高次元場の理論の量子化と新しい正則化法

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[1]flat: arXiv:0801.3064 PTP121(2009)
warped: arXiv:0812.1253; arXiv:0903.4971

1. Introduction

Casimir Energy of 4D Electromagnetism

$$
\begin{aligned}
&\text{Harmonic Oscillator } (\omega : \ \text{frequency}) \\
E = \frac{1}{2}\omega + \frac{\omega}{e^{\beta\omega}-1} &= \frac{1}{2}\omega + \omega \cdot \frac{e^{-\beta\omega}}{1-e^{-\beta\omega}} \quad , \\
\omega \quad --> \quad F(\omega) &: \ \text{energy density operator} \\
e^{-\beta\omega} \quad --> \quad W(\omega,\beta) &: \ \text{Weight function, Wien's damping factor} \\
\omega \cdot \frac{1}{1-e^{-\beta\omega}} \sim \frac{1}{\beta} \quad --> &\ \text{Rayley-Jeans region}
\end{aligned}
\tag{1}
$$

$$\frac{E_{Cas}}{(2L)^2} = \frac{\pi^2}{(2l)^3}\frac{B_4}{4!} = -\frac{\pi^2}{720}\frac{1}{(2l)^3} \quad , \quad l \ : \ \text{boundary parameter} \quad , \tag{2}$$

Figure 1: Graph of Planck's radiation formula. $\mathcal{P}(\beta, k) = \frac{1}{(c\hbar)^3}\frac{1}{\pi^2}k^3/(\mathrm{e}^{\beta k} - 1)$ $(1 \leq \beta \leq 2,\ 0.01 \leq k \leq 10)$.

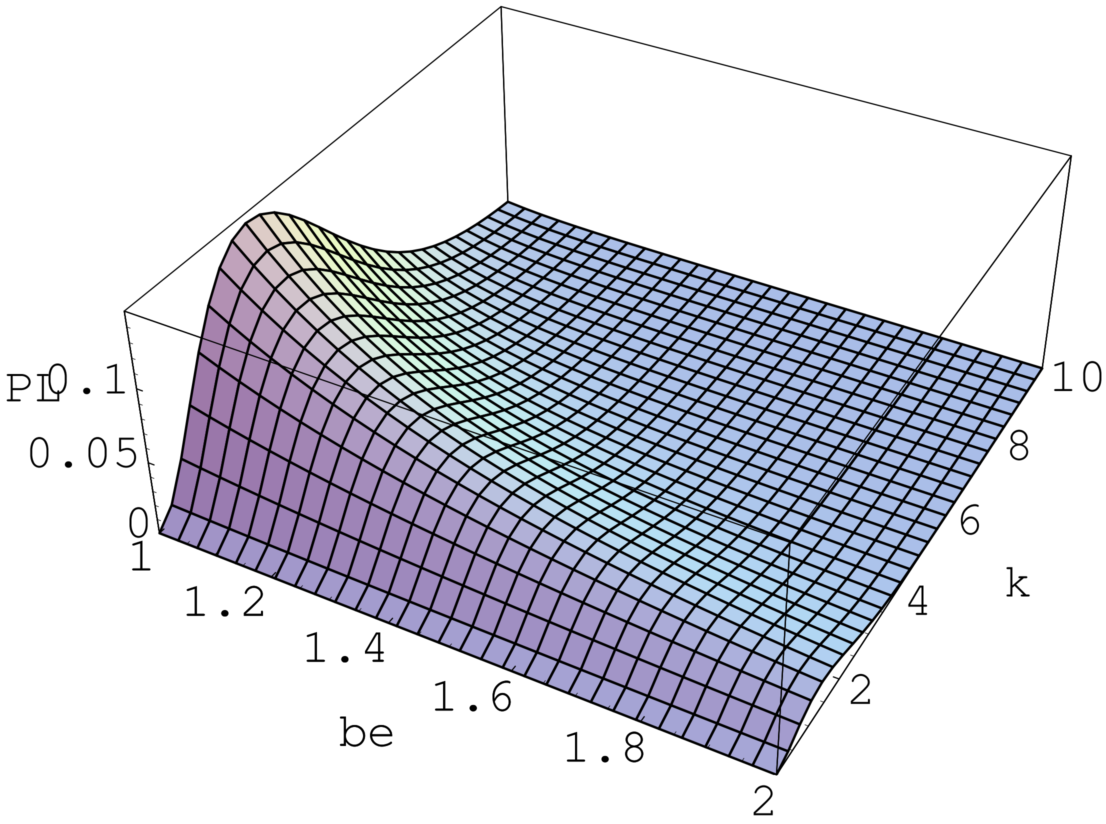
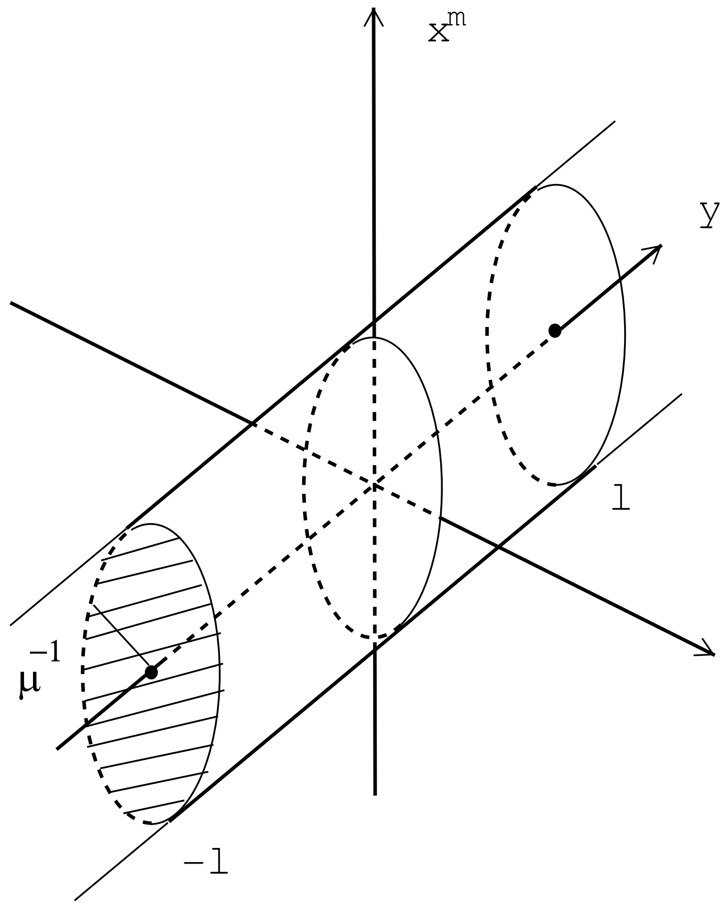


Figure 2: Regularized Flat Geometry, Appelquist and Chodos (1983)



5D Electromagnetism on the *flat* geometry

The extra space is *periodic* (periodicity $2l$) and Z_2-parity

$$ds^2 = \eta_{\mu\nu}dx^\mu dx^\nu + dy^2 \quad , \quad -\infty < x^\mu, y < \infty \quad , \quad y \to y + 2l,\ y \leftrightarrow -y \quad ,$$

$$(\eta_{\mu\nu}) = \mathrm{diag}(-1,1,1,1)\ ,(X^M) = (X^\mu = x^\mu, X^5 = y) \equiv (x,y)\ ,$$

$$M,N = 0,1,2,3,5;\ \ \mu,\nu = 0,1,2,3. \quad (3)$$

The Casimir energy

$$E_{Cas}(\Lambda,l) = \frac{2\pi^2}{(2\pi)^4}\int_{1/l}^{\Lambda} d\tilde{p}\int_{1/\Lambda}^{l} dy\ \tilde{p}^3 W(\tilde{p},y)F(\tilde{p},y) \quad ,$$

$$F(\tilde{p},y) \equiv F^-(\tilde{p},y) + 4F^+(\tilde{p},y) = \int_{\tilde{p}}^{\Lambda} d\tilde{k}\frac{-3\cosh\tilde{k}(2y-l) - 5\cosh\tilde{k}l}{2\sinh(\tilde{k}l)} \quad . \quad (4)$$

Λ the 4D-momentum cutoff; $W(\tilde{p},y)$ the *weight function*

1) Un-weighted case: $W=1$

Un-restricted integral region :

$$E_{Cas}(\Lambda,l)=\frac{1}{8\pi^2}\left[-0.1249l\Lambda^5-(1.41,0.706,0.353)\times 10^{-5}\ l\Lambda^5\ln(l\Lambda)\right] \quad ,$$

Randall-Schwartz integral region :

$$E^{RS}_{Cas}=\frac{1}{8\pi^2}[-0.0894\ \Lambda^4] \quad . \quad (5)$$

2) Weighted case

$$E^{W}_{Cas}/\Lambda l=$$

$$
\begin{cases}
-2.50\frac{1}{l^4} + (-0.142, 1.09, 1.13)\times 10^{-4}\frac{\ln(l\Lambda)}{l^4} & \text{for } \ W_1 = (1/N_1)\mathrm{e}^{-(1/2)l^2\tilde{p}^2-(1/2)y^2/l^2} \\
-6.04\times 10^{-2}\frac{1}{l^4} & \text{for } \ W_2 = (1/N_2)\mathrm{e}^{-\tilde{p}y} \\
-2.51\frac{1}{l^4} + (19.5, 11.6, 6.68)\times 10^{-4}\frac{\ln(l\Lambda)}{l^4} & \text{for } \ W_8 = (1/N_8)\mathrm{e}^{-(l^2/2)(\tilde{p}^2+1/y^2)}
\end{cases}
$$

(W_1: elliptic, W_2: hyperbolic, W_8: reciprocal).

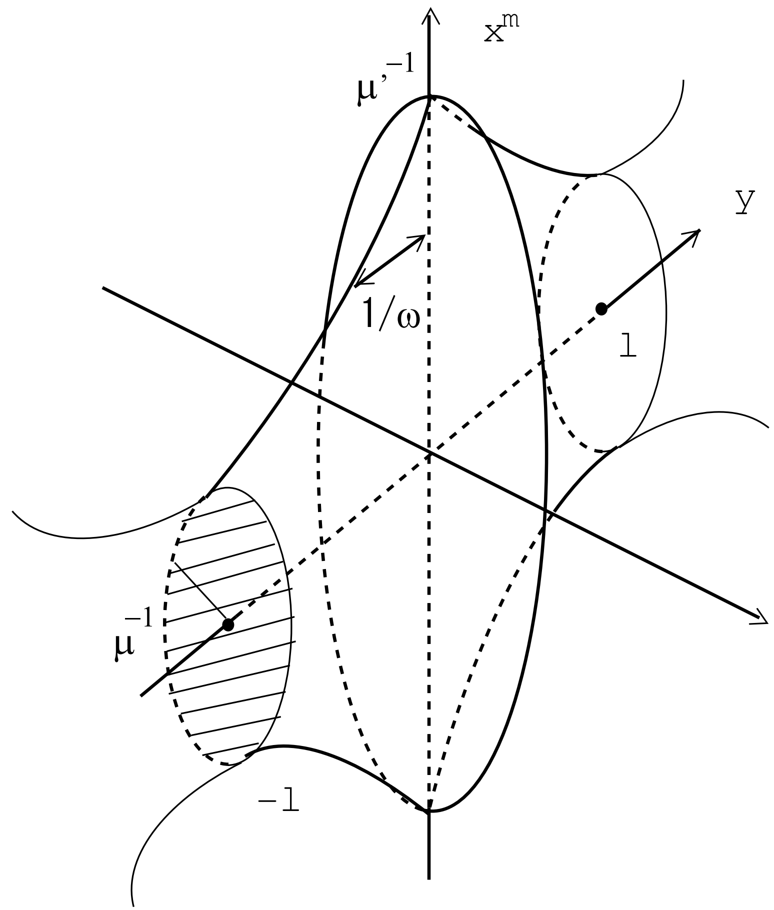
The renormalization of the compactification size l.

$$
E^W_{Cas}/\Lambda l = -\frac{\alpha}{l^4}(1-4c\ln(l\Lambda)) = -\frac{\alpha}{l'^4} \quad , \qquad (7)
$$

$$
- - - - - > \quad \text{attractive Casimir force}
$$

♡ Λl: the normalization factor ♡ c: pure number ♡ β-func of l

Figure 3: Regularized Warped Geometry



2. Heat-Kernel Approach and Position/Momentum Propagator

$$G_p^{\mp}(z,z') = \mp\frac{\omega^3}{2}z^2{z'}^2\frac{\{\mathbf{I}_0(\frac{\tilde{p}}{\omega})\mathbf{K}_0(\tilde{p}z)\mp\mathbf{K}_0(\frac{\tilde{p}}{\omega})\mathbf{I}_0(\tilde{p}z)\}\{\mathbf{I}_0(\frac{\tilde{p}}{T})\mathbf{K}_0(\tilde{p}z')\mp\mathbf{K}_0(\frac{\tilde{p}}{T})\mathbf{I}_0(\tilde{p}z')\}}{\mathbf{I}_0(\frac{\tilde{p}}{T})\mathbf{K}_0(\frac{\tilde{p}}{\omega})-\mathbf{K}_0(\frac{\tilde{p}}{T})\mathbf{I}_0(\frac{\tilde{p}}{\omega})} \quad .$$

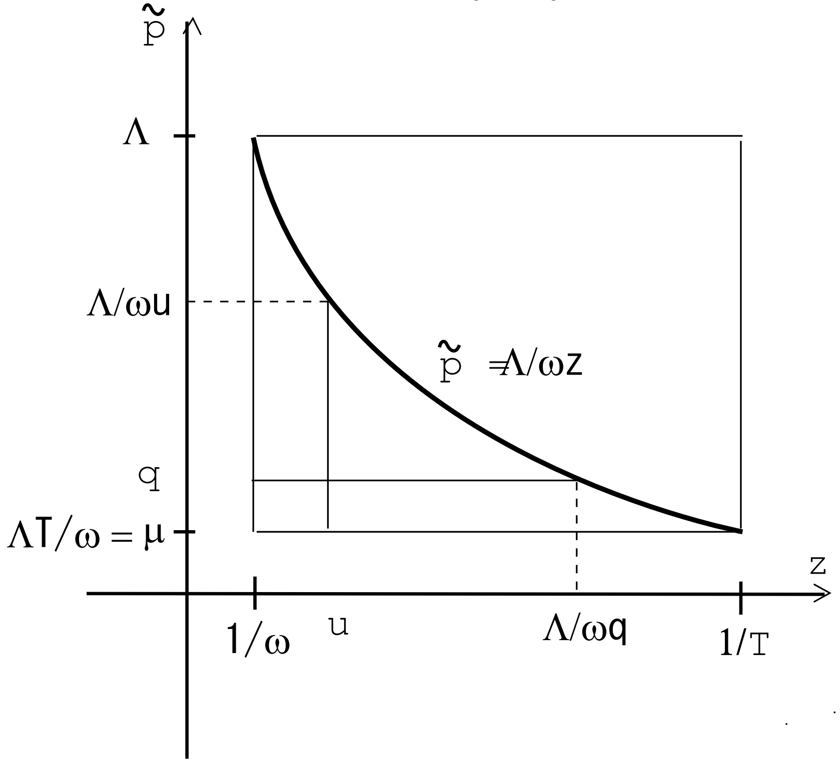
Λ-regularized Casimir energy.

$$E_{Cas}^{\Lambda,\mp}(\omega,T) = \int \frac{d^4p}{(2\pi)^4}\bigg|_{\tilde{p}\leq\Lambda}\int_{1/\omega}^{1/T} dz\ F^{\mp}(\tilde{p},z) \quad ,$$

$$F^{\mp}(\tilde{p}, z) = \frac{2}{(\omega z)^3} \int_{\tilde{p}}^{\Lambda} \tilde{k}\; G_k^{\mp}(z, z) d\tilde{k} \equiv \int_{\tilde{p}}^{\Lambda} \mathcal{F}^{\mp}(\tilde{k}, z) d\tilde{k} \quad . \tag{9}$$

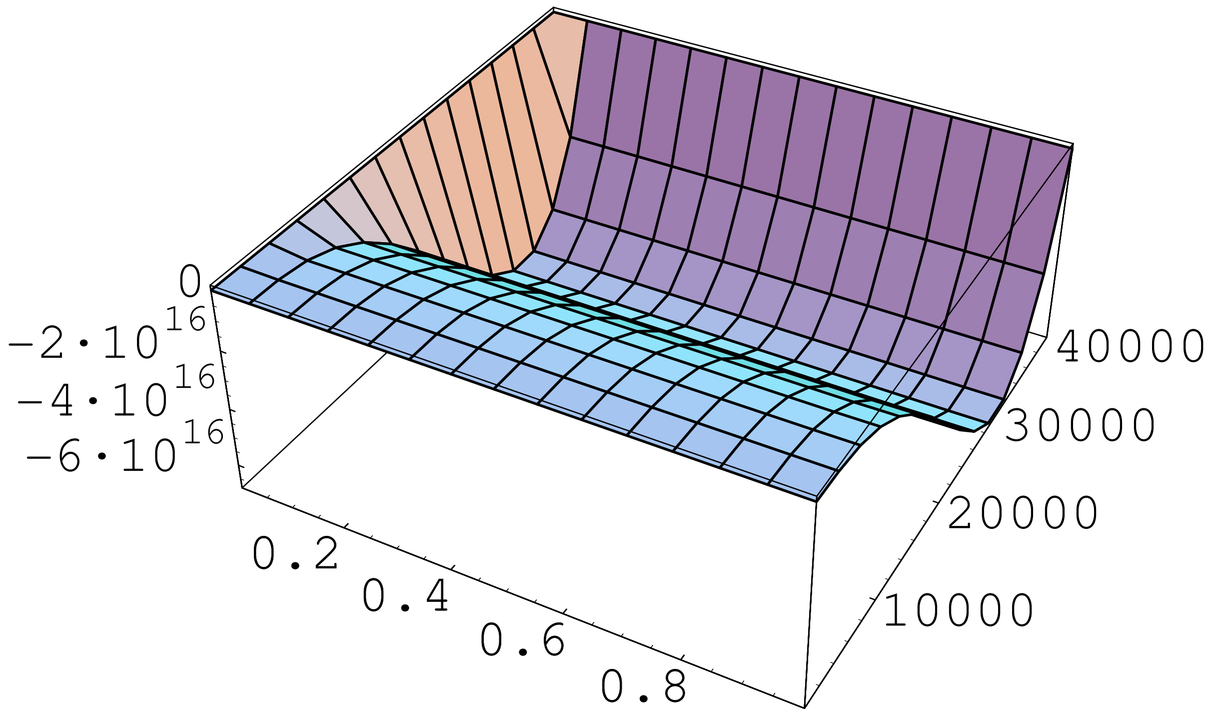
3. UV and IR Regularization Parameters and Evaluation of Casimir Energy

Figure 4: Space of (z,$\tilde{p}$) for the integration. The hyperbolic curve will be used.



(Λ, T)-*regularized* value of (9). The integral region: *rectangle* shown in Fig.4.

Figure 5: Behaviour of $(-1/2)\tilde{p}^3 F^-(\tilde{p}, z)$ (9). $T = 1, \omega = 10^4, \Lambda = 4 \cdot 10^4$. $1.0001/\omega \leq z < 0.9999/T$, $\Lambda T/\omega \leq \tilde{p} \leq \Lambda$.



$$E_{Cas}^{\Lambda,-}(\omega,T)=\frac{2\pi^2}{(2\pi)^4}\times\left[-0.0250\frac{\Lambda^5}{T}\right]\quad , \tag{10}$$

which does not depend on ω. No log-term. (Note: $0.025=1/40.$)

4. UV and IR Regularization Surfaces, Principle of Minimal Area and Renormalization Flow

One proposal of this was presented by Randall and Schwartz('01).
ポイント：　高い対称性を要求しないで自由度を下げる

$$E_{Cas}^{-RS}(\omega, T) = \frac{2\pi^2}{(2\pi)^4}\frac{\Lambda^5}{\omega}\left\{-1.58\times 10^{-2} - 1.69\times 10^{-4}\ln\frac{\Lambda}{\omega}\right\} \quad , \qquad (11)$$

which is *independent* of T .

Figure 6: Space of ($\tilde{p}$,z) for the integration (present proposal).

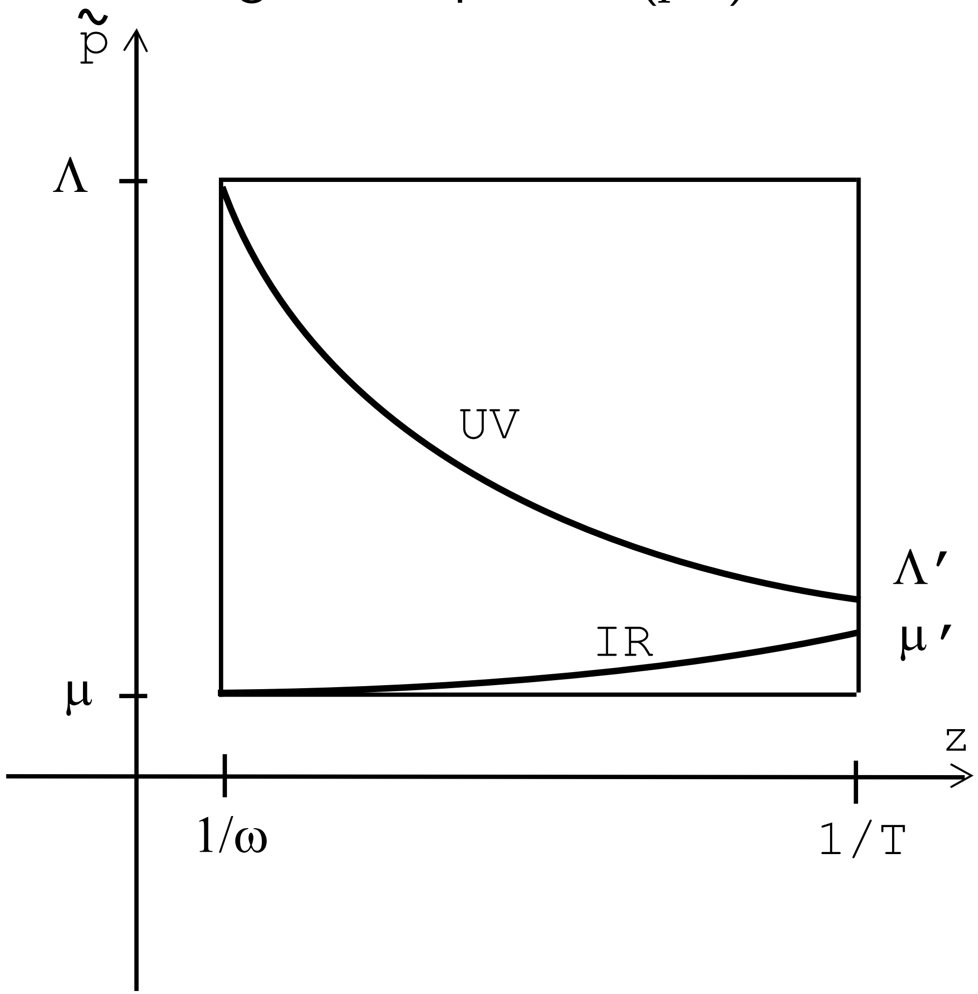


Figure 7: Reg. Surf. B_{IR} and B_{UV} in 5D (x^μ, z). Sphere Lattice

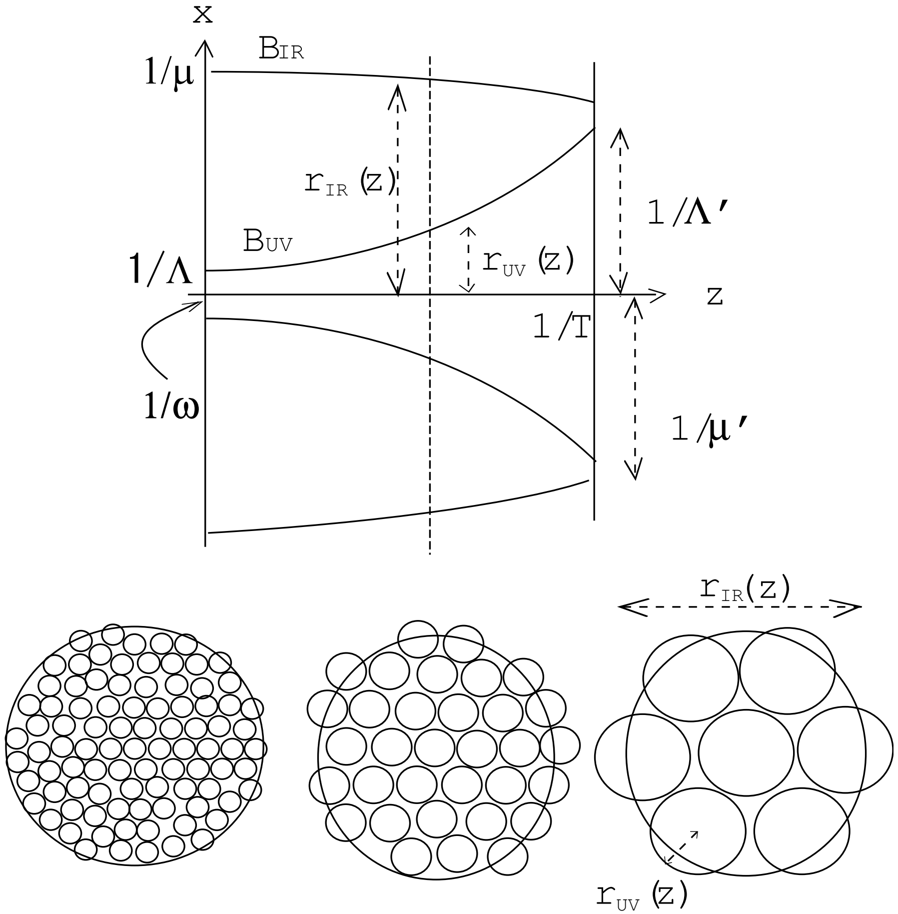


Figure 8: UV regularization surface in 5D coordinate space.

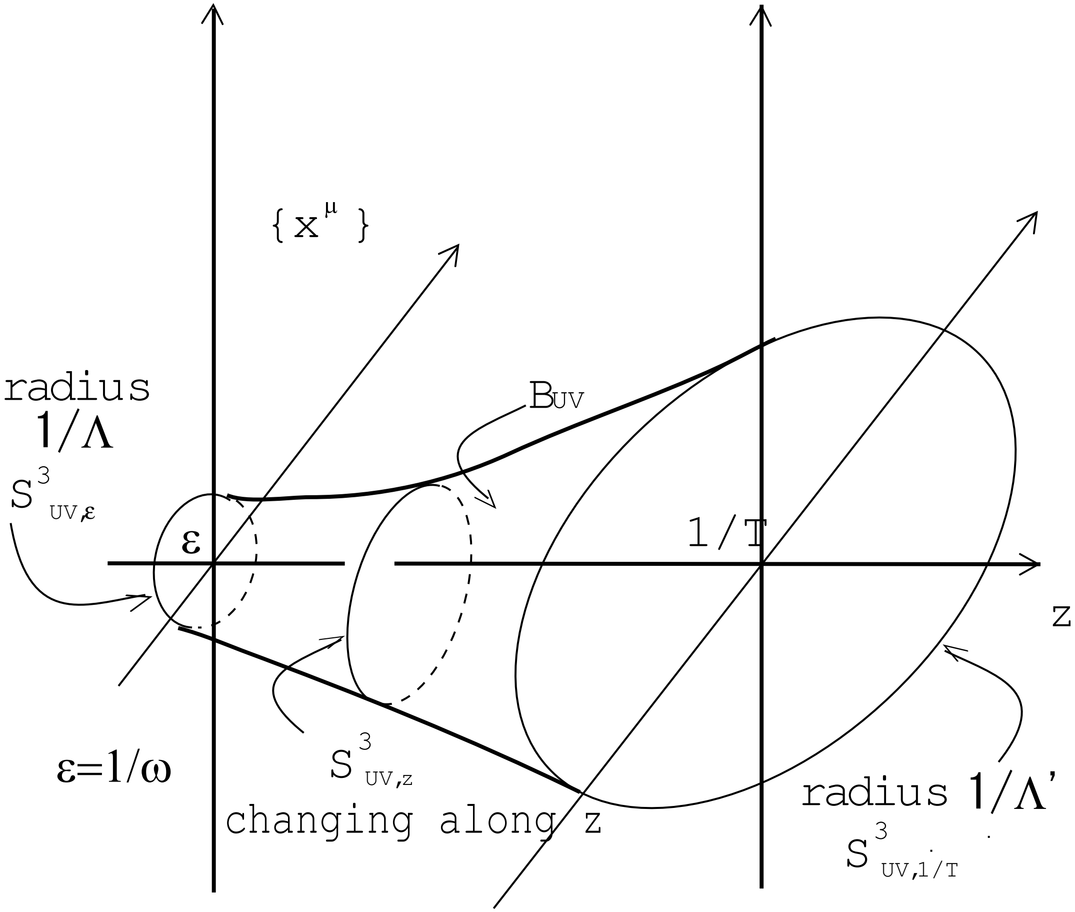
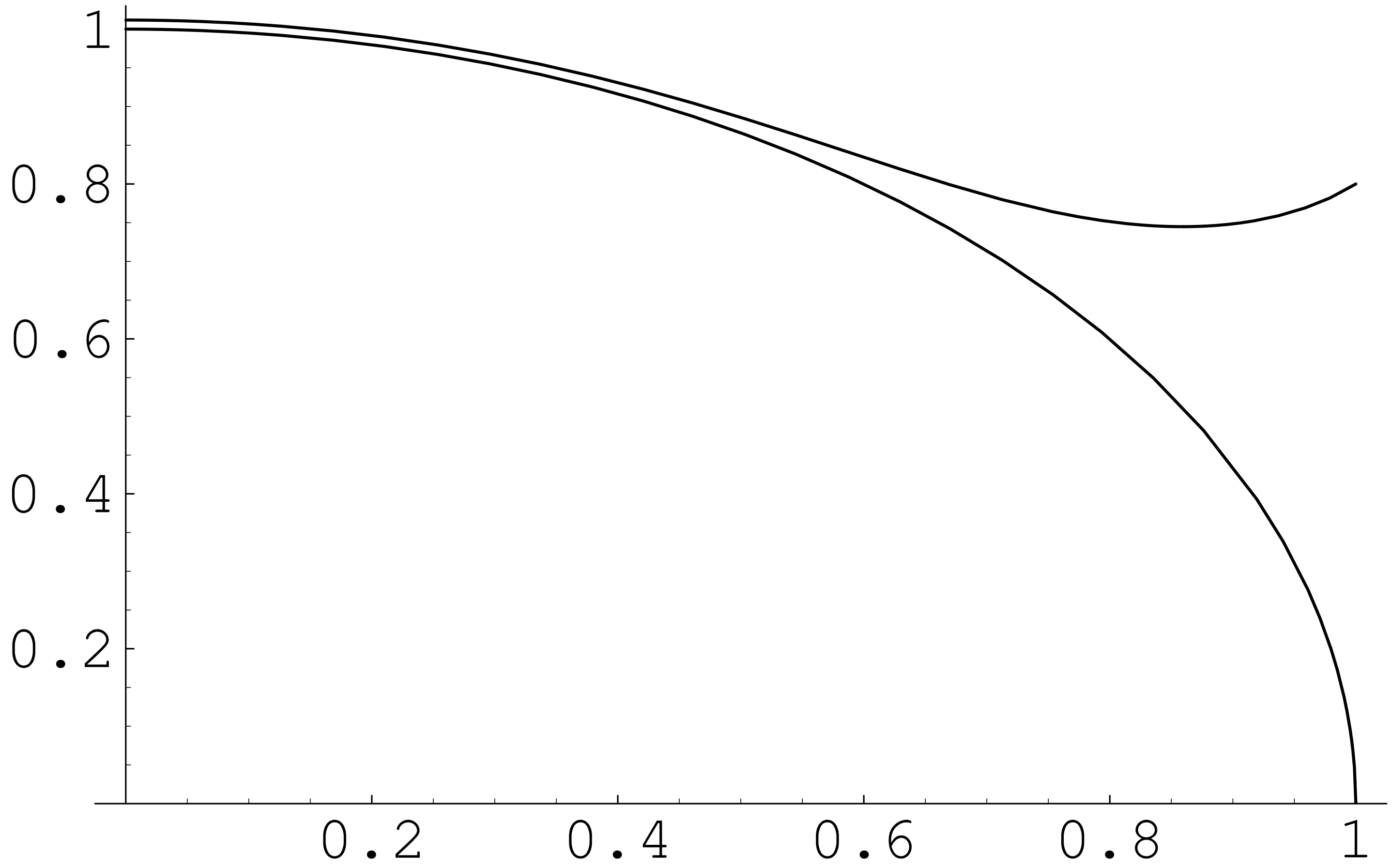


Figure 9: Numerical Solution of Minimal Surface. Vertical axis: r; Horizontal axis: z. $T = 1, \omega = 10^4, 10^{-4} \leq z \leq 1.0$. Upper ($\mathsf{B}_{IR}$): $r(1) = 0.8, r'(1) = 1.0$,; Lower (B_{UV}): $r(1) = 10^{-4}, r'(1) = -1.0$. Both curves are Graph Type (ia).



5. Weight Function and Casimir Energy Evaluation

$$E^{\mp\ W}_{Cas}(\omega, T) \equiv \int \frac{d^4p}{(2\pi)^4} \int_{1/\omega}^{1/T} dz\ W(\tilde{p}, z) F^{\mp}(\tilde{p}, z)$$

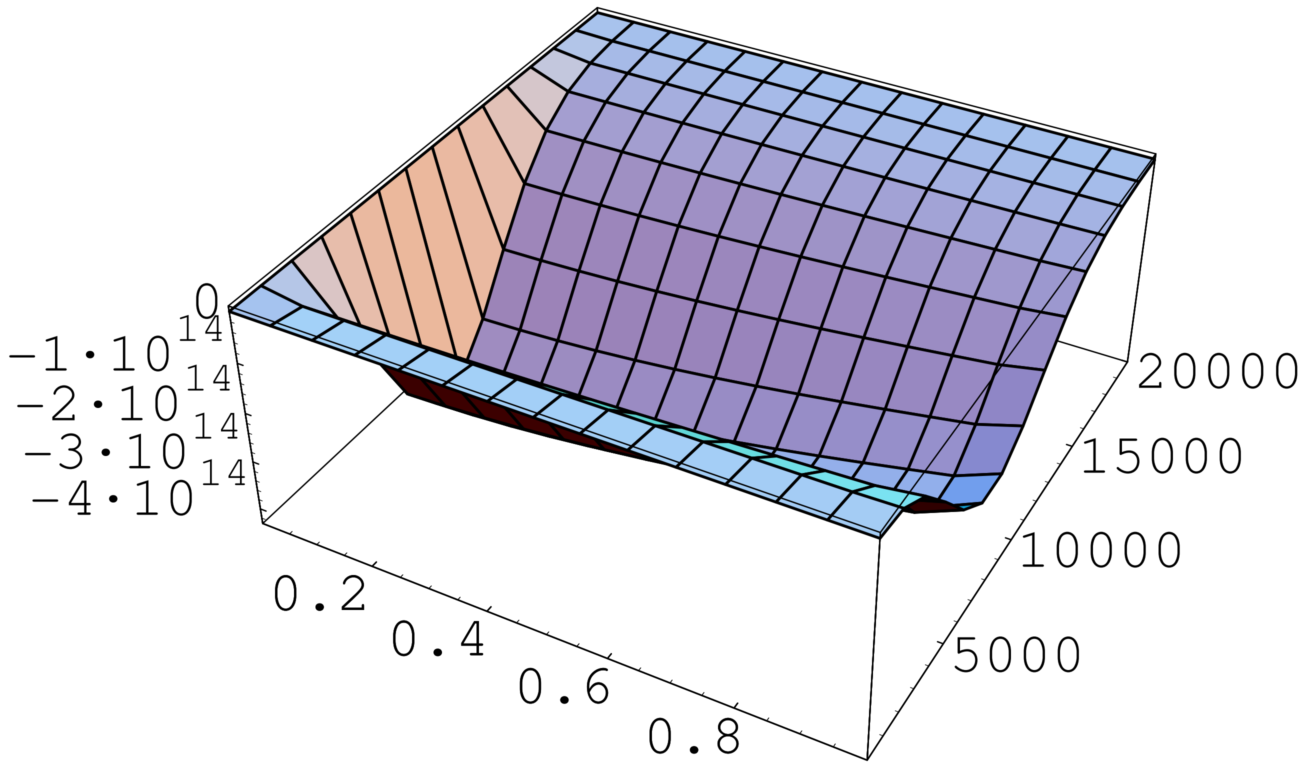
$$F^{\mp}(\tilde{p}, z) = s(z) \int_{p^2}^{\infty} \{G^{\mp}_k(z, z)\} dk^2 = \frac{2}{(\omega z)^3} \int_{\tilde{p}}^{\infty} \tilde{k}\ G^{\mp}_k(z, z) d\tilde{k}$$

Examples of $W(\tilde{p}, z)$: $\quad W(\tilde{p}, z) =$

$$\begin{cases} (N_1)^{-1} e^{-(1/2)\tilde{p}^2/\omega^2 - (1/2)z^2T^2} \equiv W_1(\tilde{p}, z),\ N_1 = 1.711/8\pi^2 & \text{elliptic suppression} \\ (N_2)^{-1} e^{-\tilde{p}zT/\omega} \equiv W_2(\tilde{p}, z),\ N_2 = 2\frac{\omega^3}{T^3}/8\pi^2 & \text{hyperbolic suppression1} \\ (N_8)^{-1} e^{-1/2(\tilde{p}^2/\omega^2 + 1/z^2T^2)} \equiv W_8(\tilde{p}, z),\ N_8 = 0.4177/8\pi^2 & \text{reciprocal suppression1} \end{cases}$$

where $G_k^{\mp}(z,z)$ are defined in (8). N_i are normalization constants. We show the shape of the energy integrand $(-1/2)\tilde{p}^3 W_1(\tilde{p},z)F^{-}(\tilde{p},z)$ in Fig.10.

Figure 10: Behavior of $(-1/2)\tilde{p}^3 W_1(\tilde{p}, z) F^-(\tilde{p}, z)$(elliptic suppression). $\Lambda = 20000,\ \omega = 5000,\ T = 1$. $1.0001/\omega \leq z \leq 0.9999/T,\ \mu = \Lambda T/\omega \leq \tilde{p} \leq \Lambda$.



We can check the divergence (scaling) behavior of $E_{Cas}^{\mp\ W}$ by $numerically$ evaluating the $(\tilde{p}, z)$-integral (12) for the rectangle region of Fig.4.

$$-E_{Cas}^{W} =$$

$$\left\{ \begin{array}{ll} \frac{\omega^4}{T}\Lambda \cdot 1.2\left\{1 + 0.11\ \ln\frac{\Lambda}{\omega} - 0.10\ \ln\frac{\Lambda}{T}\right\} & \text{for} \quad W_1 \\ \frac{T^2}{\omega^2}\Lambda^4 \cdot 0.062\left\{1 + 0.03\ \ln\frac{\Lambda}{\omega} - 0.08\ \ln\frac{\Lambda}{T}\right\} & \text{for} \quad W_2 \\ \frac{\omega^4}{T}\Lambda \cdot 1.6\left\{1 + 0.09\ \ln\frac{\Lambda}{\omega} - 0.10\ \ln\frac{\Lambda}{T}\right\} & \text{for} \quad W_8 \end{array} \right. \quad (13)$$

They give, after normalizing the factor Λ/T, only the log-divergence.

$$E_{Cas}^{W}/\Lambda T^{-1} = -\alpha\omega^4\left(1 - 4c\ln(\Lambda/\omega) - 4c'\ln(\Lambda/T)\right) \quad , \quad (14)$$

This means the 5D Casimir energy is finitely obtained by the ordinary renormalization of the warp factor ω. In the above result of the warped case, the IR parameter l in the flat result (7) is replaced by the inverse of the warp factor ω.

6. Meaning of Weight Function and Quantum Fluctuation of Coordinates and Momenta

In order to most naturally accomplish the above requirement, we can go to a *new step*. Namely, we *propose* to *replace* the 5D space integral with the weight W, by the following path-integral. We newly define the Casimir energy in the higher-dimensional theory as follows.

$$\mathcal{E}_{Cas}(\omega, T, \Lambda) \equiv \int_{1/\Lambda}^{1/\mu} d\rho \int_{\tilde{p}(1/\omega)=\tilde{p}(1/T)=1/\rho} \prod_{a,z} \mathcal{D}p^a(z) F(\tilde{p}, z)$$
$$\times \exp\left[-\frac{1}{2\alpha'} \int_{1/\omega}^{1/T} \frac{1}{\omega^4 z^4} \frac{1}{\tilde{p}^3} \sqrt{\frac{\tilde{p}'^2}{\tilde{p}^4} + 1}\, dz \right]$$

$$= \int_{1/\Lambda}^{1/\mu} d\rho \int_{r(1/\omega)=r(1/T)=\rho} \prod_{a,z} \mathcal{D}x^a(z) F(\frac{1}{r}, z)$$

$$\times \exp\left[-\frac{1}{2\alpha'} \int_{1/\omega}^{1/T} \frac{1}{\omega^4 z^4} \sqrt{r'^2 + 1}\, r^3 dz \right] \quad , \qquad (15)$$

where $\mu = \Lambda T/\omega$ and the limit $\Lambda T^{-1} \to \infty$ is taken. The string (surface) tension parameter $1/2\alpha'$ is introduced. (Note: Dimension of α' is [Length]^4.) The square-bracket ($[\cdots]$)-parts of (15) are $-\frac{1}{2\alpha'}$Area $= -\frac{1}{2\alpha'} \int \sqrt{\mathrm{det} g_{ab}} d^4x$ where g_{ab} is the induced metric on the 4D surface. $F(\tilde{p}, z)$ is defined in (12) or (9) and shows the *field-quantization* of the bulk scalar (EM) fields.

The proposed definition, (15), clearly shows the 4D space-coordinates x^a or the 4D momentum-coordinates p^a are quantized (quantum-statistically, not field-theoretically) with the Euclidean time z and the "area Hamiltonian" $A =$

$\int \sqrt{\det g_{ab}}\ d^4x$. Note that $F(\tilde{p}, z)$ or $F(1/r, z)$ appears, in (15), as the energy density operator in the quantum statistical system of $\{p^a(z)\}$ or $\{x^a(z)\}$.
cf. T. Yoneya, 1987, "Duality and Indeterminancy Principle in String Theory". PTP103(2000)1081

7. Discussion and Conclusion

$$E^W_{Cas}/\Lambda T^{-1} = -\alpha\omega^4\,(1 - 4c\ln(\Lambda/\omega) - 4c'\ln(\Lambda/T)) = -\alpha(\omega_r)^4 \quad ,$$
$$\omega_r = \omega\ \sqrt[4]{1 - 4c\ln(\Lambda/\omega) - 4c'\ln(\Lambda/T)} \quad . \qquad (16)$$

we find the renormalization group function for the warp factor ω as

$$|c| \ll 1 \quad , \quad |c'| \ll 1 \quad , \quad \omega_r = \omega(1 - c\ln(\Lambda/\omega) - c'\ln(\Lambda/T)) \quad ,$$
$$\beta(\beta\text{-function}) \equiv \frac{\partial}{\partial(\ln\Lambda)}\ln\frac{\omega_r}{\omega} = -c - c' \quad . \qquad (17)$$

We should notice that, in the flat geometry case, the IR parameter (extra-space size) l is renormalized. In the present warped case, however, the corresponding parameter T is *not renormalized*, but the *warp parameter* ω *is renormalized*. Depending on the sign of $c+c'$, the 5D bulk curvature ω *flows* as follows. When $c+c'>0$, the bulk curvature ω decreases (increases) as the the measurement energy scale Λ increases (decreases). When $c+c'<0$, the flow goes in the opposite way.

Application to Cosmological Constant Problem

$$\frac{1}{G_N}\lambda_{obs} \sim \frac{1}{G_N {R_{cos}}^2} \sim m_\nu^4 \sim (10^{-3}eV)^4 \quad , \tag{18}$$

where R_{cos} is the cosmological size (Hubble length), m_ν is the neutrino mass.

$$\frac{1}{G_N}\lambda_{th} \sim \frac{1}{{G_N}^2} = {M_{pl}}^4 \sim (10^{28}eV)^4 \quad . \tag{19}$$

The famous huge discrepancy factor: $\lambda_{th}/\lambda_{obs} \sim 10^{124}$. If we apply the present approach, we have the warp factor ω, and the result (16) strongly suggests the following choice:

$$\frac{1}{G_N}\tilde{\lambda}_{th} = -\alpha_1 \omega^4 \quad , \quad \alpha_1 \ : \ \text{some coefficient} \quad ,$$

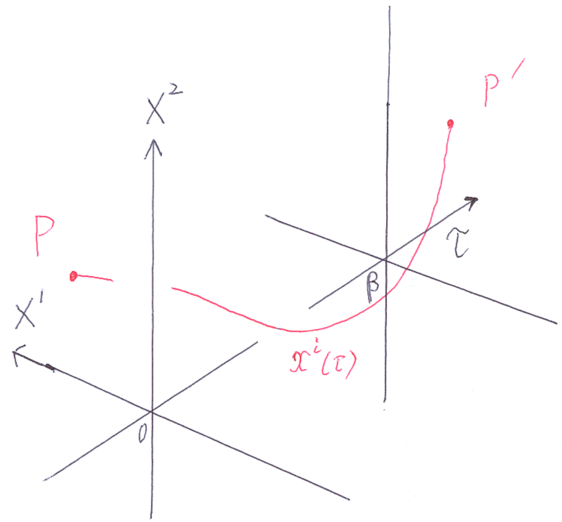
$$\omega \sim \frac{1}{\sqrt[4]{G_N {R_{cos}}^2}} = \sqrt{\frac{M_{pl}}{R_{cos}}} \sim m_\nu \sim 10^{-3}\text{eV} \quad . \qquad (20)$$

We succeed in obtaining the finiteness and its gross absolute value of the cosmological constant. Now we understand that the smallness of the cosmological constant comes from the renormalization flow for the non asymptotic-free case ($c + c' < 0$ in (17)).

Further Meaning of the Proposed Casimir Energy

Geometrical Approach to N Harmonic Oscillators

Figure 11: A path $\{x^i(\tau)|i = 1, 2, \cdots, N\}$ in N(=2)+1 dim space.



N+1 dim Euclidean space $(X^i, \tau);\ i = 1, 2, \cdots N$

$$ds^2 = d\tau^{-2}\{\sum_{i=1}^{N}(dX^i)^2\}^2 + \omega^4\{\sum_{i=1}^{N}(X^i)^2\}^2 d\tau^2 + 2\omega^2\{\sum_{i=1}^{N}(X^i)^2\}\{\sum_{j=1}^{N}(dX^j)^2\} \quad (21)$$

On a path $\{x^i(\tau)\}$, we have the induced metric and the length L is

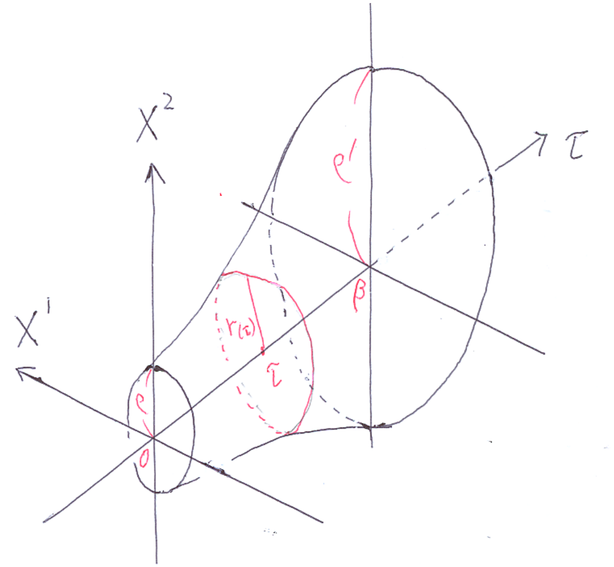
$$L[x^i(\tau)] = \int ds = \int_0^\beta \sum_{i=1}^{N}((\dot{x}^i)^2 + \omega^2(x^i)^2)d\tau \quad . \quad (22)$$

Taking $\frac{1}{2}L$ as the Hamiltonian, the free energy F is

$$\mathrm{e}^{-\beta F} = \left(\prod_i \int_{-\infty}^{\infty} d\rho_i \right) \int_{\substack{x^i(0) = \rho_i \\ x^i(\beta) = \rho_i}} \prod_{i,\tau} \mathcal{D}x^i(\tau) \exp\left[-\frac{1}{2} \int_0^\beta \sum_{i=1}^{N} ((\dot{x}^i)^2 + \omega^2 (x^i)^2) d\tau \right]$$

This is the free energy of N HOs.

Figure 12: N(=2) dim hypersurface in N+1 dim space $(X^1, X^2, \cdots, X^N, \tau)$.



Instead of (21), we start from

$$ds^2 = \omega^4 \{\sum_{i=1}^{N} (X^i)^2\}^2 d\tau^2 + 2\omega^2 \{\sum_{i=1}^{N} (X^i)^2\}\{\sum_{i=1}^{N} (dX^i)^2\} \quad , \tag{24}$$

On the hypersurface $\sum_{i=1}^{N} (X^i)^2 = r^2(\tau)$, the induced metric gives us the area

$$A_N = \int \sqrt{\det g_{ij}} d^N X = \int_0^\beta (\omega r)^N \sqrt{4\dot{r}^2 + 1}\ r^{N-1} d\tau \quad . \tag{25}$$

Taking A_N as the Hamiltonian, the free energy is

$$\mathrm{e}^{-\beta F} = \int_0^\infty d\rho \int_{\substack{r(0)=\rho \\ r(\beta)=\rho}} \prod_{\tau, i} \mathcal{D}x^i(\tau) \exp\left[-\frac{1}{2}\int (\omega r)^N \sqrt{4\dot{r}^2 + 1}\ r^{N-1} d\tau\right] \quad . \tag{26}$$

Similar to the proposed 5D Casimir energy (Warped case).