非可換インスタントンのADHM構成法

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Goal

- **Extension of ADHM construction of instantons to non-commutative (NC) spaces (with full proof).**

$$[x^\mu, x^\nu] = i\underline{\theta^{\mu\nu}}$$

NC parameter (real const.)

1. Introduction

Anti-Self-Dual Yang-Mills (ASDYM) eqs. play important roles in elementary particle theory, geometry and integrable systems.

- **Finite-action solutions (instantons) reveal non-perturbative effects in QFT ←ADHM**
- **a master eq. of lower-dim integrable eqs such as KdV, NLS, Toda, Liouville……..**
  ←twistor theory, Ward's conjecture
  [Mason-Woodhouse,…], [NC extension: MH-Toda, MH, Gilson-MH-Nimmo,…]
- **NC extension ←→ background (ele-mag) flux**

ASDYM eq. with G=U(N)

- **ASDYM eq. (real rep.)**

$$\mu, \nu = 0,1,2,3$$

$$F_{01} = -F_{23},$$
$$F_{02} = -F_{31},$$
$$F_{03} = -F_{12}.$$

$$F_{\mu\nu} := \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$$

Field strength

A_μ: **Gauge field**
(N × N anti-Hermitian)

- **There are two descripitions of NC extension:**
  - **Moyal-product formalism**
  - **Operator formalism**

NC ASDYM eq. with G=U(N) in Moyal

- **NC ASDYM eq. (real rep.)**

$$F^*_{01} = -F^*_{23}, \quad F^*_{02} = -F^*_{31}, \quad F^*_{03} = -F^*_{12}$$

$$(F^*_{\mu\nu} := \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]_*)$$

$$\theta^{\mu\nu} = \left[\begin{array}{cc|cc} 0 & \theta^1 & & \\ -\theta^1 & 0 & \multicolumn{2}{c}{O} \\ \hline & & 0 & \theta^2 \\ \multicolumn{2}{c|}{O} & -\theta^2 & 0 \end{array}\right]$$

(Spell: All products are Moyal products.)

Under the spell, we get a theory on NC spaces:

$$f(x) * g(x) := f(x) \exp\left(\frac{i}{2}\theta^{\mu\nu}\overleftarrow{\partial}_\mu \overrightarrow{\partial}_\nu\right) g(x)$$

$$= f(x)g(x) + i\frac{\theta^{\mu\nu}}{2}\partial_\mu f(x)\partial_\nu g(x) + O(\theta^2)$$

$$[x^\mu, x^\nu]_* := x^\mu * x^\nu - x^\nu * x^\mu = i\theta^{\mu\nu}$$

NC ASDYM eq. with G=U(N) in operator

- Take coordinates as operators (in 2dim):

$$[\hat{x},\hat{y}]=i\theta \xrightarrow{\text{complex}} [\hat{z},\hat{\bar{z}}]=2\theta \xrightarrow{\text{rescale}} [\hat{a},\hat{a}^{+}]=1$$

Ann.op. Cre. Op.
acting on Fock space:

Fields (infinite matrix):

$$\hat{F}(\hat{z},\hat{\bar{z}})=\sum_{m,n}^{\infty}F_{mn}\left|m\right\rangle\left\langle n\right|$$

$$H=\oplus C\left|n\right\rangle \quad n=0,1,2,\ldots$$

Occupation number basis

- NC ASDYM eq. (real rep.)

$$\hat{F}_{01}=-\hat{F}_{23}\,,\qquad \hat{F}_{02}=-\hat{F}_{31}\,,\qquad \hat{F}_{03}=-\hat{F}_{12}$$

$$\theta^{\mu\nu}=\left[\begin{array}{cc|cc} 0 & \theta^{1} & \multicolumn{2}{c}{O} \\ -\theta^{1} & 0 & & \\ \hline \multicolumn{2}{c|}{O} & 0 & \theta^{2} \\ & & -\theta^{2} & 0 \end{array}\right] \begin{array}{c} \curvearrowleft H_1 \\ \curvearrowleft H_2 \end{array}$$

2. ADHM Construction of NC instantons

- ADHM construction is one of the most useful methods to generate all instanton solutions just by solving matrix equations.

  (ADHM=Atiyah-Drinfeld-Hitchin-Manin)

- ADHM construction is based on a duality between an instanton moduli space specified by ASDYM eq. (PDE) and a dual moduli space specified by ADHM eq. (Matrix eq.)

ADHM construction of (NC) instantons

[Atiyah–Drinfeld–Hitchin–Manin]

ADHM eq. (G=``U(k)''): k × k matrix eq.

$$[B_1, B_1^+]+[B_2, B_2^+]+I\,I^+ - J^+J = 0$$

$$[B_1, B_2]+IJ = 0$$

↕ **1:1**

ADHM data $B_{1,2} : k\times k,$

$I : k\times N, \quad J : N\times k$

Instantons $A_\mu : N\times N$

ASDYM eq. (G=U(N), C_2=-k): N × N PDE

$$F_{z_1\bar{z}_1} + F_{z_2\bar{z}_2} = 0$$

$$F_{z_1 z_2} = 0$$

D-brane interpretation of ADHM construction

[Witten, Douglas]

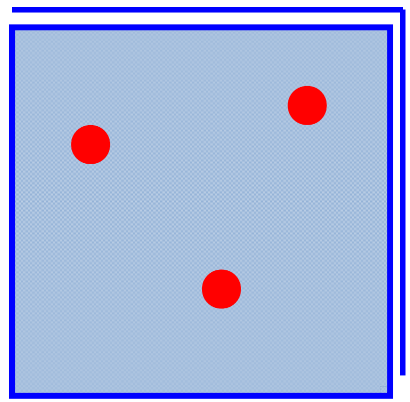
ADHM eq. (G=``U(k)''): k × k matrix eq.

$$[B_1, B_1^+] + [B_2, B_2^+] + I\,I^+ - J^+ J = 0$$
$$[B_1, B_2] + IJ = 0$$

0-0 strings ⇔ k × k: $B_{1,2}$
0-4 strings ⇔ k × N : I,J

D-term conditions

k D0 branes



N D4 branes

ADHM data $B_{1,2} : k \times k,$

$I : k \times N, \quad J : N \times k$

1:1

Instantons $A_\mu : N \times N$

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SUSY trf. of gaugino

String theory is a treasure house of duality!

ADHM construction of (NC) instantons

ADHM eq. (G=``U(k)''): k × k matrix eq.

$$[B_1, B_1^+] + [B_2, B_2^+] + I I^+ - J^+ J = 0$$
$$[B_1, B_2] + IJ = 0$$

Strategy:
(i) Solve ADHM eq.
(ii) Solve dualty map eq.
(iii) Construct gauge fields

(i) ADHM data $B_{1,2} : k \times k, \quad I : k \times N, \quad J : N \times k$

↓ (ii) **Duality map** $\nabla^+ V = 0, \quad V^+ V = 1_N$

$$\nabla = \begin{pmatrix} I^+ & J \\ \bar{z}_2 - B_2^+ & -(z_1 - B_1) \\ \bar{z}_1 - B_1^+ & z_2 - B_2 \end{pmatrix}$$
$$(N + 2k) \times 2k$$

(iii) Instantons $A_\mu = V^+ \partial_\mu V : N \times N$

ASDYM eq. (G=U(N), C_2=-k): N × N PDE

$$F_{z_1 \bar{z}_1} + F_{z_2 \bar{z}_2} = 0$$
$$F_{z_1 z_2} = 0$$

ADHM construction of BPST instanton (N=2,k=1)

ADHM eq. (G=``U(1)'')

$$[B_1, B_1^+] + [B_2, B_2^+] + I\,I^+ - J^+ J = 0$$
$$[B_1, B_2] + IJ = 0$$

(i) $B_{1,2} = \alpha_{1,2}, \quad I = (\rho, 0),\ J = \begin{pmatrix} 0 \\ \rho \end{pmatrix}$

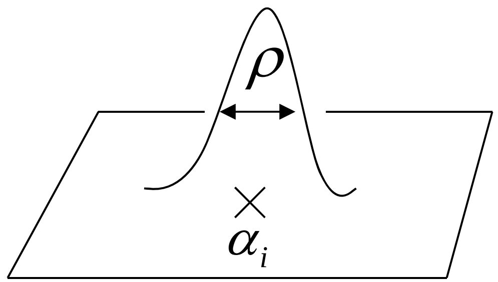
$B_{1,2} \updownarrow$ **position**; $I, J \updownarrow$ **size**

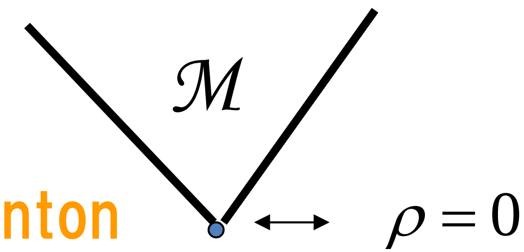
(iii) $A_\mu = \dfrac{i(x-b)^\nu \eta^{(-)}_{\mu\nu}}{(x-b)^2 + \rho^2}, \quad F_{\mu\nu} = \dfrac{2i\rho^2}{((x-b)^2+\rho^2)^2}\eta^{(-)}_{\mu\nu}$ $\xrightarrow{\rho \to 0}$ **singular**

ASDYM eq. (G=U(2), C_2=-1)

$$F_{z_1\bar{z}_1} + F_{z_2\bar{z}_2} = 0$$
$$F_{z_1 z_2} = 0$$

Final remark: matrices B and coords. z always appear in pair: z−B





Small instanton singularity

ADHM construction of NC BPST instanton (N=2,k=1)

[Nekrasov&Schwarz, hep-th/9802068]

ADHM eq. (G=``U(1)'') 1 × 1 matrix eq.

$$[B_1, B_1^+] + [B_2, B_2^+] + I I^+ - J^+ J = \zeta$$
$$[B_1, B_2] + IJ = 0$$

const. in RHS comes from $-[z_1, \bar{z}_1] - [z_2, \bar{z}_2]$

$$B_{1,2} = \alpha_{1,2}, \quad I = (\sqrt{\rho^2 + \zeta}, 0), \; J = \begin{pmatrix} 0 \\ \rho \end{pmatrix}$$

position　　size → slightly fat?

$A_\mu, F_{\mu\nu}$: something smooth

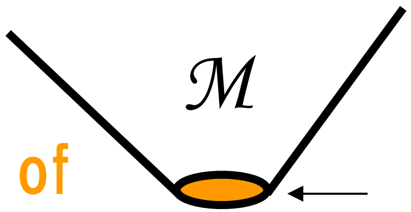
$\rho \to 0$

Regular!
(U(1) instanton!)

[Hiraku Nakajima]

ASDYM eq. (G=U(2), C_2=-1)

$$F_{z_1\bar{z}_1} + F_{z_2\bar{z}_2} = 0$$
$$F_{z_1 z_2} = 0$$



$\rho = 0$

Resolution of the singularity

ADHM construction of NC U(1) instanton (N=1,k=1)

ADHM eq. (G=``U(1)'') 1 × 1 matrix eq.

$$[B_1, B_1^+]+[B_2, B_2^+]+I I^+ - J^+ J = \zeta$$
$$[B_1, B_2]+IJ = 0$$

$B_{1,2} = 0,\quad I = \sqrt{\zeta},\ J = 0$

$B_{1,2}$ ↕ position; $\sqrt{\zeta}$ ↕ size

$A_\mu, F_{\mu\nu}$: something smooth

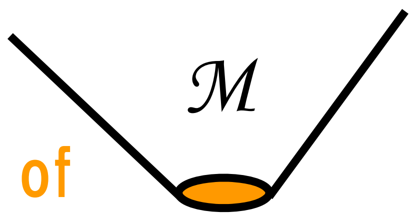
ASDYM eq. (G=U(1), C_2=-1)

$$F_{z_1\bar{z}_1} + F_{z_2\bar{z}_2} = 0$$
$$F_{z_1 z_2} = 0$$

Regular U(1) instanton

↕

Resolution of the singularity



Look more closely on the duality map

ADHM eq. (G=``U(1)'') 1 × 1 matrix eq.

$$[B_1, B_1^+] + [B_2, B_2^+] + I I^+ - J^+ J = \zeta$$
$$[B_1, B_2] + IJ = 0$$

$$B_{1,2} = \alpha_{1,2}, \quad I = \sqrt{\zeta}, \ J = 0$$

↓ **Duality map** $\hat{\nabla}^+ \hat{V} = 0, \quad \hat{V}^+ \hat{V} = 1, \quad \hat{\nabla} = \begin{pmatrix} \sqrt{\zeta} & 0 \\ \hat{a}_2^+ & -\hat{a}_1 \\ \hat{a}^+{}_1 & \hat{a}_2 \end{pmatrix}$

$$\hat{A}_\mu = \hat{V}^+ \partial_\mu \hat{V}$$

ASDYM eq. (G=U(1), C_2=-1)

$$F_{z_1 \bar{z}_1} + F_{z_2 \bar{z}_2} = 0$$
$$F_{z_1 z_2} = 0$$

Naive (unnormalized) solution:

$$\hat{V}_0 = \begin{pmatrix} \hat{a}_1^+ \hat{a}_1 + \hat{a}_2^+ \hat{a}_2 \\ -\sqrt{\zeta}\, \hat{a}_2 \\ -\sqrt{\zeta}\, \hat{a}_1 \end{pmatrix} |0,0\rangle$$

V_0 has zero-mode!

(unhappy)

Look more closely on the duality map

ADHM eq. (G=``U(1)'') 1 × 1 matrix eq.

$$[B_1,B_1^+]+[B_2,B_2^+]+I\,I^+ - J^+J = \zeta$$
$$[B_1,B_2]+IJ = 0$$

$$B_{1,2} = \alpha_{1,2}, \qquad I = \sqrt{\zeta},\ J = 0$$

↓ **Duality map**

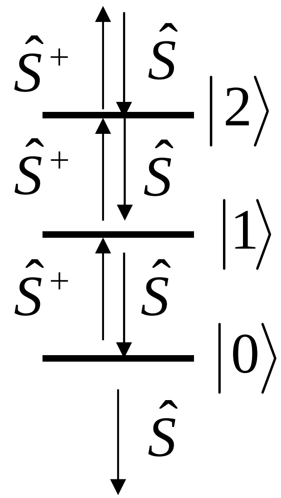
$$\hat{A}_\mu = \hat{V}^+ \partial_\mu \hat{V}$$

ASDYM eq. (G=U(1), C_2=-1)

$$F_{z_1\bar{z}_1} + F_{z_2\bar{z}_2} = 0$$
$$F_{z_1 z_2} = 0$$

(Ex) 2-dim case

$$\hat{S} = \sum_{n=0}^{\infty} |n\rangle\langle n+1|$$
$$\hat{S}\hat{S}^+ = 1$$
$$\hat{S}^+\hat{S} = 1 - |0\rangle\langle 0|$$



V has no zero-mode! (happy(^-^))

$$\hat{V} = \hat{V}_0 \hat{N}_0 \hat{S}, \qquad \hat{S}\hat{S}^+ = 1,\ \hat{S}^+\hat{S} = 1 - |0,0\rangle\langle 0,0|$$

Shift operator

$$\hat{V}_0 = \begin{pmatrix} \hat{a}_1^+\hat{a}_1 + \hat{a}_2^+\hat{a}_2 \\ -\sqrt{\zeta}\,\hat{a}_2 \\ -\sqrt{\zeta}\,\hat{a}_1 \end{pmatrix} |0,0\rangle$$

V_0 has zero-mode! (unhappy)

- **Furuuchi-san first pointed out such problem on operator zero-modes for V and solved it by using the shift operators.**

  [hep-th/9912047], [hep-th/0005199]

- **We apply Furuuchi-san's observation and other properties on NC field theories to all ingredients in the ADHM construction and prove several missing pieces to complete the beautiful NC ADHM duality.**

3. Origin of instanton number from ADHM

We can prove the following NC formula:

$$\int d^4x \, Tr_N F_{\mu\nu} * F^{\mu\nu} = -\int d^4x \, \partial^2 \partial_\mu Tr_k f^{-1} * \partial^\mu f$$

$$\left(\xrightarrow{\theta \to 0} Tr_N F_{\mu\nu} F^{\mu\nu} = -\partial^2 \partial^2 \log \det f \right)$$ [Corrigan–Goddard–Osborn–Templeton]

$$f := (\nabla^+ \nabla)^{-1}$$ **We can get this inverse in the whole Fock space by using shift operators !**

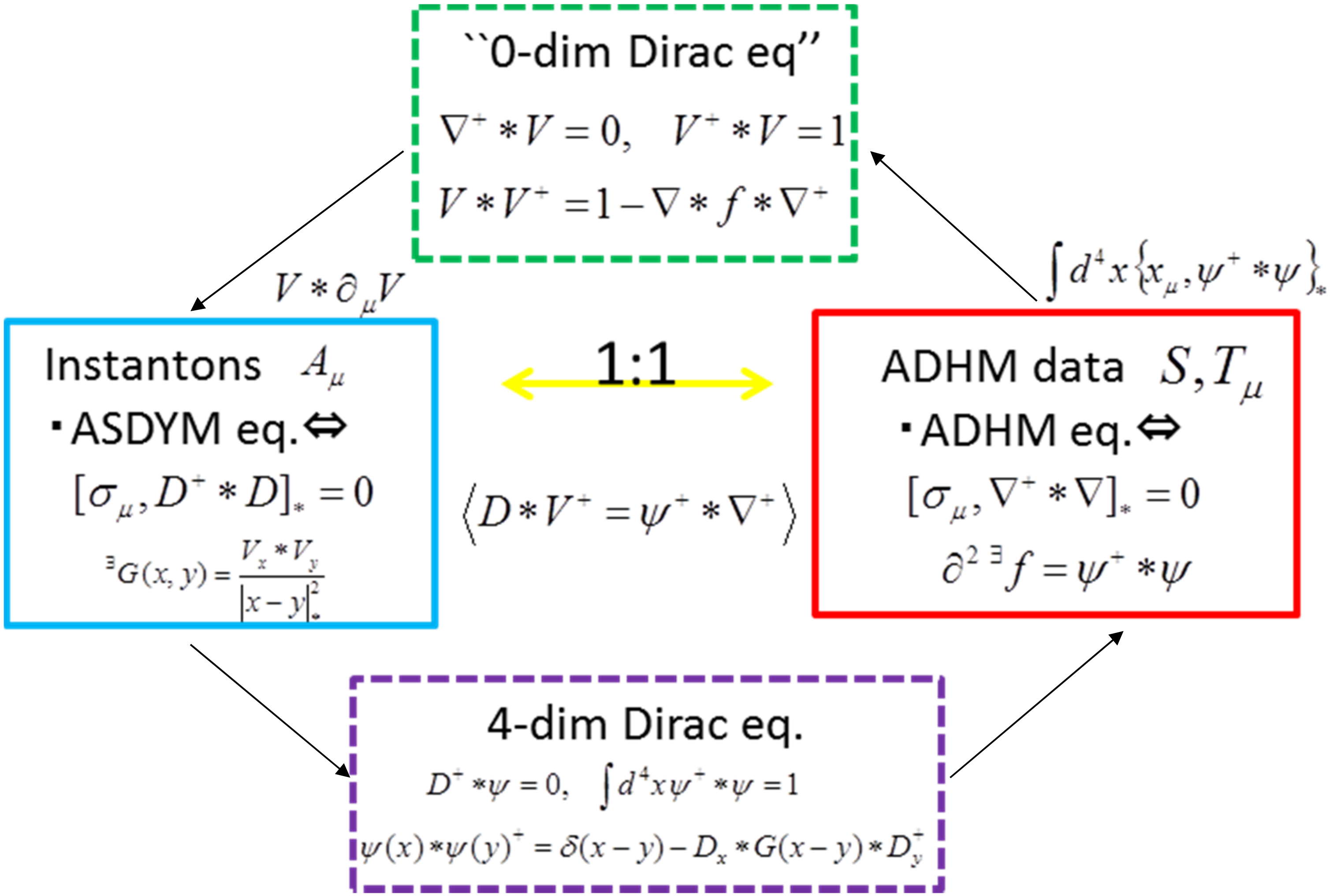
Then we have:

$$C_2 = -\frac{1}{16\pi^2} \int d^4x \, Tr_N F_{\mu\nu} * F^{\mu\nu} = -\frac{1}{16\pi^2} \int d^4x \, \partial^2 \partial_\mu Tr_k f^{-1} * \partial^\mu f$$

$$= -\frac{8}{16\pi^2} \int d\Omega \, Tr_k \underline{1_k} = -k$$

comes from the size of ADHM data!

4. Conclusion



[MH-Nakatsu, arXiv:1208.nnnn [hep-th]]