Vortex counting and exact superpotential in N=(2,2) theories

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1. Introduction

Similarity between 4d gauge theory and 2d sigma model

asymptotically free, anomaly, mass gap,

⋮

instantons → non-perturbative effects

4d gauge theory	2d sigma model
• Yang-Mills instantons	• sigma model instantons

→ exact results in supersymmetric theories

instanton/vortex conting in 4d/2d supersymmetric gauge theories

4d $\mathcal{N}=2$ instanton counting, [Nekrasov, 2002] , etc

- **instanton partition function** $Z_{inst}(\tau, a, m, \epsilon_i)$
- **prepotential** $\mathcal{F}_{inst}(\tau, a, m) = \lim_{\epsilon_i \to 0} \epsilon_1 \epsilon_2 \log Z_{inst}(\tau, a, m, \epsilon_i)$

2d $\mathcal{N}=(2,2)$ vortex counting

- **vortex partition function** $Z_{vortex}(\tau, m, \epsilon)$
- **twisted superpotential** $\widetilde{W}_{vortex}(\tau, m) = \lim_{\epsilon \to 0} \epsilon \log Z_{vortex}(\tau, m, \epsilon)$

2d $\mathcal{N} = (2, 2)$ non-linear sigma model

sigma model instantons

small instanton singularity

2d $\mathcal{N}=(2,2)$ non-linear sigma model

low energy imit

2d $\mathcal{N}=(2,2)$ gauged linear sigma model

sigma model instantons

small instanton singularity

vortices

non-singular

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2d $\mathcal{N}=(2,2)$ gauged linear sigma model

vortices

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Fayet-Iliopoulos parameter and 2d theta angle

$$\tau(\mu) = \frac{\theta}{2\pi} + i\xi = \frac{n}{2\pi i} \log \frac{\Lambda}{\mu}$$

Λ : dynamical scale

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Fayet-Iliopoulos parameter and 2d theta angle

$$\tau(\mu) \;=\; \frac{\theta}{2\pi}+i\xi \;=\; \frac{n}{2\pi i}\log\frac{\Lambda}{\mu}$$

Λ : dynamical scale

on-shell value of twisted superpotential

$$\widetilde{W}=i\langle\sigma\rangle\tau+\widetilde{W}_{vortex}$$

$|\widetilde{W}_A-\widetilde{W}_B|$: BPS kink mass

VEV of vector multiplet scalar

2. CP $^{\text{N-1}}$ sigma model

U(1) gauge theory with N charged chiral multiplets

$$H = (H_1, H_2, \cdots, H_N)$$

classical vacua

$$\left.\begin{array}{l} HH^\dagger = \xi \\ \sigma H - HM = 0 \end{array}\right\} \Rightarrow$$

N vacua

$$\langle H \rangle = \sqrt{\xi}(1, 0, \cdots, 0)$$

$$\langle \sigma \rangle = m_1$$

$$\cdots$$

$$M = \mathrm{diag}\,(m_1, m_2, \cdots, m_N)$$

twisted masses

invariant k-vortex moduli matrix

$$H_0 = (z^k, 0, \cdots, 0), \qquad V H_0(e^{i\epsilon} z) e^{iM} = H_0(z), \quad V = e^{-im_1 - ik\epsilon}$$

invariant k-vortex moduli matrix

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neighborhood around the fixed point

$$\delta H_0 = (p_1, p_2, \cdots, p_N) \qquad p_a(z) = \sum_{j=0}^{k-1} c_{a,j} z^j$$

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$$V \delta H_0(e^{i\epsilon} z) e^{iM} \Rightarrow c_{a,j} \to \exp\left[i(m_a - m_1) + i(k-j)\epsilon\right] c_{a,j}$$

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k-vortex contribution

$$Z_k = \prod_{a=1}^{N} \prod_{j=0}^{k-1} \frac{1}{m_{a1} - (k-j)\epsilon} \qquad m_{a1} \equiv m_a - m_1$$

k vortex contribution

$$Z_k = \prod_{a=1}^{N} \prod_{j=1}^{k} \frac{1}{m_{a1} - j\epsilon}$$

vortex counting parameter

$$\Lambda^N = \mu^N e^{2\pi i \tau}$$

$$Z_{vortex} = \sum_{k=0}^{\infty} \Lambda^{kN} Z_k = {}_0F_{N-1}\left(\{\} \, , \, \left\{ 1 - \frac{m_{a1}}{\epsilon} \right\}_{a=2}^{N} ; \left(\frac{\Lambda}{\epsilon} \right)^N \right)$$

hypergeometric function

vortex partition function in b-th vacua of CP^{N-1} sigma model

$$Z_{vortex,b} = {}_0F_{N-1}\left(\{\} \, , \, \left\{ 1 - \frac{m_{ab}}{\epsilon} \right\}_{a \neq b} ; \left(\frac{\Lambda}{\epsilon} \right)^N \right)$$

twisted superpotential

$$\widetilde{W}_{vortex,b}(\tau, m) = \lim_{\epsilon \to 0} \epsilon \log Z_{vortex,b}(\tau, m, \epsilon)$$

integrating out

(1) $$\widetilde{W} = i\sigma\tau - \frac{1}{2\pi}\sum_{a=1}^{N}(\sigma - m)\left[\log\frac{\sigma - m}{\mu} - 1\right]$$

(2) $$\widetilde{W} = \frac{1}{2\pi}\sum_{a=1}^{N} m_a Y_a + \mu \prod_{a=1}^{N} e^{Y_a}$$

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recursion relation

$$\prod_{a=1}^{N}\left[m_{ab} - k\epsilon\right] Z_{k,b} = Z_{k-1,b}$$

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$$\hat{\sigma} \equiv \epsilon\Lambda^N \frac{\partial}{\partial \Lambda^N} = \frac{\epsilon}{2\pi i}\frac{\partial}{\partial \tau}$$

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differential equation for $Z_{vortex,b}$

$$\prod_{a=1}^{N}\left[m_{ab}-\hat{\sigma}\right]Z_{vortex,b}\;=\;\prod_{a=1}^{N}\left[m_{ab}-\hat{\sigma}\right]\sum_{k=0}^{\infty}\Lambda^{Nk}Z_{k,b}$$

twisted superpotential

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differential equation for $Z_{vortex,b}$

$$\prod_{a=1}^{N}\left[m_{ab}-\hat{\sigma}\right]Z_{vortex,b}\;=\;\Lambda^N Z_{vortex,b}$$

full partition function

perturbative part

$$Z_b = \exp\left(2\pi i \frac{m_b}{\epsilon}\tau\right) Z_{vortex,b}$$

differential equation for full partition function

$$P(\hat{\sigma})\, Z_b = 0, \qquad P(\hat{\sigma}) \equiv \prod_{a=1}^{N}(m_a - \hat{\sigma}) - \Lambda^N$$

- $P(\hat{\sigma})$ **is independent of choice of vacuum** full partition function

perturbative part

$$Z_b \;=\; \exp\left(2\pi i\frac{m_b}{\epsilon}\tau\right) Z_{vortex,b}$$

differential equation for full partition function

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- $P(\hat{\sigma})$ **is independent of choice of vacuum**

⬇

$Z_b\ (b = 1, \cdots, N)$ **are linearly independent solutions of** $P(\hat{\sigma})\, Z_b = 0$

“classical limit”

$$Z_b = \exp\left(\frac{2\pi}{\epsilon}\widetilde{W}_{\epsilon,b}\right)$$

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differential equation

$$0 \;=\; P(\hat{\sigma})\, Z_b \;=\; P(-i\partial_\tau \widetilde{W}_{\epsilon,b}) + \mathcal{O}(\epsilon)$$

“classical limit”

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differential equation

$$0 \;=\; P(\hat{\sigma})\,Z_b \;=\; P(-i\partial_\tau \widetilde{W}_{\epsilon,b}) + \mathcal{O}(\epsilon)$$

$$\epsilon \to 0$$

$$\Rightarrow \quad 0 \;=\; P(-i\partial_\tau \widetilde{W}_b)$$

“classical limit”

$$Z_b = \exp\left(\frac{2\pi}{\epsilon}\widetilde{W}_{\epsilon,b}\right)$$

differential equation

$$0 \;=\; P(\hat{\sigma})\, Z_b \;=\; P(-i\partial_\tau \widetilde{W}_{\epsilon,b}) + \mathcal{O}(\epsilon)$$

$$\xrightarrow{\epsilon \to 0} \quad 0 \;=\; P(-i\partial_\tau \widetilde{W}_b)$$

F-term condition $\partial_\sigma \widetilde{W} = 0$

$$\widetilde{W} = i\sigma\tau - \frac{1}{2\pi}\sum_{a=1}^{N}(\sigma - m_a)\left[\log\frac{\sigma - m_a}{\mu} - 1\right] \longrightarrow P(\sigma) = 0$$

“classical limit”

$$Z_b = \exp\left(\frac{2\pi}{\epsilon}\widetilde{W}_{\epsilon,b}\right)$$

differential equation

$$0 \;=\; P(\hat{\sigma})\, Z_b \;=\; P(-i\partial_\tau \widetilde{W}_{\epsilon,b}) + \mathcal{O}(\epsilon)$$

$\epsilon \to 0$

$$\Rightarrow \quad 0 \;=\; P(-i\partial_\tau \widetilde{W}_b)$$

F-term condition $\partial_\sigma \widetilde{W} = 0$

$$\widetilde{W} = i\sigma\tau - \frac{1}{2\pi}\sum_{a=1}^{N}(\sigma - m_a)\left[\log\frac{\sigma - m_a}{\mu} - 1\right] \longrightarrow \quad P(\sigma) = 0$$

$$\Rightarrow \quad P(\hat{\sigma})\, Z_b = 0$$

“quantized” F-term condition

$$[\hat{\tau}, \hat{\sigma}] = i\hbar = -\frac{\epsilon}{2\pi i}$$

U(1) gauge theory with N chiral multiplets of charge +1

U(1) gauge theory with N chiral multiplets of charge +1

+ $\tilde{N}$ chiral multiplets of charge -1

U(1) gauge theory with N chiral multiplets of charge +1

+ $\tilde{N}$ chiral multiplets of charge -1

k-vortex contribution

$$Z_k = \prod_{a=1}^{N}\prod_{j=1}^{k}\frac{1}{m_{a1}-j\epsilon} \times \prod_{a=1}^{\tilde{N}}\prod_{j=1}^{k}(\tilde{m}_{a1}-j\epsilon)$$

U(1) gauge theory with N chiral multiplets of charge +1

+ $\tilde{N}$ chiral multiplets of charge -1

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recursion relation

$$\prod_{a=1}^{N}\prod_{j=1}^{k}(m_{a1}-k\epsilon)Z_k = \prod_{a=1}^{\tilde{N}}\prod_{j=1}^{k}(\tilde{m}_{a1}-k\epsilon)\,Z_{k-1}$$

U(1) gauge theory with N chiral multiplets of charge +1

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k-vortex contribution

$$Z_k = \prod_{a=1}^{N}\prod_{j=1}^{k}\frac{1}{m_{a1} - j\epsilon} \times \prod_{a=1}^{\tilde{N}}\prod_{j=1}^{k}(\tilde{m}_{a1} - j\epsilon)$$

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differential equation

$$P(\hat{\sigma})\, Z_b = 0,$$

$$P(\hat{\sigma}) = \prod_{a=1}^{N}(m_a - \hat{\sigma}) - \Lambda^{N-\tilde{N}}\prod_{\tilde{a}=1}^{\tilde{N}}(\tilde{m}_{\tilde{a}} - \hat{\sigma})$$

masses of BPS kinks in 2d $\mathcal{N}=(2,2)$ theory

$\widetilde{W}(m,\tilde{m},\Lambda)$

masses of BPS monopoles in 4d $\mathcal{N}=2$ SU(N) gauge theory with N fund. and $\tilde{N}$ anti-fund. hypermultiplets

$\mathcal{F}(m,\tilde{m},\Lambda)$

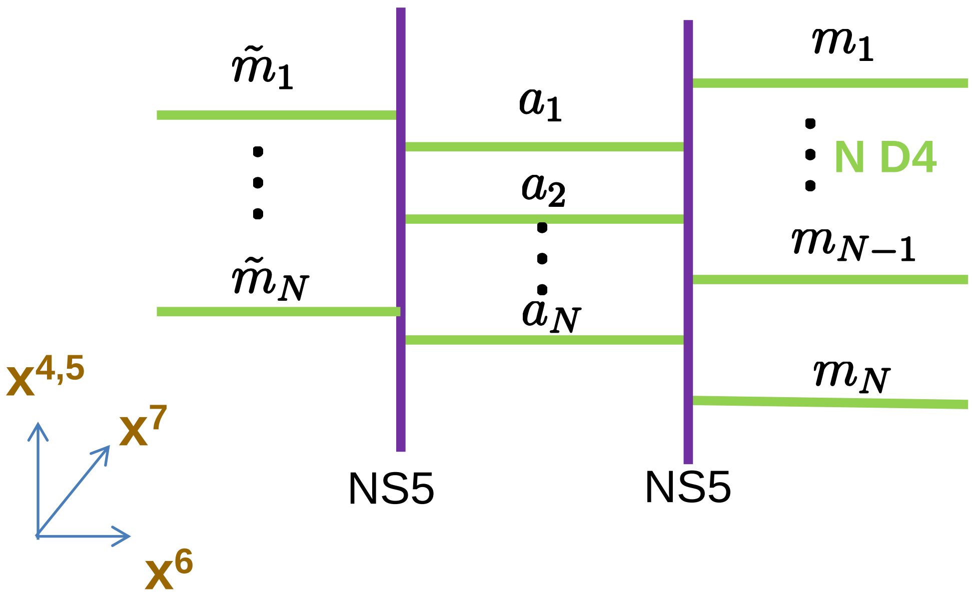
root of Higgs branch

$a_i = m_i$ **(classical)**

$$F(t,v) = \left[t\prod_{\tilde{a}=1}^{\tilde{N}}(v-\tilde{m}_a) - \prod_{a=1}^{N}(v-m_a)\right](t-\Lambda^{N-\tilde{N}}) = 0$$

[Dorey, 1998], [Shifman-Yung, 2004], [Hanany-Tong, 2004], etc

4d gauge theory at root of Higgs branch



$$F(t,v) = A(v)t^2 + B(v)t + C(v) = 0$$

$$A(v) = \Lambda^{N-\tilde{N}} \prod_{\tilde{a}=1}^{\tilde{N}} (v - \tilde{m}_{\tilde{a}})$$

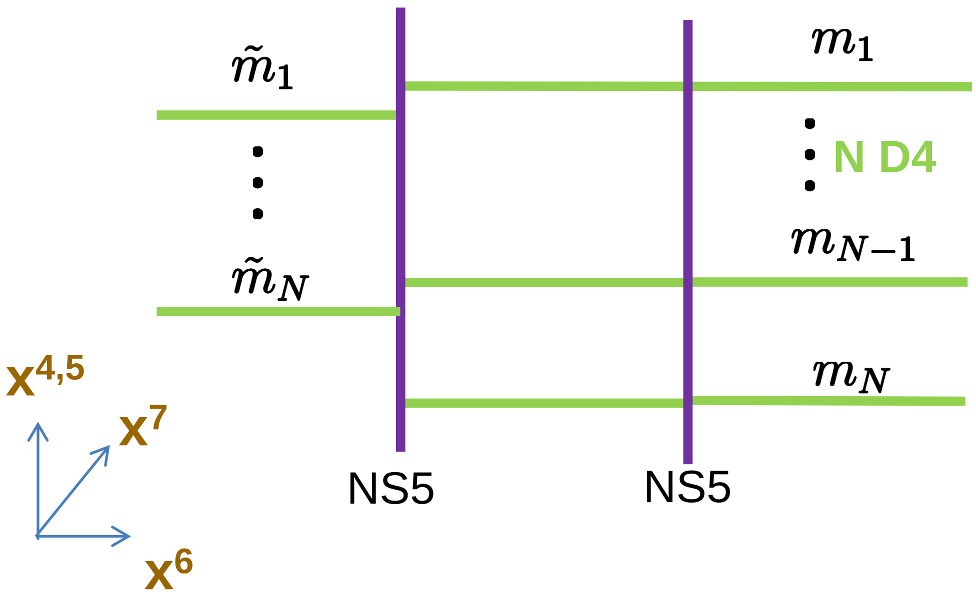
$$B(v) = \prod_{i=1}^{N} (v - a_i)$$

$$C(v) = \prod_{a=1}^{N} (v - m_a)$$

D4, NS5 ➡ **M5** $\quad v = x_4 + ix_5, \quad t = \exp(x_6 + ix_{10})/R,$

	0	1	2	3	4	5	6	7	8	9
N D4	●	●	●	●			●			
NS5	●	●	●	●	●	●				

4d gauge theory at root of Higgs branch



$$B^2 - 4AC = \prod_{a=1}^{N} (v - e_a)^2$$

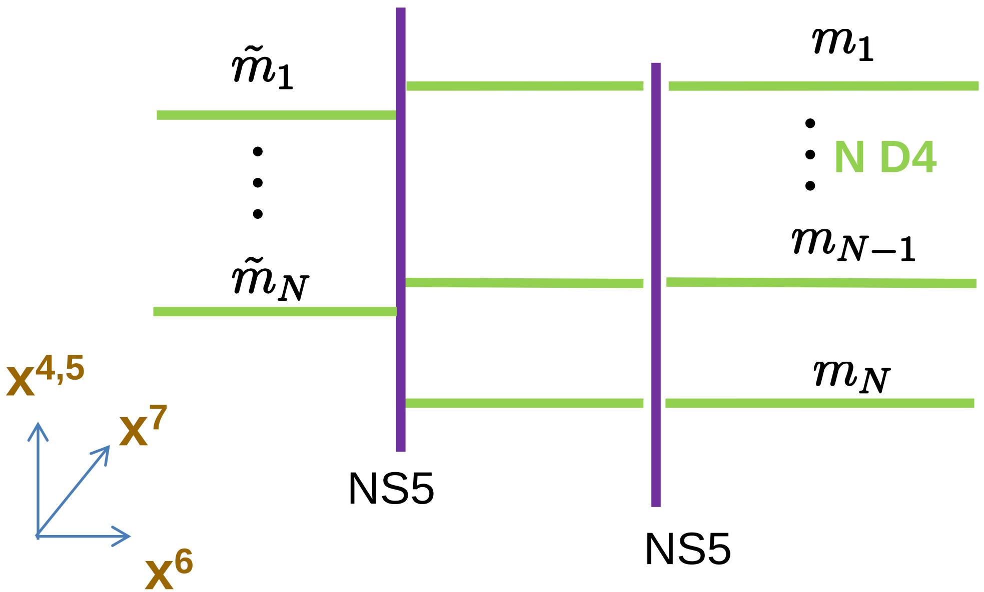
$$F(t,v) = \left[t \prod_{\tilde{a}=1}^{\tilde{N}} (v - \tilde{m}_a) - \prod_{a=1}^{N} (v - m_a) \right]$$

$$\times (t - \Lambda^{N-\tilde{N}}) = 0$$

two M5

	0	1	2	3	4	5	6	7	8	9
N D4	●	●	●	●			●			
NS5	●	●	●	●	●	●				

4d gauge theory at root of Higgs branch

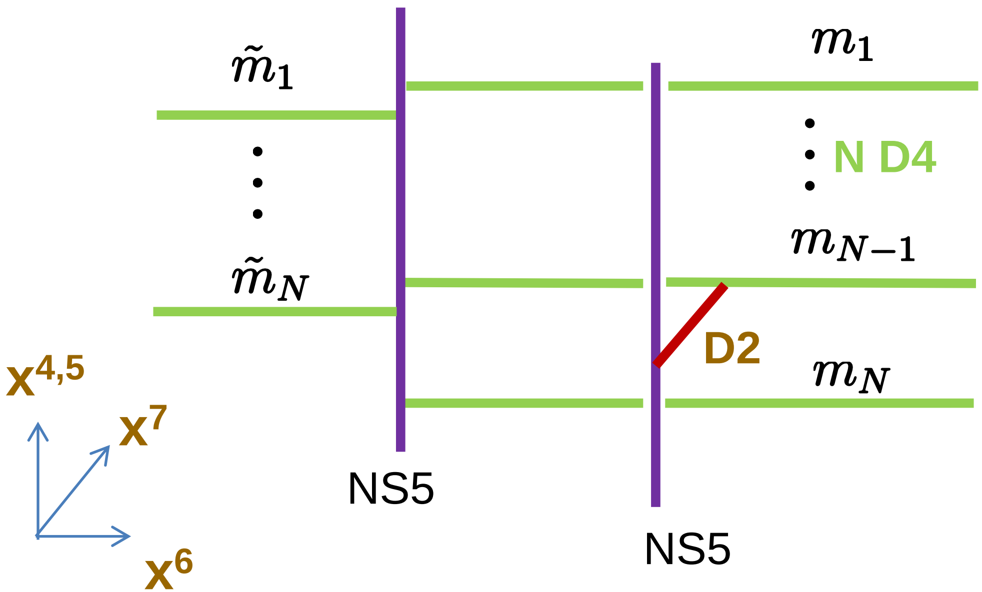


$\Delta x^7_{NS5} \propto$ **4d FI parameter**

Higgs phase

	0	1	2	3	4	5	6	7	8	9
N D4	●	●	●	●			●			
NS5	●	●	●	●	●	●				

4d gauge theory at root of Higgs branch



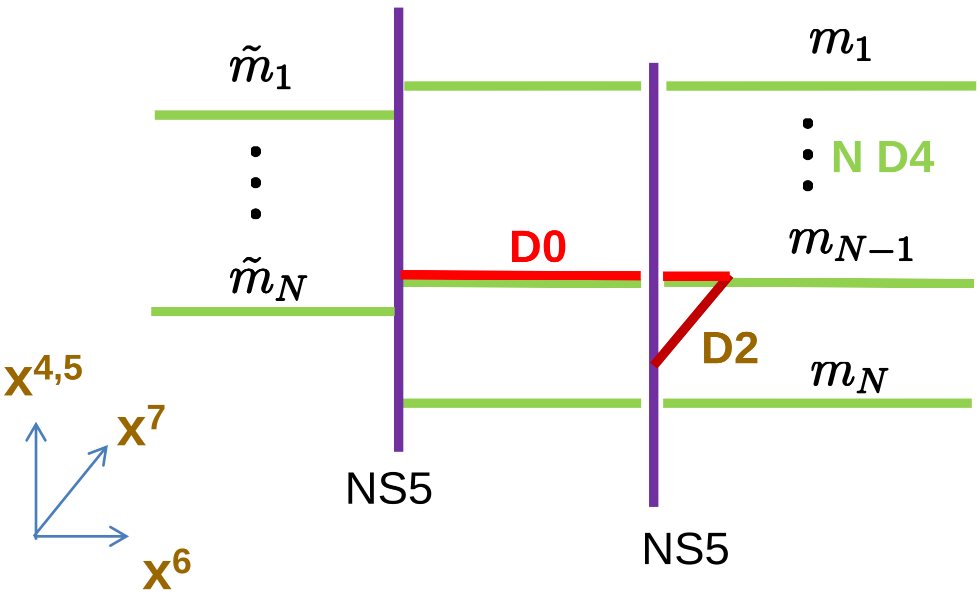
$\Delta x^7_{NS5} \propto$ **4d FI parameter**

Higgs phase

D2: vortex (2d theory)

	0	1	2	3	4	5	6	7	8	9
N D4	●	●	●	●			●			
NS5	●	●	●	●	●	●				
D2	●	●						●		

4d gauge theory at root of Higgs branch



$\Delta x^7_{NS5} \propto$ **4d FI parameter**

➡ **Higgs phase**

D2: vortex (2d theory)

D0: vortices on vortex

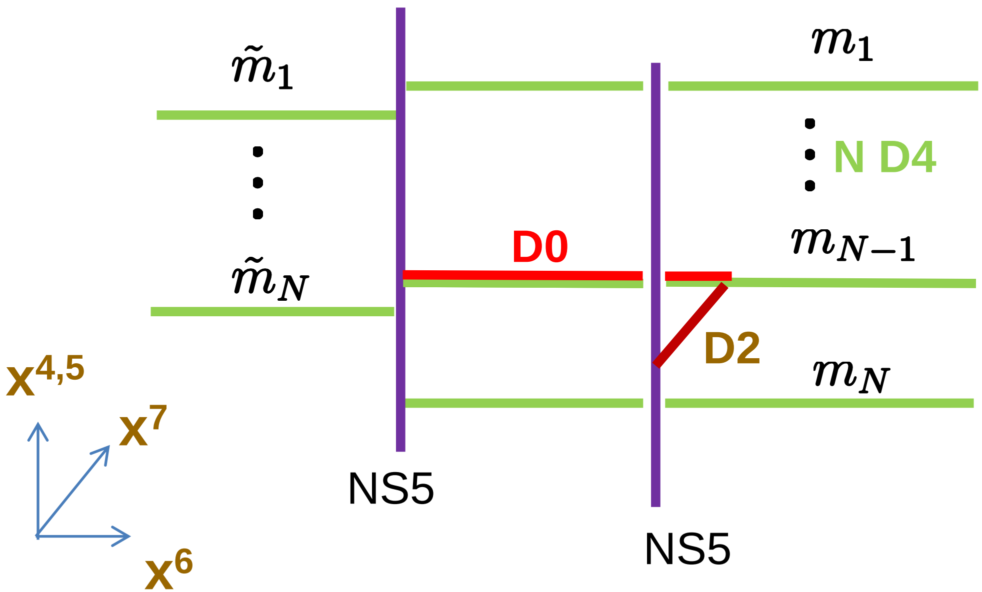
||

instantons trapped inside vortex

[Eto-Isozumi-Nitta-Ohashi-Sakai, 2005]

	0	1	2	3	4	5	6	7	8	9
N D4	●	●	●	●			●			
NS5	●	●	●	●	●	●				
D2	●	●						●		

4d gauge theory at root of Higgs branch



$$\begin{cases} t\prod_{\tilde{a}=1}^{\tilde{N}}(v-\tilde{m}_a)-\prod_{a=1}^{N}(v-m_a)=0 \\ t-\Lambda^{N-\tilde{N}}=0 \end{cases}$$

$$\Rightarrow \quad P(v)=0$$

σ : v of M2 $\quad$ τ : $\log t$ of M2

$[\hat{\sigma},\hat{\tau}]=\frac{\epsilon}{2\pi}$ **: non-commutativity**

	0	1	2	3	4	5	6	7	8	9
N D4	●	●	●	●			●			
NS5	●	●	●	●	●	●				
D2	●	●						●		

3. Grassmaniann sigma model

U(N) gauge theory with Nf charged chiral multiplets

fixed point

$$H_0^{\vec{k}} = \left(\begin{array}{ccc|ccc} z^{k_1} & & & 0 & \cdots & 0 \\ & \ddots & & \vdots & \ddots & \vdots \\ & & z^{k_N} & 0 & \cdots & 0 \end{array}\right)$$

tangent space

$$\delta H_0^{\vec{k}} = \left(\begin{array}{ccc|ccc} \sum_{j=1}^{k_1} c_{11,j} z^{j-1} & \cdots & \sum_{j=1}^{k_N} c_{1N,j} z^{j-1} & \sum_{j=1}^{k_1} d_{11,j} z^{j-1} & \cdots & \sum_{j=1}^{k_1} d_{1\tilde{N}} z^{j-1} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \sum_{j=1}^{k_1} c_{N1,j} z^{j-1} & \cdots & \sum_{j=1}^{k_N} c_{NN,j} z^{j-1} & \sum_{j=1}^{k_N} d_{N1,j} z^{j-1} & \cdots & \sum_{j=1}^{k_N} d_{N\tilde{N}} z^{j-1} \end{array}\right)$$

contribution from the fixed point

$$Z_{(k_1,\cdots,k_n)} = \left(\prod_{l=1}^{N}\prod_{m=1}^{N}\prod_{j=1}^{k_m}\frac{1}{m_{ml}+(k_l-j+1)\epsilon}\right)\times\left(\prod_{l=1}^{N}\prod_{s=1}^{\tilde{N}}\prod_{j=1}^{k_l}\frac{1}{m_{n+s,l}+(k_l-j+1)\epsilon}\right)$$

contribution from the fixed point

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vortex counting parameters

Λ **:overall** $U(1)$ $\longrightarrow$ Λ_i **:** $U(1)^N$ $\Leftrightarrow$ $\hat{\sigma}_i \equiv \epsilon\Lambda_i^{N_\mathrm{F}}\dfrac{\partial}{\partial\Lambda_i^{N_\mathrm{F}}} = \dfrac{\epsilon}{2\pi i}\dfrac{\partial}{\partial\tau_i}$

contribution from the fixed point

$$Z_{(k_1,\cdots,k_n)} = \left(\prod_{l=1}^{N}\prod_{m=1}^{N}\prod_{j=1}^{k_m}\frac{1}{m_{ml}+(k_l-j+1)\epsilon}\right)\times\left(\prod_{l=1}^{N}\prod_{s=1}^{\tilde{N}}\prod_{j=1}^{k_l}\frac{1}{m_{n+s,l}+(k_l-j+1)\epsilon}\right)$$

vortex counting parameters

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vortex partition function of Grassmaniann sigma model

$$Z_{vortex} = \sum_{k_1=0}^{\infty}\cdots\sum_{k_N=0}^{\infty}\Lambda_1^{N_f k_1}\cdots\Lambda_N^{N_f k_N} Z_{(k_1,\cdots,k_N)}$$

$$= \prod_{l>m}\frac{m_{ml}+\hat{\sigma}_l-\hat{\sigma}_m}{m_{ml}}\prod_{l=1}^{N}{}_0F_{N_f-1}\left(\{\};\left\{1+\frac{m_{la}}{\epsilon}\right\}_{a\neq l};\frac{\Lambda_l^{N_f}}{\epsilon^{n_f}}\right)$$

recursion relation

$$Z_{(\cdots,k_i+1,\cdots)} = (-1)^{n-1}\prod_{m=1}^{N_f}\frac{1}{m_{mi}+(k_i+1)\epsilon}\prod_{m\neq i}^{N}\frac{m_{mi}+(k_i-k_m+1)\epsilon}{m_{mi}+(k_i-k_m)\epsilon}Z_{(\cdots,k_i,\cdots)}$$

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differential equation for Z

$$P_i(\hat{\sigma}_1,\cdots,\hat{\sigma}_N)\,Z=0 \quad (i=1,\cdots,N)$$

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= F-term equation of $\widetilde{W}(\tau_i,\sigma_i)=\sum_{i=1}^{N}\left[i\sigma_i\tau_i-\frac{1}{2\pi}\sum_{l=1}^{N_f}(\sigma_i+m_l)\left(\log\frac{\sigma_i+m_l}{\mu}-1\right)\right]$

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- **non-Abelian case: worldvolume theory of multiple vortices?**

vortex worldvolume theory

2d N=(2,2) U(k) theory with N fund. + $\tilde{N}$ anti-fund. + one adjoint

$$I \qquad\qquad J \qquad\qquad B$$

ϵ_1 **: mass for B (vortex position)** $\quad$ ϵ_2 **: 2d Ω-deformation**