannot be distinguished from the spontaneous SUSY breaking)
- (local) SUSY Ward-Takahashi (WT) identity would be the best
- In the target continuum theory, we expect

$$\partial_\mu \langle s_\mu(x)\, \mathcal{O}(y_1,\ldots,y_n)\rangle \qquad s_\mu\text{: supercurrent}$$
$$= \frac{\mu^2}{g^2} \langle f(x)\, \mathcal{O}(y_1,\ldots,y_n)\rangle - i \frac{\delta}{\delta\epsilon(x)} \langle \mathcal{O}(y_1,\ldots,y_n)\rangle$$

- Here, we introduced a SUSY breaking scalar mass term

$$S_{\text{mass}} = \frac{\mu^2}{g^2} \int d^2x \,\operatorname{tr}\left[\bar{\phi}\phi\right], \qquad f \equiv 4iC\left(\Gamma_\uparrow \operatorname{tr}\left[\phi\Psi\right] + \Gamma_\downarrow \operatorname{tr}\left[\bar{\phi}\Psi\right]\right),$$

where

$$\Gamma_{\uparrow,\downarrow} \equiv \frac{i}{2}(\Gamma_2 \mp i\Gamma_3)$$

SUSY WT identity on the lattice

- We can show that, on the lattice

$$\partial_\mu^* \langle s_\mu(x)\mathcal{O}(y_1,\ldots,y_n)\rangle$$
$$= \frac{\mu^2}{g^2}\langle f(x)\mathcal{O}(y_1,\ldots,y_n)\rangle - i\frac{1}{a^2}\frac{\delta}{\delta\epsilon(x)}\langle\mathcal{O}(y_1,\ldots,y_n)\rangle$$
$$+ \langle B(x)\mathcal{O}(y_1,\ldots,y_n)\rangle$$

- $B(x)$ is $O(a)$, gauge invariant, fermionic and mass dimension $5/2$
- Under $U(1)_A$: $B(x) \to \exp(-\alpha\Gamma_2\Gamma_3)B(x)$
- Under the reflection: $B(x) \to \mathcal{R}B(\tilde{x})$ ($\mathcal{R} \equiv \frac{1}{2}(i+\Gamma_5)(\Gamma_0-\Gamma_1)$)

Breaking term $B(x)$

- When $\mathcal{O}$ is a product of elementary fields,

$$B(x) \xrightarrow{a\to 0} c_\uparrow C\Gamma_\uparrow \operatorname{tr}[\phi\Psi] + c_\downarrow C\Gamma_\downarrow \operatorname{tr}\left[\bar{\phi}\Psi\right],$$

- Further assuming that SUSY has no intrinsic anomaly (i.e., the breaking can be removed by local counterterms),

$$B(x) \xrightarrow{a\to 0} cC\left(\Gamma_\uparrow \operatorname{tr}[\phi\Psi] + \Gamma_\downarrow \operatorname{tr}\left[\bar{\phi}\Psi\right]\right).$$

- However, because of the lattice Q-symmetry

$$B(x) = \begin{pmatrix} * \\ * \\ * \\ 0 \end{pmatrix} \Rightarrow c = 0 \Rightarrow B(x) \xrightarrow{a\to 0} 0$$

SUSY WT identity

- When $\mathcal{O}$ is a product of elementary fields, in the continuum limit,

$$\partial_\mu \langle s_\mu(x)\mathcal{O}(y_1,\ldots,y_n)\rangle = \frac{\mu^2}{g^2}\langle f(x)\mathcal{O}(y_1,\ldots,y_n)\rangle - i\frac{1}{a^2}\frac{\delta}{\delta\epsilon(x)}\langle\mathcal{O}(y_1,\ldots,y_n)\rangle$$

- $s_\mu(x)$ is a correctly normalized supercurrent that generates SUSY transf. on elementary fields
- Definition the supercurrent is arbitrary as long as $s'_\mu(x) = s_\mu(x) + O(a)$
- Even when $\mathcal{O}$ contains composite operators, if $x \neq y_i$,
  - $B(x) \xrightarrow{a\to 0} 0$
  - Definition the supercurrent is again arbitrary
- This is precisely the case studied in Kanamori-Suzuki, in which

$$\mathcal{O}(y) = f_\nu(y) \equiv -2i\frac{1}{g^2}\Gamma_\nu\left(\Gamma_\uparrow \operatorname{tr}[\phi\Psi] + \Gamma_\downarrow \operatorname{tr}\left[\bar{\phi}\Psi\right]\right)$$

Nonperturbative check (Kanamori, H.S., NPB 811 (2009))

- Continuum limit of the ratio

$$\frac{\partial_\mu \left\langle (s'_\mu)_i(x)(f_0)_i(y)\right\rangle}{\langle (f)_i(x)(f_0)_i(y)\rangle} \xrightarrow{a\to 0} \frac{\mu^2}{g^2} \qquad \text{for } x \neq y?$$

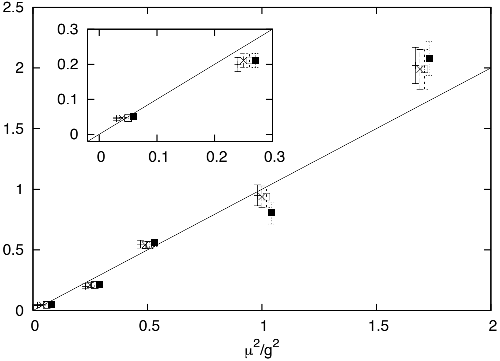


Figure: $G = SU(2)$. antiperiodic BC. $\beta = 2L$, $Lg = \sqrt{2}$. $i = 1$ $(+)$, $i = 2$ $(\times)$, $i = 3$ $(\square)$, $i = 4$ $(\blacksquare)$

SUSY current algebra and hamiltonian density

- When $\mathcal{O}$ is a supercurrent itself, SUSY WT identity provides a SUSY current algebra among correctly normalized current operators
- In particular, for $\mu^2 = 0$,

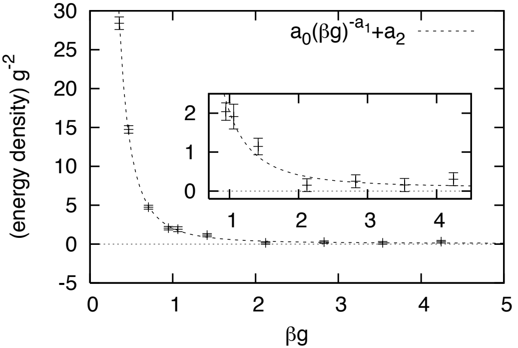
$$\partial_\mu^* \left\langle (s_\mu)_{i=4}(x) \left(s_0'\right)_{i=1}(y)\right\rangle = i\frac{1}{a^2}\delta_{x,y}\left\langle Q\left(s_0'\right)_{i=1}(x)\right\rangle \equiv 2\left\langle \mathcal{H}(x)\right\rangle$$

  even for $a \neq 0$
- This is precisely the prescription for the hamiltonian density advocated by Kanamori-Sugino-Suzuki in the context of the dynamical SUSY breaking
- provides the Nambu-Goldstone theorem for the spontaneous SUSY breaking

Vacuum energy density (Kanamori, PRD 79 (2009))

- can be obtained from the zero temperature limit $\beta \to \infty$ of $\langle \mathcal{H} \rangle$

$$\mathcal{E}_0/g^2 = 0.09 \pm 0.09(\text{sys})\,^{+0.10}_{-0.08}(\text{stat})$$



- Dynamical spontaneous SUSY breaking in this system (Hori-Tong) is unlikely. . .

Less ambiguous confirmation of SUSY WT identity?

- Since we have a small parameter,

$$ag \ll 1$$

  we may use lattice Perturbation Theory (PT)
- Free from Monte Carlo errors. Periodic BC case
- However,
  - PT in infinite volume $L = \infty$ generally suffers from infrared divergences
  - Constant (zero-momentum) modes do not allow perturbative expansion
- Onogi and Takimi (PRD 72 (2005)): scalar 2-point function in lattice model by Kaplan et al.
  - Perturbative integration over non-zero momentum modes
  - Nonperturbative numerical integration over constant modes

Semi-perturbative analysis (periodic BC)

- Finite box with the size $L = Na$ (N: 1-dimensional number of lattice points)
- Gauge potential

$$U_\mu(x) = \exp\left(iA_\mu(x)\right)$$

- Momentum decomposition on a finite lattice

$$A_\mu(x) = \sum_k e^{ikx/a}\,\tilde{A}_\mu(k),$$

$$\phi(x) = \sum_k e^{ikx/a}\,\tilde{\phi}(k), \quad \bar{\phi}(x) = \sum_k e^{ikx/a}\,\tilde{\bar{\phi}}(k),$$

$$\Psi(x) = \sum_k e^{ikx/a}\,\tilde{\Psi}(k),$$

where

$$k_\mu \equiv \frac{2\pi n_\mu}{N}, \quad n_\mu = 0, 1, 2, \dots, N-1$$

Semi-perturbative analysis of SUSY WT identity

- Constant modes are special, because

$$S_{\text{2DSYM}}^{\text{LAT}} = \frac{N^2}{a^2 g^2} \,\mathrm{tr}\left[-\frac{1}{2}[\tilde{A}_\mu(0), \tilde{A}_\nu(0)]^2 + \tilde{\Psi}(0)^T C\Gamma_\mu i[\tilde{A}_\mu(0), \tilde{\Psi}(0)] + \cdots \right.$$

- Therefore,

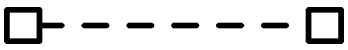
$$\square = \tilde{\Psi}(0) = O((ag)^{3/4}) \qquad \circ = \tilde{A}_\mu(0) \text{ or } \tilde{\phi}(0) = O((ag)^{1/2})$$

and modifies a naive order counting

- The lowest nontrivial order for the SUSY WT identity

$$\partial_\mu \langle s_\mu(x) f_\nu(y)\rangle \overset{?}{=} \frac{\mu^2}{g^2} \langle f(x) f_\nu(y)\rangle \qquad \text{for } x \neq y$$

is $O((ag)^{3/2})$ and, schematically,

∂_μ  $= \dfrac{\mu^2}{g^2}$  $+\,\mathcal{C}$ 

- The blob denotes the scalar 1-loop self-energy

$\mathcal{C} \equiv$ 

- A somewhat lengthy one-loop calculation yields (λ: the gauge parameter)

$$\mathcal{C} = k\frac{2}{N^2}\sum_{(n_0,n_1)\neq(0,0)}\left[\frac{1}{2}\left(1+\frac{1}{\lambda}\right)\frac{1}{\mathring{k}^2}+\frac{1}{2}\left(1-\frac{1}{\lambda}\right)\frac{1}{\hat{k}^2+a^2\mu^2}-\frac{1}{\hat{k}^2}\right]+O(a^2)$$

where

$$\mathring{k}^2 \equiv \sum_{\mu=0}^{1}\left(\mathring{k}_\mu\right)^2, \qquad \hat{k}^2 \equiv \sum_{\mu=0}^{1}\left(\hat{k}_\mu\right)^2, \qquad \mathring{k}_\mu \equiv \sin k_\mu, \qquad \hat{k}_\mu \equiv 2\sin\frac{k_\mu}{2}$$

- We may further neglect $a^2\mu^2$ in the denominator and then

$$\mathcal{C} = 0$$

and shows the SUSY WT identity in the first nontrivial order

Caveat on PT in 2D gauge theory

- In the continuum limit, we have $ag \ll 1$
- Then, everything in the continuum limit can be studied by PT?

No!

- There is a hidden parameter N, the number of lattice points

$$ag \ll 1, \text{ but } Lg = N \times ag \text{ is not necessarily small}$$

- It turns out that (when there are no other high energy scales)

PT is an asymptotic expansion w.r.t. Lg, not simply ag

and reliable only for $Lg \ll 1$, small physical volume ($\Leftrightarrow$ infrared divergence in $L = \infty$)

- If one is interested in low energy physics in large volume, PT is useless. Instead use exact solution, Monte Carlo simulation, etc.

Summary

We

- clarified the implication of SUSY WT identity
- argued the restoration of SUSY WT identity in formal PT
- confirmed the SUSY WT identity in the first nontrivial order by using a semi-perturbative analysis (a small volume expansion)

- Similar analysis is called for 4D SUSY gauge theories...

2D $\mathcal{N} = (2,2)$ SYM (we assume $G = SU(k)$)

- Continuum action (dimensional reduction of 4D $\mathcal{N} = 1$ SYM to 2D)

$$S_{\text{2DSYM}} = \frac{1}{g^2} \int d^2x \ \text{tr} \left[\frac{1}{2} F_{MN} F_{MN} + \Psi^T C \Gamma_M D_M \Psi + \tilde{H}^2 \right]$$

- SUSY

$$\delta A_M = i\epsilon^T C \Gamma_M \Psi, \qquad \delta\Psi = \frac{i}{2} F_{MN} \Gamma_M \Gamma_N \epsilon + i\tilde{H} \Gamma_5 \epsilon$$

$$\delta\tilde{H} = -i\epsilon^T C \Gamma_5 \Gamma_M D_M \Psi$$

- We set $\Gamma_0 = \begin{pmatrix} -i\sigma_1 & 0 \\ 0 & i\sigma_1 \end{pmatrix}$, $\Gamma_1 = \begin{pmatrix} i\sigma_3 & 0 \\ 0 & -i\sigma_3 \end{pmatrix}$, $\Gamma_2 = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}$, $\Gamma_3 = C = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

$$\Psi^T \equiv (\psi_0, \psi_1, \chi, \eta/2), \qquad \epsilon^T \equiv -(\varepsilon^{(0)}, \varepsilon^{(1)}, \tilde{\varepsilon}, \varepsilon),$$

and first focus on a particular fermionic transf. Q

$$\delta \equiv \varepsilon^{(0)} Q^{(0)} + \varepsilon^{(1)} Q^{(1)} + \tilde{\varepsilon} \tilde{Q} + \varepsilon Q$$

2D $\mathcal{N} = (2,2)$ SYM

- Q-transformation

$$
\begin{aligned}
QA_\mu &= \psi_\mu, & Q\psi_\mu &= iD_\mu\phi, \\
Q\phi &= 0, & & \\
Q\bar{\phi} &= \eta, & Q\eta &= [\phi, \bar{\phi}], \\
Q\chi &= H, & QH &= [\phi, \chi],
\end{aligned}
$$

where

$$
\phi \equiv A_2 + iA_3, \qquad \bar{\phi} = A_2 - iA_3, \qquad H \equiv \tilde{H} + iF_{01},
$$

is nilpotent

$$
Q^2 = \delta_\phi \simeq 0 \text{ on gauge invariant combinations}
$$

δ_ϕ: an infinitesimal gauge transformation with the parameter ϕ

- Continuum action is moreover Q-exact

$$
S_{\text{2DSYM}} = Q\frac{1}{g^2}\int d^2x \,\text{tr}\left[-2i\chi F_{01} + \chi H + \frac{1}{4}\eta[\phi, \bar{\phi}] - i\psi_\mu D_\mu \bar{\phi}\right]
$$