Lattice Supersymmetry with a Deformed Superalgebra

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in collaboration with

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Motivation

Why Supersymmetry on a Lattice (Ultimate Goal)?

Nonperturbative/Constructive/Strong-coupling formulation of SUSY QFT with the 1st principle calculations

- Rigid regularization scheme independent of perturbation
- Numerical simulations

Possible Applications?

- Gauge/gravity duals
- SUSY breaking, phenomenology beyond SM

Difficulties

Symmetries on a Lattice: Always Nontrivial

- Poincaré invariance $\Longrightarrow$ Discretized version is enough
- Gauge symmetry $\Longrightarrow$ Wilson's link formulation
- Chiral symmetry $\Longrightarrow$ Ginsparg–Wilson fermion, etc.
- Supersymmetry $\Longrightarrow$ Lattice version as well??

Immediate Obstacles for SUSY on a Lattice

- Doubling phenomena $\Longrightarrow$ mismatch of fermion & boson d.o.f.
- Leibniz rule failure

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Symmetries on a Lattice: Always Nontrivial

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Immediate Obstacles for SUSY on a Lattice

- Doubling phenomena $\Longrightarrow$ mismatch of fermion & boson d.o.f.
  $\Longrightarrow$ avoided with extended SUSY, or G–W fermions, etc.
- Leibniz rule failure $\Longrightarrow$ more crucial

Leibniz Rule Failure

Leibniz Rule Failure of "Derivative" Op.

- Superalgebra contains momentum op.:

$$\{Q_A, Q_B\} = P_{AB} = i\gamma^\mu \partial_\mu .$$

- On the lattice, $\partial_\mu \to \partial_\mu^{\mathrm{lat}}$: "derivative" on the lattice?
- Natural candidate $\partial_\mu^{\mathrm{lat}} = \partial_{+\mu}$: finite difference op. would obey slightly modified Leibniz rule: [Dondi–Nicolai, Fujikawa, . . .]

$$\partial_{+\mu}(\varphi \cdot \varphi')(x) = \partial_{+\mu}\varphi(x) \cdot \varphi'(x) + \varphi(x + a\hat{\mu}) \cdot \partial_{+\mu}\varphi'(x).$$

- No-go theorem: no local "derivative" on the lattice can obey the exact Leibniz rule. [Kato–Sakamoto–So]

Leibniz Rule Problem

Problem

- Due to the Grassmann-odd nature, supercharge would obey the exact Leibniz rule even on the lattice

$$Q_A(\varphi \cdot \varphi')(x) = Q_A\varphi(x) \cdot \varphi'(x) + (-1)^{|\varphi|}\varphi(x) \cdot Q_A\varphi'(x).$$

- Simple realization of superalgebra on the lattice

$$\{Q_A, Q_B\} = i\gamma^\mu \partial_\mu^{\text{lat}}$$

isn't possible.

Leibniz Rule Problem

Solutions?

- Give up the exact algebra $\Longrightarrow$ fine-tune problem in general.

  [Curuci–Veneziano, ...]

- Keep only a subalgebra which doesn't contain the momentum operator

  $\Longrightarrow$ works without fine-tuning in low dimensions.

  [Kaplan et. al., Catterall et. al., Sugino, ...]

  $\Longrightarrow$ also manageable in four dimensions? [Elliott–Giedt–Moore, ...]

Our Approach

- Deform the Leibniz rule for the supercharge.

  [D'Adda–Kawamoto–Kanamori–Nagata, Arianos–D'Adda–Feo–Kawamoto–J. S.]

Deformed-Algebra Approach

Deformed Leibniz Rule for Supercharges Let us introduce the deformed rule

$$Q_A^{\mathrm{lat}}(\varphi\cdot\varphi')(x) = Q_A^{\mathrm{lat}}\varphi(x)\cdot\varphi'(x) + (-1)^{|\varphi|}\varphi(x+a_A)\cdot Q_A^{\mathrm{lat}}\varphi'(x).$$

- This extends the notion of Lie superalgebra.
- Really a symmetry of a QFT?

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  $\Longrightarrow$ rigourous treatment: Hopf algebra.
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- This extends the notion of Lie superalgebra.
  $\Longrightarrow$ rigourous treatment: Hopf algebra.
- Really a symmetry of a QFT?
  $\Longrightarrow$ QFT with mildly generalized statistics and corresponding Ward–Takahashi identities. [Oeckl, Sasai–Sasakura]

Plan of Talk

Hopf Algebra

Hopf Algebra

$$\boxed{\text{Hopf Algebra } H}$$

$$\|$$

Algebra

- associative product $\cdot : H \otimes H \longrightarrow H$
- unit $\eta : \mathbb{C} \longrightarrow H$

$$+$$

Coalgebra

- coassociative coproduct $\Delta : H \longrightarrow H \otimes H$
- counit $\epsilon : H \longrightarrow \mathbb{C}$

$$+$$

Antipode

- $S : H \longrightarrow H$

Leibniz Rule $\Longrightarrow$ Coproduct

Leibniz Rules $\Longrightarrow$ Coproduct

- Specifying Leibniz rules amounts to determining coproducts:

$$Q_A^{\mathrm{lat}}(\varphi\cdot\varphi')(x) = Q_A^{\mathrm{lat}}\varphi(x)\cdot\varphi'(x) + (-1)^{|\varphi|}\varphi(x+a_A)\cdot Q_A^{\mathrm{lat}}\varphi'(x)$$

$$\Downarrow$$

$$\begin{cases} Q_A^{\mathrm{lat}} \triangleright (\varphi\cdot\varphi')(x) = m\big(\Delta(Q_A^{\mathrm{lat}})\triangleright(\varphi\otimes\varphi')\big)(x), \\ \Delta(Q_A^{\mathrm{lat}}) = Q_A^{\mathrm{lat}}\otimes\mathbb{1} + (-1)^{\mathcal{F}}T_{a_A}\otimes Q_A^{\mathrm{lat}}, \end{cases}$$

where

$$m(\varphi\otimes\varphi') = \varphi\cdot\varphi',$$

$$T_{a_A}\triangleright\varphi(x) = \varphi(x+a_A),$$

$$(-1)^{\mathcal{F}}\triangleright\varphi(x) = (-1)^{|\varphi|}\varphi(x).$$

Coproducts Formulae

Coproducts

$$\Delta(Q_A^{\mathrm{lat}}) = Q_A^{\mathrm{lat}} \otimes \mathbb{1} + (-1)^{\mathcal{F}} T_{a_A} \otimes Q_A^{\mathrm{lat}},$$
$$\Delta(P_\mu^{\mathrm{lat}}) = P_\mu^{\mathrm{lat}} \otimes \mathbb{1} + T_{a\hat{\mu}} \otimes P_\mu^{\mathrm{lat}},$$
$$\Delta(T_b) = T_b \otimes T_b, \quad \Delta((-1)^{\mathcal{F}}) = (-1)^{\mathcal{F}} \otimes (-1)^{\mathcal{F}}.$$

Cf. Coproducts for the Normal Leibniz Rules

$$\Delta(Q_A) = Q_A \otimes \mathbb{1} + (-1)^{\mathcal{F}} \mathbb{1} \otimes Q_A,$$
$$\Delta(P_\mu) = P_\mu \otimes \mathbb{1} + \mathbb{1} \otimes P_\mu,$$
$$\Delta(T_b) = T_b \otimes T_b, \quad \Delta((-1)^{\mathcal{F}}) = (-1)^{\mathcal{F}} \otimes (-1)^{\mathcal{F}}.$$

Consistency

Products of More Fields

- Associativity $\Longrightarrow$ coassociativity:

$$(\varphi_1 \cdot \varphi_2) \cdot \varphi_3 = \varphi_1 \cdot (\varphi_2 \cdot \varphi_3)$$

$$\Downarrow$$

$$Q_A^{\mathrm{lat}} \triangleright (\varphi_1 \cdot \varphi_2) \cdot \varphi_3 = Q_A^{\mathrm{lat}} \triangleright \varphi_1 \cdot (\varphi_2 \cdot \varphi_3)$$

$$\Downarrow$$

$$(\Delta \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \Delta) \circ \Delta.$$

- This holds for our explicit formulae.

Counit Consistency

Trivial Representation $\Longrightarrow$ Counit

- $c \in \mathbb{C}$: constant field,

$$Q_A \triangleright c \equiv \epsilon(Q_A)c$$

Consistency

$$\varphi = 1 \cdot \varphi = \varphi \cdot$$

$$\Downarrow$$

$$Q_A \triangleright \varphi = Q_A \triangleright (1 \cdot \varphi) = Q_A \triangleright (1 \cdot \varphi)$$

$$\Downarrow$$

$$(\epsilon \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \epsilon) \circ \Delta = \mathrm{id}$$

- Counit ϵ has to be determined to satisfy this consistency.

Counit & Antipode Formulae

Counits

$$\epsilon(Q_A^{\mathrm{lat}}) = 0, \quad \epsilon(P_\mu^{\mathrm{lat}}) = 0, \quad \epsilon(T_b) = 1, \quad \epsilon\big((-1)^{\mathcal{F}}\big) = 1.$$

- These satisfy the previous consistency conditions.

Antipodes

$$S(Q_A^{\mathrm{lat}}) = -T_{a_A}^{-1} \cdot (-1)^{\mathcal{F}} \cdot Q_A^{\mathrm{lat}},$$

$$S(P_\mu^{\mathrm{lat}}) = -T_{a\hat{\mu}}^{-1} \cdot P_\mu^{\mathrm{lat}},$$

$$S(T_b) = T_b^{-1}, \quad S\big((-1)^{\mathcal{F}}\big) = (-1)^{\mathcal{F}}.$$

Superalgebra on the Lattice

Hopf-Algebraic Superalgebra

- A consistent superalgebra on the lattice can be introduced as a Hopf algebra, with the algebraic structure

$$\{Q_A^{\mathrm{lat}}, Q_B^{\mathrm{lat}}\} = 2\tau^{\mu}_{AB} P_{\mu}^{\mathrm{lat}},$$

$$[Q_A^{\mathrm{lat}}, P_{\mu}^{\mathrm{lat}}] = [P_{\mu}^{\mathrm{lat}}, P_{\nu}^{\mathrm{lat}}] = 0,$$

$$[Q_A^{\mathrm{lat}}, T_b] = [P_{\mu}^{\mathrm{lat}}, T_b] = [T_b, T_c] = 0,$$

$$\{Q_A^{\mathrm{lat}}, (-1)^{\mathcal{F}}\} = [P_A^{\mathrm{lat}}, (-1)^{\mathcal{F}}] = [T_b, (-1)^{\mathcal{F}}] = 0,$$

plus the algebra maps Δ, ϵ, S.

Noncommutative Representation?

"Commutative" Representation & Statistics

- Hopf algebra is generally represented on a noncommutative thus nonlocal field space.
- In fact, we can reduce the noncommutativity to commutativity up to a generalized statistics (braiding) by adding a grading structure.

$$
\begin{array}{ccc}
\varphi \otimes \varphi' & \xrightarrow{\Psi} & \varphi' \otimes \varphi \\
\downarrow & & \downarrow \\
Q_A(\varphi \otimes \varphi') & \xrightarrow{\Psi} & Q_A(\varphi' \otimes \varphi).
\end{array}
$$

Generalized Statistics

Generalized Statistics

- General braiding formula

$$\Psi\Big(\varphi_{A_0\cdots A_p}(x)\otimes\varphi'_{B_0\cdots B_q}(y)\Big)$$
$$=(-1)^{pq}\varphi'_{B_0\cdots B_q}\left(y+\sum_{i=1}^{p}(a^{\mathrm{l}}_{A_i}-a^{\mathrm{r}}_{A_i})\right)$$
$$\otimes\,\varphi_{A_0\cdots A_p}\left(x-\sum_{i=1}^{q}(a^{\mathrm{l}}_{B_i}-a^{\mathrm{r}}_{B_i})\right),$$

where

$$\varphi_{A_0\cdots A_p}:=Q^{\mathrm{lat}}_{A_p}\cdots Q^{\mathrm{lat}}_{A_1}\varphi_{A_0},\quad \varphi_{A_0}:=\phi.$$

Quantization

Braided QFT [Oeckl]

- The braiding structure allows us to construct a QFT perturbatively with a formal path integral quantization. Cf. Fock space representation & deformed CCR.

Braided Functional Derivative

$$\frac{\delta}{\delta\varphi(x)}(\varphi_1\cdot\varphi_2)=\frac{\delta}{\delta\varphi(x)}\varphi_1\cdot\varphi_2+(-1)^{|\varphi||\varphi_1|}T_{\varphi}\varphi_1\cdot\frac{\delta}{\delta\varphi(x)}\varphi_2$$

Path Integral

$$\int\frac{\delta}{\delta\varphi(x)}=0$$

Ward–Takahashi Identity

Correlation Functions

- The formal expression is enough to define & compute

$$Z = \int e^{-S}, \quad \langle \varphi_1 \cdots \varphi_n \rangle = \frac{1}{Z} \int \varphi_1 \cdots \varphi_n e^{-S}.$$

Ward–Takahashi Identity [Sasai–Sasakura]

- The Hopf-algebraic supersymmetry is expressed by the corresponding Ward–Takahashi identities

$$a \triangleright \langle \varphi_1 \cdots \varphi_n \rangle = \epsilon(a) \langle \varphi_1 \cdots \varphi_n \rangle.$$

Summary and Discussion

Summary

- We can introduce a Hopf algebra on a lattice as a lattice-deformed superalgebra.
- A consistency requires that the fields representing the Hopf algebra acquire a generalized statistics.
- The corresponding QFT can be constructed at least perturbatively.

Discussion

- Nonperturbative/“simulationable” path integral definition?
- Gauge theory extension, & its strong coupling expansion?
- Connection with the other regularization approaches?