On BPS vortices in gauge theories with general non-Abelian groups.

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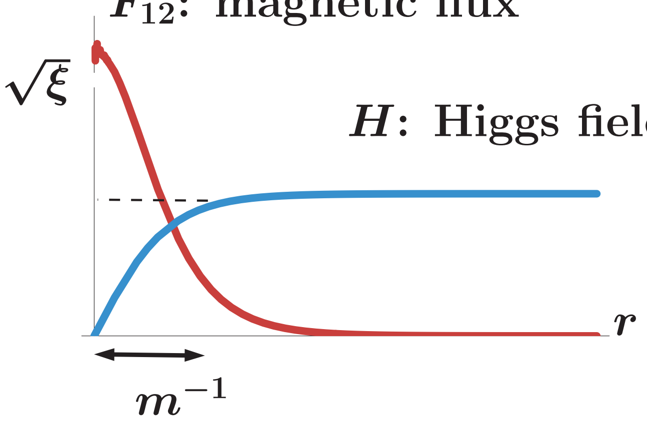
§1. Introduction & Motivation

- Vortex is a magnetic flux (with two codim.) in Higgs phase.

We consider a topological BPS soliton supported by $\pi_1(U(1)) = \mathbb{Z}$.

- A vortex in Abelian gauge theory

Abrikosov('57) and Nielsen-Olesen('73) (ANO vortex).
Moduli space for multi vortices: Taubes('80).



no known analytic solution

exponential tails e^{-mr}(local-type), $(m = g\sqrt{\xi})$

- Extension to vortices in the $U(N_c)$ gauge theory

  Hanany-Tong, Konishi et al.

orientational moduli

$$F_{12} = U\mathrm{diag}(F_{12}^{\mathrm{ANO}}, 0, \cdots, 0)U^{\dagger}, \to \mathbb{C}P^{N_c-1} \simeq \frac{SU(N)}{SU(N-1) \otimes U(1)}$$

- The moduli matrix formalizm determines moduli spaces for 1/2 BPS vortices in the $U(N_c)$ gauge theory completely. (EINOS('05))

- Today's topic:

Extention the moduli matrix formalizm into

arbitrary gauge group $G = [U(1) \times G']/\mathbb{Z}_{n_0}$

$\Rightarrow$ non-trivial (anisotropic) vacuum moduli

$$\Rightarrow \begin{cases} \text{polynomial tails } 1/r^n \text{ (semilocal-type)} \\ \text{fractional vortices: multi peaks even for the minimal vortex solution} \end{cases}$$

§2. 1/2 BPS Equations for Vortices

- Our model: ($\mathcal{N}=2$ SUSY) $G=(U(1)\times G')/\mathbb{Z}_{n_0}$ gauge theory
  with N_f fundamental fields and FI term

The scalar potential of this model

$$V_{\rm pot} = \frac{e^2}{4}\left(\xi - {\rm Tr}[H^\dagger H]\right)^2 + \frac{g^2}{4}\sum_{I\geq 1}\left({\rm Tr}[H^\dagger T_I H]\right)^2$$

$(H)^{rA} \equiv H^{rA}$: chiral fields, complex $N_c \times N_f$ matrix

color $r=1,\cdots,N_c(=2N)$, flavor $A=1,2,\cdots N_f(=2N)$

$\xi(>0)$: Fayet-Illiopoulos term

e, g: gauge couplings for $U(1)$ and G'

T_I $(I=1,\cdots)$: generators of G'

- Dimension of the vacuum moduli space $\mathcal{M}_{\rm vac}$

$$\dim_{\mathbb{C}}\mathcal{M}_{\rm vac} = N_{\rm c}N_{\rm f} - \dim G \geq 0, \quad N_{\rm f}\geq N_{\rm c}$$

$\Rightarrow$ non-trivial vaccum moduli except for $G=U(N_{\rm c}), N_{\rm f}=N_{\rm c}$ case

Vacuum moduli space $\mathcal{M}_{\rm vac}$ in the Higgs phase

For $G' = SO(N_{\rm c}), N_{\rm f} = N_{\rm c}$,

the vacuum is described by holomorphic G' invariants

$$M \equiv H^{\rm T} J H, \quad B \equiv \det H$$

with G' invariant tensor J. ($J^{\rm T} = J$ for SO, $J^{\rm T} = -J$ for USp)

$$\begin{aligned} \mathcal{M}_{\rm vac} &= \{H | D\text{-term condition}\}/G \\ &\simeq \{M, B | M : \text{symmetric}, \det(J) B^2 = \det M\}/\mathbb{C}^* \end{aligned}$$

$\mathrm{rank}(M) < N_c - 1 \Rightarrow$ (partial) Coulomb phase!

Scalar curvature of $\mathcal{M}_{\rm vac}$ $(g, e \to \infty)$

$$R = \frac{2\sum_l \mu_l}{\xi} \sum_{i>j} \left(\frac{1}{\mu_i + \mu_j} + \sum_k \frac{\mu_k}{(\mu_i + \mu_k)(\mu_j + \mu_k)} \right) + \text{const.}$$

with fixing the flavor symmetry,

$$M = \mathrm{diag}(\mu_1, \mu_2, \cdots, \mu_{N_c}), \quad \mu_i \in \mathbb{R}_{\geq 0}$$

Ref: In the case of $G' = SU(N_c)$, $\mathcal{M}_{\rm vac}$ is just a point for $N_c = N_f$,

and a Grassmannian ($R = $ const.) for $N_c < N_f$.

•1/2 BPS equations for vortices

$$
\begin{aligned}
0 &= \mathcal{D}_1 H + i\mathcal{D}_2 H, \\
0 &= F^0_{12} + \frac{e^2}{2}(\xi - \mathrm{Tr}[H^\dagger H]) \\
0 &= F^I_{12} - \frac{g^2}{2}\mathrm{Tr}[H^\dagger T_I H], \quad \text{for } I \geq 1.
\end{aligned}
$$

General solution for the first eq. ($z \equiv x_1 + ix_2, \quad \bar{\partial} = \partial/\partial z^*$)

$$
\begin{aligned}
H &= S^{-1}(z, z^*) H_0(z), \quad \bar{\partial} H_0(z) = 0 \\
A_1 + iA_2 &= -i2S^{-1}\bar{\partial} S
\end{aligned}
$$

with an arbitrary $N_c \times N_f$ matrix $H_0(z)$, and an $S(z, z^*) \in G^{\mathbb{C}}$.
The last two equations uniquely determine the gauge invariant $SS^\dagger$ with a given $H_0(z)$.

$$
H_0 \longrightarrow S \longrightarrow A_{1,2},\ H
$$

$H_0(z)$ parameterizes the moduli space for vortices.
We call $H_0(z) = H_0(z, \varphi^\alpha)$ a 'moduli matrix' containing complex moduli φ^α.

- **Topological Charge and Weak Condition for $G' = SO(2N)$**

BPS bound of energy, $E \geq 2\pi\xi\nu$,

$$\nu \equiv -\frac{1}{2\pi}\int d^2x F^0_{12} = \frac{1}{4\pi}\oint dz\partial \log \det SS^{\dagger 2}$$

$$\Rightarrow \quad H_0(z)^{\mathrm{T}} J H_0(z) = (SH)^{\mathrm{T}} J (SH) = \sqrt[N]{\det S}\, M \sim \mathcal{O}(z^{2\nu})$$

We ontain a (weak) condition for the moduli matrix $H_0(z)$

$$H_0(z)^{\mathrm{T}} J H_0(z) = M_{\mathrm{vev}} z^k + \mathcal{O}(z^{k-1})$$

with a 'vortex number' $k \equiv 2\nu \in \mathbb{Z}_{>0}$ and $\langle M \rangle = M_{\mathrm{vev}}$.

Ref; $\det H_0(z) = \mathcal{O}(z^{N_c\nu}), k \equiv N_c\nu \in \mathbb{Z}$ for $G' = SU(N_c), N_f = N_c$.

- Moduli space for $SO(2N), USp(2N)$ semilocal vortices

There is an equivalence relation with $V(z) \in G^{\mathbb{C}}$, $\bar{\partial} V(z) = 0$,

$$\{H_0(z),\ S(z,z^*)\} \simeq \{V(z)H_0(z),\ V(z)S(z,z^*)\}.$$

Therefore the moduli space with a vortex number $k \in Z_{>0}$ is

$$\mathcal{M}_k^{\text{semilocal}} = \frac{\{H_0(z)|H_0(z) \in \mathrm{Pol}(z),\ {H_0}^{\mathrm{T}} J H_0 = \mathcal{O}(z^k)\}}{\{V(z)|V(z) \in G^{\mathbb{C}},\ \bar{\partial} V(z) = 0\}}$$

We find $\dim_{\mathbb{C}}(\mathcal{M}_k^{\text{semilocal}}) = 2kN^2$ which coincides with result of the index theorem.

For instance, the moduli matrix for the minimal vortex solution is given by

$$H_0(z) = \begin{pmatrix} z\mathbf{1}_N - A & C \\ B & \mathbf{1}_N \end{pmatrix}, \quad \text{with a choice } M_{\text{vev}} = J = \begin{pmatrix} & \mathbf{1}_N \\ \mathbf{1}_N & \end{pmatrix}.$$

Here A, B, C are arbitrary $N \times N$ matrices with $B^{\mathrm{T}} = -B, C^{\mathrm{T}} = C$.

§3. Semilocal vortex v.s. Local vortex

In a region where $|z| \gg (e\sqrt{\xi})^{-1}, (g\sqrt{\xi})^{-1}$,
vortex solution in the gauge theory
$\simeq$ lump solution in an NLσM with a target space $\mathcal{M}_{\rm vac}$

For $G' = SO(N), N_{\rm f} = N$ case

$$M = M(z) = \frac{H_0^T J H_0}{{\rm Tr} H_0^T J H_0 / N} \sim M_{\rm vev} + \mathcal{O}(z^{-1}).$$

$\Rightarrow$ Generic voretx solutions have polynomial tails:(semilocal type)
Ref. $G' = SU(N_c)$ with $N_f > N_c$.

Iff $H_0(z)$ satisfies the strong condition,

$$H_0(z)^T J H_0(z) = M_{\rm vev} \times \prod_{i=1}^{k} (z - z_i), \quad \Rightarrow \quad M(z) = M_{\rm vev}$$

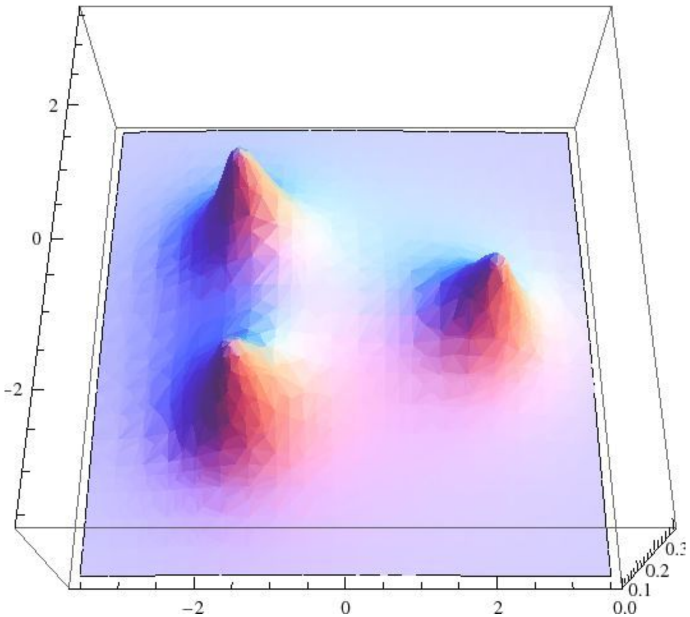
$\Rightarrow$ exponential tail $e^{-e\sqrt{\xi}|z|}, e^{-g\sqrt{\xi}|z|}$ (local vortex)

§4. Local vortex: Confined Fractional Vortices

- Semilocal vortex solution with $k = 1$

A typical form of $H_0(z)$ for $G' = SO(2N)$ with $N_f = N_c$ is

$$H_0(z) = \begin{pmatrix} z\mathbf{1}_N - A & C \\ 0 & \mathbf{1}_N \end{pmatrix}, \quad A = \mathrm{diag}(z_1, z_2, \cdots,), \quad C = \mathrm{diag}(c_1, c_2, \cdots)$$



Energy density for $G' = SO(6)$
at the limit $e, g \to \infty$

z_1, z_2, z_3: positions of three peaks
c_1, c_2, c_3: size of three peaks

There exist N peaks ('fractional' vortices) around $z = z_i$ $(i = 1, \cdots, N)$!!.

$$E \sim 2\xi\partial\bar{\partial}\log\left(\sum_{i=1}^{N}\sqrt{|z-z_i|^2+|c_i|^2}\right) \overset{z\approx z_i, {}^\forall c_i\approx 0}{\approx} \frac{\xi}{2\sum_{j\neq i}|z_i - z_j|} \times \frac{1}{|z-z_i|}.$$

Large distances $|z_i - z_j|$ $\Rightarrow$ The energy density is diluted.

- Natural realization of a local vortex

Lift of all continuous directions of the vacuum by a super potential

$$W = m\mathrm{Tr}[XMJ], \quad M = H^{\mathrm{T}}JH,$$

with additional fields X: a traceless $N_c \times N_c$ matrix, $X^T J = JX$.

F-term condition $\Rightarrow$ the strong condition

$$\frac{\partial W}{\partial X} = 0 \quad \rightarrow M \propto J \quad \rightarrow \quad H_0^T(z) J H_0(z) = J \prod_i (z - z_i).$$

$\Rightarrow z_i - z_j = c_i = 0.$

A local vortex can be regarded as confined fractional vortices.

§5. Moduli spaces for Local vortices

Moduli space for the local vortices

$$\mathcal{M}_k^{\text{local}} = \frac{\{H_0(z) | H_0(z) \in \text{Pol}(z),\ {H_0}^{\mathrm{T}} J H_0 = J P(z), P(z) = \mathcal{O}(z^k)\}}{\{V(z) | V(z) \in G^{\mathbb{C}},\ \bar{\partial} V(z) = 0\}}$$

Examples for $k = 1$

$$G' = SO(2N)\ :\quad \mathcal{M}_{k=1}^{\text{local}} = \mathbb{C} \times \frac{SO(2N)}{U(N)} \times \mathbb{Z}_2$$

$$G' = USp(2N)\ :\quad \mathcal{M}_{k=1}^{\text{local}} = \mathbb{C} \times \frac{USp(2N)}{U(N)}$$

$$G' = SO(3)\ :\ \mathcal{M}_{k=1}^{\text{local}} = \mathbb{C} + \mathbb{C} \times \mathbb{C}P^1$$

$$G' = SO(5)\ :\ \mathcal{M}_{k=1}^{\text{local}} = \mathbb{C} \times \frac{USp(4)}{U(2)} + \mathbb{C} \times W\mathbb{C}P^4_{(2,1,1,1,1)}$$

Moduli spaces can have singular subspaces.

§5. Summary and Discussion

- We extended the moduli matrix formalizm for BPS vortices into the arbitary gauge group $G = U(1) \times G'/\mathbb{Z}_{n_0}$.
- Vortex solutions are generically semilocal type in the model without super potentials, and consist of fractional vortices.
- With an appropriate super potential, a local vortex arises as confined fractional vortices.
- We determined the moduli spaces of the minimal local vortex for $SO(2N)$ and $USp(2N)$ and some cases for $SO(2N+1)$.

Future problems

- Effective action of (semi)local vortices
- Non BPS vortices
- Moduli matching between coinsident vortices and monopoles.
- method for approximation of solutions
  to calculate the metric of the moduli space
- Proof of existence and uniqueness of $SS^\dagger$