非可換インスタントンのADHM構成法

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Goal

• Extension of ADHM construction of instantons
to non-commutative (NC) spaces (with full proof).

\[ [x^\mu, x^\nu] = i \theta^{\mu \nu} \]

NC parameter (real const.)
1. Introduction

Anti-Self-Dual Yang-Mills (ASDYM) eqs. play important roles in elementary particle theory, geometry and integrable systems.

- Finite-action solutions (instantons) reveal non-perturbative effects in QFT $\leftrightarrow$ ADHM
- A master eq. of lower-dim integrable eqs such as KdV, NLS, Toda, Liouville........ $\leftarrow$ twistor theory, Ward’s conjecture
  

- NC extension $\leftrightarrow$ background (ele-mag) flux
ASDYM eq. with G=U(N)

- **ASDYM eq. (real rep.)**

\[
F_{01} = - F_{23} , \quad F_{02} = - F_{31} , \quad F_{03} = - F_{12} .
\]

\[
F_{\mu\nu} := \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]
\]

\(\mu, \nu = 0, 1, 2, 3\)

Field strength

- **Gauge field**

\(A_\mu: (N \times N \text{ anti-Hermitian})\)

- **There are two descriptions of NC extension:**
  - Moyal-product formalism
  - Operator formalism
NC ASDYM eq. with G=U(N) in Moyal

• NC ASDYM eq. (real rep.)

\[ F_{01}^* = - F_{23}^* , \quad (F_{\mu\nu}^* := \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]^\ast) \]
\[ F_{02}^* = - F_{31}^* , \]
\[ F_{03}^* = - F_{12}^* \]

(Spell: All products are Moyal products.)

\[
\theta^{\mu\nu} = \begin{bmatrix} 0 & \theta^1 & O \\ -\theta^1 & 0 & O \\ O & 0 & \theta^2 \end{bmatrix}
\]

Under the spell, we get a theory on NC spaces:

\[
f(x)^* g(x) := f(x) \exp \left( \frac{i}{2} \theta^{\mu\nu} \tilde{\partial}_\mu \tilde{\partial}_\nu \right) g(x)
\]
\[
= f(x)g(x) + i \frac{\theta^{\mu\nu}}{2} \partial_\mu f(x) \partial_\nu g(x) + O(\theta^2)
\]

\[
[x^{\mu} , x^{\nu}]^\ast := x^{\mu}^* x^{\nu} - x^{\nu}^* x^{\mu} = i \theta^{\mu\nu}
\]
NC ASDYM eq. with $G=U(N)$ in operator

- Take coordinates as operators (in 2dim):

$$\left[ \hat{x}, \hat{y} \right] = i \theta \xrightarrow{\text{complex}} \left[ \hat{z}, \hat{\bar{z}} \right] = 2 \theta \xrightarrow{\text{rescale}} \left[ \hat{a}, \hat{a}^+ \right] = 1$$

Fields (infinite matrix):

$$\hat{F}(\hat{z}, \hat{\bar{z}}) = \sum_{m,n} F_{mn} \left| m \right\rangle \left\langle n \right|$$

Ann.op. Cre. Op. acting on Fock space:

$$H = \oplus C \left| n \right\rangle \quad n = 0, 1, 2, \ldots$$

Occupation number basis

- NC ASDYM eq. (real rep.)

$$\begin{align*}
\hat{F}_{01} &= -\hat{F}_{23}, \\
\hat{F}_{02} &= -\hat{F}_{31}, \\
\hat{F}_{03} &= -\hat{F}_{12}
\end{align*}$$

$$\theta^{\mu\nu} = \begin{bmatrix}
0 & \theta^1 \\ -\theta^1 & 0 \\
\theta^2 & 0 \end{bmatrix} \quad \leftrightarrow \quad H_1$$

$$\theta^2 = \begin{bmatrix}
0 & 0 \\ 0 & \theta^2 \end{bmatrix} \quad \leftrightarrow \quad H_2$$
2. ADHM Construction of NC instantons

- ADHM construction is one of the most useful methods to generate all instanton solutions just by solving matrix equations. (ADHM=Atiyah-Drinfeld-Hitchin-Manin)

- ADHM construction is based on a duality between an instanton moduli space specified by ASDYM eq. (PDE) and a dual moduli space specified by ADHM eq. (Matrix eq.)
ADHM construction of (NC) instantons

**ADHM eq. (G=``U(k)’’):** $k \times k$ matrix eq.

\[
[B_1, B_1^+] + [B_2, B_2^+] + I I^+ - J^+ J = 0 \\
[B_1, B_2] + IJ = 0
\]

**ADHM data**

- $B_{1,2} : k \times k$
- $I : k \times N$
- $J : N \times k$
- $A_{\mu} : N \times N$

**1:1**

**Instantons**

**ASDYM eq. (G=U(N), C_2=-k):** $N \times N$ PDE

\[
F_{\overline{z}_1 \overline{z}_1} + F_{\overline{z}_2 \overline{z}_2} = 0 \\
F_{\overline{z}_1 \overline{z}_2} = 0
\]
D-brane interpretation of ADHM construction

**ADHM eq. (G="U(k)")**: $k \times k$ matrix eq.

\[
[B_1, B_1^+] + [B_2, B_2^+] + I I^+ - J^+ J = 0
\]
\[
[B_1, B_2] + IJ = 0
\]

**ADHM data**

$B_{1,2} : k \times k$, $I : k \times N$, $J : N \times k$

$1:1$

**Instantons**

$A_\mu : N \times N$

**ASDYM eq. (G=U(N), C_2=-k)**: $N \times N$ PDE

\[
F_{z_1 \overline{z}_1} + F_{z_2 \overline{z}_2} = 0
\]
\[
F_{z_1 z_2} = 0
\]

[0-0 strings $\Leftrightarrow k \times k$: B_{1,2}]

[0-4 strings $\Leftrightarrow k \times N$: I,J]

**D-term conditions**

k D0 branes

**SUSY trf. of gaugino**

N D4 branes

**String theory is a treasure house of duality!**
ADHM construction of (NC) instantons

ADHM eq. \((G=\text{``U}(k)'')\): \(k \times k\) matrix eq.

\[
[B_1, B_1^+] + [B_2, B_2^+] + II^+ - JJ^+ = 0
\]

\[
[B_1, B_2] + IJ = 0
\]

Strategy:
(i) Solve ADHM eq.
(ii) Solve duality map eq.
(iii) Construct gauge fields

(i) ADHM data \(B_{1,2} : k \times k, \quad I : k \times N, \quad J : N \times k\)

(ii) Duality map \(\nabla^+ V = 0, \quad V^+ V = 1_N \quad \nabla = \begin{pmatrix}
I^+ & J \\
\bar{z}_2 - B_2^+ & -(z_1 - B_1) \\
\bar{z}_1 - B_1^+ & z_2 - B_2
\end{pmatrix}
\)

\((N + 2k) \times 2k\)

(iii) Instantons \(A_\mu = V^+ \partial_\mu V : N \times N\)

ASDYM eq. \((G=\text{U}(N), \quad C_2=-k)\): \(N \times N\) PDE

\[
F_{z_1 \bar{z}_1} + F_{z_2 \bar{z}_2} = 0
\]

\[
F_{z_1 \bar{z}_2} = 0
\]
ADHM construction of BPST instanton (N=2, k=1)

**ADHM eq. (G=``U(1)’’)**

\[
[B_1, B_1^+] + [B_2, B_2^+] + II^+ - J^+ J = 0
\]

\[
[B_1, B_2] + IJ = 0
\]

(i) \(B_{1,2} = \alpha_{1,2}, \quad I = (\rho, 0), J = \begin{pmatrix} 0 \\ \rho \end{pmatrix} \)

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(iii) \(A_\mu = \frac{i(x-b)\nu}{(x-b)^2 + \rho^2} \eta_{\mu\nu}^{(-)}, \quad F_{\mu\nu} = \frac{2i\rho^2}{((x-b)^2 + \rho^2)^2} \eta_{\mu\nu}^{(-)} \)

**ASDYM eq. (G=U(2), C_2=-1)**

\[
F_{z_1\bar{z}_1} + F_{z_2\bar{z}_2} = 0
\]

\[
F_{\bar{z}_1\bar{z}_2} = 0
\]

Final remark: matrices B and coords. z always appear in pair: z–B

Small instanton singularity

\[ \rho \to 0 \Rightarrow \text{singular} \]

\[ \rho = 0 \Rightarrow \text{singularity} \]

\[ M \]
ADHM construction of NC BPST instanton (N=2,k=1)

ADHM eq. (G=``U(1)') $1 \times 1$ matrix eq.

\[
[B_1, B_1^+] + [B_2, B_2^+] + I I^+ - J^+ J = \zeta
\]

\[
[B_1, B_2] + IJ = 0
\]

$B_{1,2} = \alpha_{1,2}, \quad I = (\sqrt{\rho^2 + \zeta}, 0), \quad J = \begin{pmatrix} 0 \\ \rho \end{pmatrix}$

\[\begin{align*}
B_1, B_2, I, J, F_{\mu\nu} & : \text{something smooth} \\
\rho \to 0 & \quad \text{Regular!} \quad \text{(U(1) instanton!)}
\end{align*}\]

ASDYM eq. (G=U(2), $C_2=-1$)

\[
F_{z_1 \bar{z}_1} + F_{z_2 \bar{z}_2} = 0
\]

\[
F_{z_1 \bar{z}_2} = 0
\]

Resolution of the singularity

[Nekrasov&Schwarz, hep-th/9802068]

[Regina Hizuka Nakajima]

const. in RHS comes from

\[-[z_1, \bar{z}_1] - [z_2, \bar{z}_2]\]
ADHM construction of NC U(1) instanton (N=1,k=1)

ADHM eq. (G=``U(1)'') 1 × 1 matrix eq.

\[
[B_1, B_1^+] + [B_2, B_2^+] + II^+ - J^+ J = \zeta \\
[B_1, B_2] + IJ = 0
\]

\[
B_{1,2} = 0, \quad I = \sqrt{\zeta}, \quad J = 0
\]

\[
A_\mu, F_{\mu
\nu} : \text{something smooth}
\]

Regular U(1) instanton

ASDYM eq. (G=U(1), C_2=-1)

\[
F_{z_1 \bar{z}_1} + F_{z_2 \bar{z}_2} = 0 \\
F_{\bar{z}_1 \bar{z}_2} = 0
\]

Resolution of the singularity
Look more closely on the duality map

ADHM eq. ($G=\text{``U(1)''}$) $1 \times 1$ matrix eq.

$[B_1, B_1^+] + [B_2, B_2^+] + JJ^+ - J^+ J = \zeta$

$[B_1, B_2] + IJ = 0$

$B_{1,2} = \alpha_{1,2}, \quad I = \sqrt{\zeta}, \quad J = 0$

\[ \downarrow \quad \text{Duality map} \quad \hat{\nabla}^+ \hat{\nabla} = 0, \quad \hat{\nabla}^+ \hat{\nabla} = 1, \quad \hat{\nabla} = \begin{pmatrix} \sqrt{\zeta} & 0 \\ \hat{a}_2^+ & -\hat{a}_1 \\ \hat{a}_1^+ & \hat{a}_2 \end{pmatrix} \]

$A_\mu = \hat{\nabla}^+ \partial_\mu \hat{\nabla}$

ASDYM eq. ($G=\text{U(1)}, \ C_2=\text{-1}$)

$F_{z_1 \bar{z}_1} + F_{z_2 \bar{z}_2} = 0$

$F_{z_1 \bar{z}_2} = 0$

Naive (unnormalized) solution:

$\hat{V}_0 = \begin{pmatrix} \hat{a}_1^+ \hat{a}_1 + \hat{a}_2^+ \hat{a}_2 \\ -\sqrt{\zeta} \hat{a}_2 \\ -\sqrt{\zeta} \hat{a}_1 \end{pmatrix} |0,0\rangle \quad \text{V}_0 \text{ has zero-mode! (unhappy)}$
Look more closely on the duality map

ADHM eq. (G=``U(1)'') 1 × 1 matrix eq.

\[
[B_1, B_1^+] + [B_2, B_2^+] + II^+ - JJ^+ = \zeta
\]

\[
[B_1, B_2] + IJ = 0
\]

\[B_{1,2} = \alpha_{1,2}, \quad I = \sqrt{\zeta}, \quad J = 0\]

\[\downarrow \text{Duality map}\]

\[\hat{A}_\mu = \hat{V}^+ \partial_\mu \hat{V}\]

(Ex) 2-dim case

\[
\hat{S} = \sum_{n=0}^{\infty} |n\rangle \langle n+1|
\]

\[
\hat{S} \hat{S}^+ = 1
\]

\[
\hat{S}^+ \hat{S} = 1 - |0\rangle \langle 0|
\]

V has no zero-mode! (happy(^-^))

ASDYM eq. (G=U(1), C_2=-1)

\[F_{z_1 \bar{z}_1} + F_{z_2 \bar{z}_2} = 0\]

\[F_{z_1 \bar{z}_2} = 0\]

\[\hat{V}_0 = \begin{pmatrix}
\hat{a}_1^+ \hat{a}_1 + \hat{a}_2^+ \hat{a}_2 \\
-\sqrt{\zeta} \hat{a}_2 \\
-\sqrt{\zeta} \hat{a}_1
\end{pmatrix}
\]

\[\text{V}_0 \text{ has zero-mode! (unhappy)}\]
• Furuuchi-san first pointed out such problem on operator zero-modes for $V$ and solved it by using the shift operators.

[hep-th/9912047], [hep-th/0005199]

• We apply Furuuchi-san’s observation and other properties on NC field theories to all ingredients in the ADHM construction and prove several missing pieces to complete the beautiful NC ADHM duality.
3. Origin of instanton number from ADHM

We can prove the following NC formula:

$$\int d^4 x \text{Tr}_N F_{\mu\nu} \ast F^{\mu\nu} = -\int d^4 x \partial^2 \partial_\mu \text{Tr}_k f^{-1} \ast \partial^\mu f$$

$$\left(\frac{\theta \to 0}{\theta \to 0}\right) \text{Tr}_N F_{\mu\nu} F^{\mu\nu} = -\partial^2 \partial^2 \log \det f$$

[$\text{Corrigan–Goddard–Osborn–Templeton}$]

$$f := (\nabla^+ \nabla)^{-1} \text{ We can get this inverse in the whole Fock space by using shift operators!}$$

Then we have:

$$C_2 = -\frac{1}{16\pi^2} \int d^4 x \text{Tr}_N F_{\mu\nu} \ast F^{\mu\nu} = -\frac{1}{16\pi^2} \int d^4 x \partial^2 \partial_\mu \text{Tr}_k f^{-1} \ast \partial^\mu f$$

$$= -\frac{8}{16\pi^2} \int d\Omega \text{Tr}_k 1_k = -k$$

comes from the size of ADHM data!
4. Conclusion

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0-dim Dirac eq
\n\n\n\n```

Instantons $A_\mu$
- ASDYM eq. $\Leftrightarrow$

$$[\sigma_\mu, D^- \ast D]_\ast = 0$$

$$\mathfrak{g}(x, y) = \frac{V_x \ast V_y}{|x - y|^2}$$

ADHM data $S, T_\mu$
- ADHM eq. $\Leftrightarrow$

$$[\sigma_\mu, \nabla^- \ast \nabla]_\ast = 0$$

$$\bar{\nabla}^2 \mathfrak{g} = \psi^+ \ast \psi$$

4-dim Dirac eq.
$$D^- \ast \psi = 0, \quad \int d^4x \psi^- \ast \psi = 1$$

$$\psi(x) \ast \psi(y)^\dagger = \delta(x - y) - D_x \ast G(x - y) \ast D_y$$