Numerical studies of the ABJM theory for arbitrary N at arbitrary coupling constant

本多正純

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西村淳氏(総研大 & KEK), 柴正太郎氏 (KEK) 、吉田豊氏 (KEK)
との共同研究に基づく。

2012/7/25

基研研究会「場の理論と弦理論」

S^3上のU (N) × U(N) ABJM理論のシミュレーション

<u>動機 :</u>

(k: Chern-Simons level)

CFT$_3$ / AdS$_4$

∃伝統的なテクあり $\lambda = \frac{N}{k} = \text{fixed}, \; N \gg 1$

Type IIA superstring on $AdS_4 \times CP^3$

ABJM理論

$(= 3\text{d } \mathcal{N} \geq 6 \text{ SCFT})$

[Aharony-Bergman-Jafferis-Maldacena '08]

（中間）

非常に難しい

最近新しいテクが台頭 $k \ll N^{1/5}$

[Herzog-Klebanov-Pufu-Tesileanu '10]

[Marino-Putrov '11, cf. 森山さんの講演]

$\text{M} - \text{theory}$ on $AdS_4 \times S^7/Z_k$

<u>数値シミュレーションで全領域を調べる</u>

今回行ったこと

S^3上のU (N) × U(N) ABJM理論において、
（全ての対称性を厳密に保ちつつ）自由エネルギーをモンテカルロ計算

↓

- 既知の全ての解析的な結果との比較・検証
（この発表では一部を紹介）
- 未知の領域での計算結果と、既知の結果との関係

ABJMの自由エネルギーの進展

・2008年6月：ABJM誕生

・2010年7月：Planar極限が強結合で計算される [Drukker-Marino-Putrov]

$$F_{\rm DMP} = -\frac{\pi\sqrt{2}}{3}\frac{(\lambda - 1/24)^{3/2}}{\lambda^2}N^2 \qquad \text{up to } O(e^{-2\pi\sqrt{2\lambda}})$$

[Cf. Cagnazzo-Sorokin-Wulff '09]

$\lambda \gg 1$ ⇒ $\boxed{F_{\rm DMP} \simeq -\frac{\pi\sqrt{2}}{3}\frac{N^2}{\sqrt{\lambda}}}$ SUGRAの結果と一致！

※CP^{3}はnontrivial な2-cycleを持つ $(R^2_{CP^3} = 4\pi\sqrt{2\lambda})$

⇒ $$\exp\left[-2\pi\sqrt{2\lambda}\right] = \exp\left[-\frac{1}{2\pi}(\pi R^2_{CP^3})\right] = \exp\left[-\frac{1}{2\pi\alpha'}\mathrm{Area}(CP^1)\right]\Big|_{\alpha'=1}$$

～CP^1⊂ CP^{3}に巻きつくstring＝**worldsheet instanton**？

・2010年11月：k=fixed, N→∞で計算される [Herzog-Klebanov-Pufu-Tesileanu]

$$\boxed{F = -\frac{\pi\sqrt{2k}}{3}N^{3/2} + o(N^{3/2})}$$

形式的に一致
(※ λ=N/k)

（続）ABJMの自由エネルギーの進展

・2011年6月：強結合でPlanar周りのall genusの足し上げが行われる

[Fuji-Hirano-Moriyama]

$$F_{\rm FHM} = \log\left[\frac{1}{\sqrt{2}}\left(\frac{4\pi^2 N}{\lambda}\right)^{1/3} {\rm Ai}\left[\left(\frac{\pi}{\sqrt{2}}\left(\frac{N}{\lambda}\right)^2\right)^{2/3}\left(\lambda - \frac{1}{24} - \frac{\lambda^2}{3N^2}\right)\right]\right] \quad \text{up to } O(e^{-2\pi\sqrt{2\lambda}})$$

形式的に一致

(※ λ=N/k)

・2011年10月：N=2で厳密に計算される [Okuyama]

・2011年10月：k<<1, k<<Nで計算される（Fermi Gas） [Marino-Putrov]

$$F_{\rm Fermi} = \log\left[\frac{1}{\sqrt{2}}(4\pi^2 k)^{1/3} {\rm Ai}\left[\left(\frac{\pi k^2}{\sqrt{2}}\right)^{2/3}\left(\frac{N}{k} - \frac{1}{24} - \frac{1}{3k^2}\right)\right]\right] + A(k) - \frac{1}{2}\log 2$$

$$\text{up to } O(e^{-2\pi\sqrt{2N/k}}), O(e^{-\pi\sqrt{2kN}})$$

ただし、 $A(k) = \dfrac{2\zeta(3)}{\pi^2 k} - \dfrac{k}{12} - \dfrac{\pi^2 k^3}{4320}$ up to $O(k^5)$

Airyへの補正

→大きいkでどうなる？？

・2012年2月：全領域での数値計算（＝この講演）

少なくともUp to instanton的効果で、**全てのkで**

自由エネルギーがkの関数としてなめらかにつながる！

（・ここ一週間：k=1での厳密計算 [Hatsuda-Moriyama-Okuyama, Putrov-Yamazaki] ）

講演の流れ

1. Introduction & Motivation
2. どうやってABJMを計算機にのせるか？
3. 結果
4. 解釈
5. まとめ（と残った疑問点）

どうやってABJMを計算機にのせるか？

～正攻法(＝格子正則化)によるアプローチ～

作用: $$S_{\text{ABJM}} = S_{\text{CS}} + S_{\text{Matter}}$$

S_{CS} : CS term from $U(N)\times U(N)$ $\mathcal{N}=2$ Vectormultiplet

S_{Matter} : From $\mathcal{N}=4$ bi−fundamental hypermultiplet$\times 2$

定式化の困難

・格子上でCS項を構成するのが容易ではない [Cf. Bietenholz-Nishimura '00]

・格子上でのSUSYの扱いは一般に難しい [Cf. Giedt '09]

実用上の困難

・∃多自由度のフェルミオン場 → 計算量が多い

・CS項が純虚 → 符号問題

ほぼ絶望的。

(Cont’d)どうやってABJMを計算機にのせるか？

格子正則化によるアプローチは絶望的･･･

分配関数に関しては局所化が適用可能：

[Kapustin-Willet-Yaakov ’09]
[Cf. Jafferis ’10, Hama-Hosomichi-Lee ’10]

$$Z_{ABJM} = \frac{1}{N!^2} \int \frac{d^N \mu}{(2\pi)^N} \frac{d^N \nu}{(2\pi)^N} \frac{\prod_{i<j} \left[2\sinh\left(\frac{\mu_i - \mu_j}{2}\right)\right]^2 \left[2\sinh\left(\frac{\nu_i - \nu_j}{2}\right)\right]^2}{\prod_{i,j} \left[2\cosh\left(\frac{\mu_i - \nu_j}{2}\right)\right]^2} \exp\left[\frac{ik}{4\pi} \sum_{i=1}^{N} (\mu_i^2 - \nu_i^2) \right]$$

符号問題

さらなる簡単化が起こる！

ABJM行列模型の簡単化

[Kapustin-Willett-Yaakov ’10, Okuyama ‘11, Marino-Putrov ‘11]

$$Z_{ABJM} = \frac{1}{N!^2}\int \frac{d^N\mu}{(2\pi)^N}\frac{d^N\nu}{(2\pi)^N}\frac{\prod_{i<j}\left[2\sinh\left(\frac{\mu_i-\mu_j}{2}\right)\right]^2\left[2\sinh\left(\frac{\nu_i-\nu_j}{2}\right)\right]^2}{\prod_{i,j}\left[2\cosh\left(\frac{\mu_i-\nu_j}{2}\right)\right]^2}\exp\left[\frac{ik}{4\pi}\sum_{i=1}^{N}(\mu_i^2-\nu_i^2)\right]$$

Cauchy identity:

$$\boxed{u_i = e^{\mu_i},\ \ v_i = e^{\nu_i}}$$

$$\frac{\prod_{i<j}(u_i-u_j)(v_i-v_j)}{\prod_{i,j}(u_i+v_j)} = \sum_\sigma (-1)^\sigma \prod_i \frac{1}{u_i+v_{\sigma(i)}} \Longleftrightarrow \frac{\prod_{i<j}\left[2\sinh\left(\frac{\mu_i-\mu_j}{2}\right)\right]\left[2\sinh\left(\frac{\nu_i-\nu_j}{2}\right)\right]}{\prod_{i,j}\left[2\cosh\left(\frac{\mu_i-\nu_j}{2}\right)\right]} = \sum_\sigma(-1)^\sigma\prod_i\frac{1}{2\cosh\left(\frac{\mu_i-\nu_{\sigma(i)}}{2}\right)}$$

$$Z_{\mathrm{ABJM}} = \frac{1}{N!}\sum_\sigma(-1)^\sigma\int\frac{d^N\mu}{(2\pi)^N}\frac{d^N\nu}{(2\pi)^N}\prod_i\frac{1}{\left[2\cosh\left(\frac{\mu_i-\nu_i}{2}\right)\right]\left[2\cosh\left(\frac{\mu_i-\nu_{\sigma(i)}}{2}\right)\right]}\exp\left[\frac{ik}{4\pi}\sum_{i=1}^{N}(\mu_i^2-\nu_i^2)\right]$$

Fourier変換:
$$\frac{1}{2\cosh p} = \frac{1}{\pi}\int dx\,\frac{e^{\frac{2i}{\pi}px}}{2\cosh x}$$

$$Z_{\mathrm{ABJM}} = \frac{1}{N!}\sum_\sigma(-1)^\sigma\frac{1}{\pi^{2N}}\int d^Nx\,d^Ny\,\frac{1}{\prod_i 2\cosh x_i\cdot 2\cosh y_i}$$
$$\int\frac{d^N\mu}{(2\pi)^N}\frac{d^N\nu}{(2\pi)^N}\exp\left[\frac{i}{\pi}\sum_i(\mu_i-\nu_i)x_i+\frac{i}{\pi}\sum_i(\mu_i y_i-\nu_i y_{\sigma(i)})+\frac{ik}{4\pi}\sum_{i=1}^{N}(\mu_i^2-\nu_i^2)\right]$$

（続）ABJM行列模型の簡単化

$$Z_{\text{ABJM}} = \frac{1}{N!}\sum_\sigma (-1)^\sigma \frac{1}{\pi^{2N}} \int d^N x d^N y \, \frac{1}{\prod_i 2\cosh x_i \cdot 2\cosh y_i}$$
$$\int \frac{d^N\mu}{(2\pi)^N}\frac{d^N\nu}{(2\pi)^N} \exp\left[\frac{i}{\pi}\sum_i (\mu_i - \nu_i)x_i + \frac{i}{\pi}\sum_i (\mu_i y_i - \nu_i y_{\sigma(i)}) + \frac{ik}{4\pi}\sum_{i=1}^N (\mu_i^2 - \nu_i^2)\right]$$

フレネル積分

$$Z_{\text{ABJM}} = \frac{1}{N!}\sum_\sigma (-1)^\sigma \frac{1}{k^N \pi^{2N}} \int d^N x d^N y \, \frac{1}{\prod_i 2\cosh x_i \cdot 2\cosh y_i} e^{-\frac{2i}{k\pi}\sum_{i=1}^N x_i (y_i - y_{\sigma(i)})}$$

Fourier変換 : $\displaystyle \frac{1}{2\cosh p} = \frac{1}{\pi}\int dx \frac{e^{\frac{2i}{\pi}px}}{2\cosh x}$

$$Z_{\text{ABJM}} = \frac{1}{N!}\sum_\sigma (-1)^\sigma \int \frac{d^N x}{(2\pi k)^N} \frac{1}{\prod_i 2\cosh\left(\frac{x_i}{2}\right) \cdot 2\cosh\left(\frac{x_i - x_{\sigma(i)}}{2k}\right)}$$

Cauchy id.: $\displaystyle \sum_\sigma (-1)^\sigma \prod_i \frac{1}{2\cosh\left(\frac{x_i - x_{\sigma(i)}}{2k}\right)} = \frac{1}{2^N}\prod_{i<j}\tanh^2\left(\frac{x_i - x_j}{2k}\right)$

$$Z(N,k) = \frac{1}{2^N N!}\int \frac{d^N x}{(2\pi k)^N} \frac{\prod_{i<j}\tanh^2\left(\frac{x_i - x_j}{2k}\right)}{\prod_i 2\cosh\left(\frac{x_i}{2}\right)}$$

どうやってABJMを計算機にのせるか？

局所化を用いて、行列模型の解析に落とす：

$$Z_{ABJM} = \frac{1}{N!^2}\int \frac{d^N\mu}{(2\pi)^N}\frac{d^N\nu}{(2\pi)^N}\frac{\prod_{i<j}\left[2\sinh\left(\frac{\mu_i-\mu_j}{2}\right)\right]^2\left[2\sinh\left(\frac{\nu_i-\nu_j}{2}\right)\right]^2}{\prod_{i,j}\left[2\cosh\left(\frac{\mu_i-\nu_j}{2}\right)\right]^2}\exp\left[\frac{ik}{4\pi}\sum_{i=1}^{N}(\mu_i^2-\nu_i^2)\right]$$

符号問題あり

N次元分厳密に積分

$$Z_{ABJM} = \frac{1}{2^N N!}\int \frac{d^N x}{(2\pi k)^N}\frac{\prod_{i<j}\tanh^2\left(\frac{x_i-x_j}{2k}\right)}{\prod_i 2\cosh\left(\frac{x_i}{2}\right)}$$

符号問題もなく、ノートPCでも計算できてしまうくらいの簡単な積分

よくある質問:なぜこんな簡単化が起きたか？

※k=1限定

$$Z=\frac{1}{2^N N!}\int\frac{d^N x}{(2\pi)^N}\prod_{i<j}\left[\frac{\sinh^2\left(\frac{x_i-x_j}{2}\right)}{\cosh^2\left(\frac{x_i-x_j}{2}\right)}\right]\frac{1}{\prod_i 2\cosh\frac{x_i}{2}}$$

[Kapustin-Willett-Yaakov ’10]
[cf.Intrilligator-Seiberg ’96
Hanany-Witten 97’]

$$\left(\begin{array}{l}\mathcal{N}=4\ \text{vector multiplet}\ +\ \mathcal{N}=4\ \text{hyper multiplet (fundamental)}\\ \qquad\qquad\qquad\qquad\qquad\quad +\ \mathcal{N}=4\ \text{hyper multiplet (adjoint)}\end{array}\right)$$

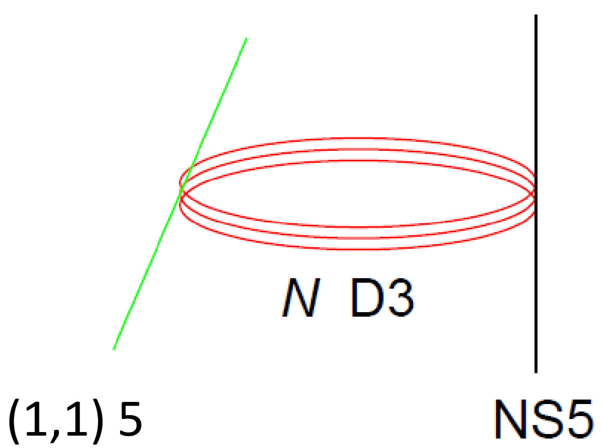
Coulomb branchと
Higgs branchの入れ替え

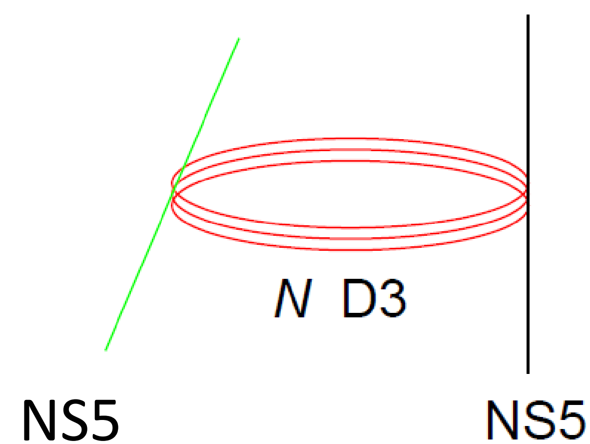
k=1 ABJM ⟺ $\mathcal{N}=4$ theory

AdS/CFT対応

[ABJM ’08]

T変換

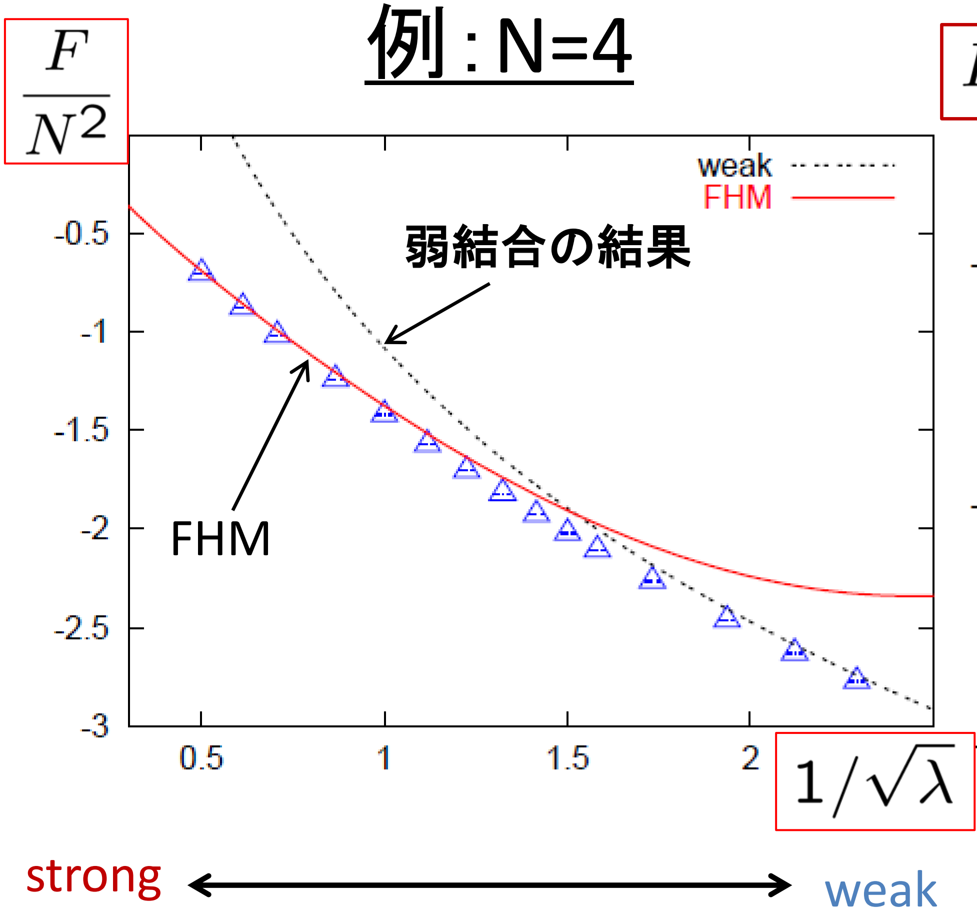


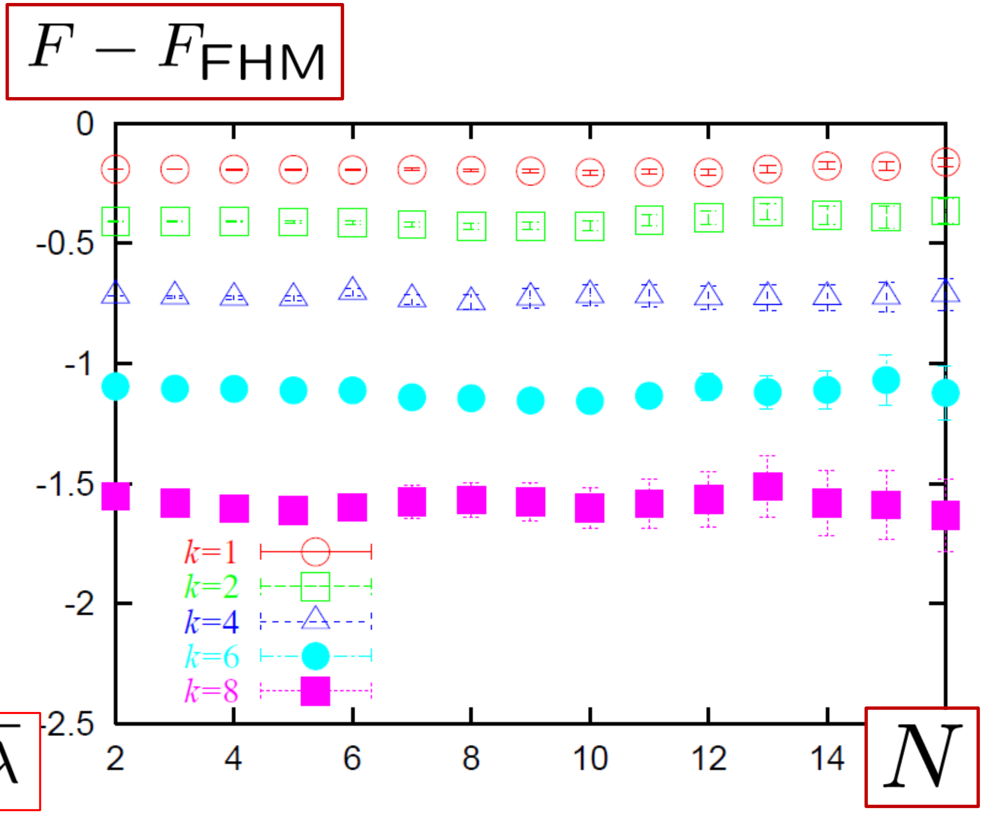


Fuji-Hirano-Moriyamaとの比較

$$F_{\rm FHM} = \log\left[\frac{1}{\sqrt{2}}\left(\frac{4\pi^2 N}{\lambda}\right)^{1/3} {\rm Ai}\left[\left(\frac{\pi}{\sqrt{2}}\left(\frac{N}{\lambda}\right)^2\right)^{2/3}\left(\lambda - \frac{1}{24} - \frac{\lambda^2}{3N^2}\right)\right]\right] \quad \text{up to } O(e^{-2\pi\sqrt{2\lambda}})$$

例 : N=4





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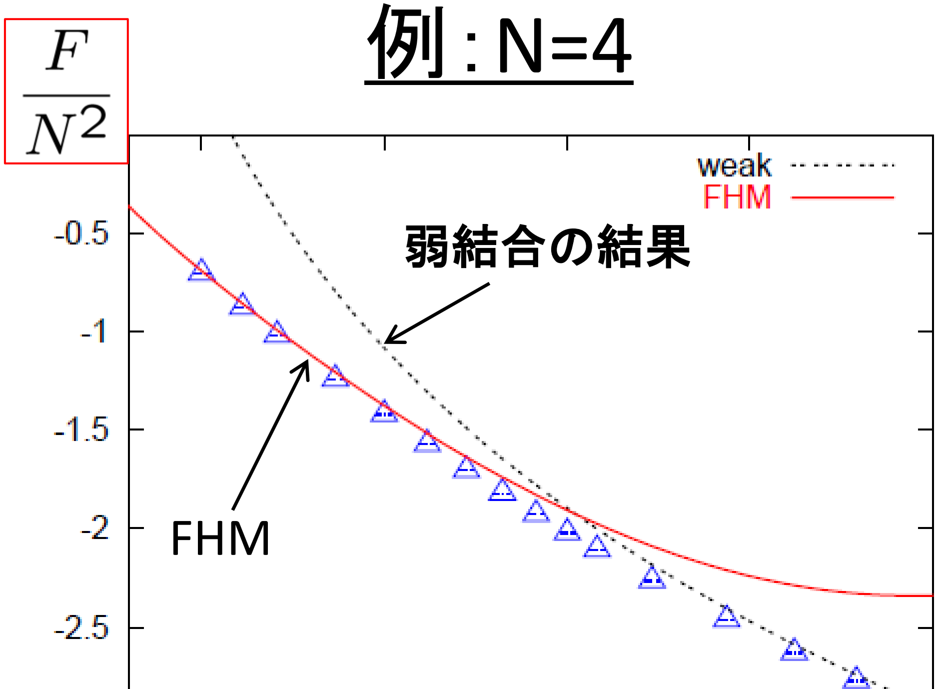

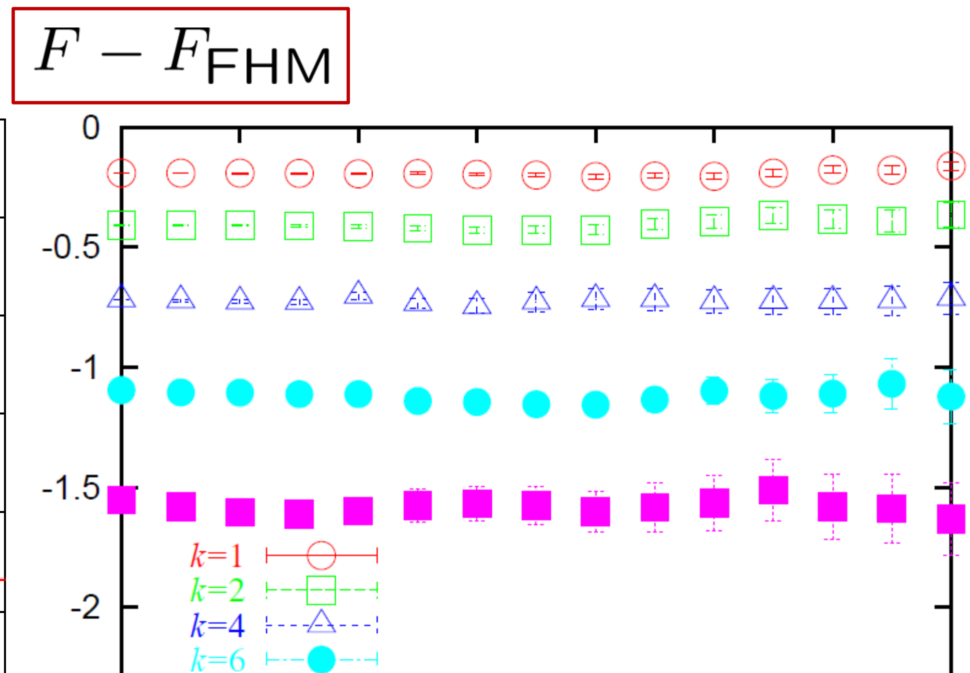

強結合でFHMとシミュレーション結果はほぼ一致
→差をとってより精密に比較

Fuji-Hirano-Moriyamaとの比較

$$F_{\mathrm{FHM}} = \log\left[\frac{1}{\sqrt{2}}\left(\frac{4\pi^2 N}{\lambda}\right)^{1/3} \mathrm{Ai}\left[\left(\frac{\pi}{\sqrt{2}}\left(\frac{N}{\lambda}\right)^2\right)^{2/3}\left(\lambda - \frac{1}{24} - \frac{\lambda^2}{3N^2}\right)\right]\right] \quad \text{up to } O(e^{-2\pi\sqrt{2\lambda}})$$

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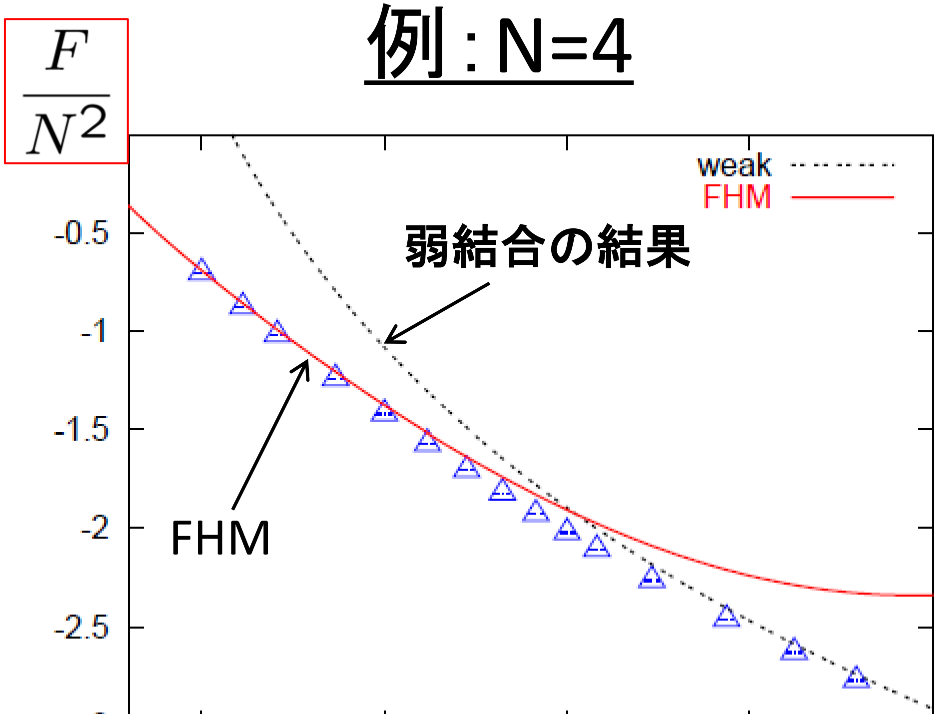

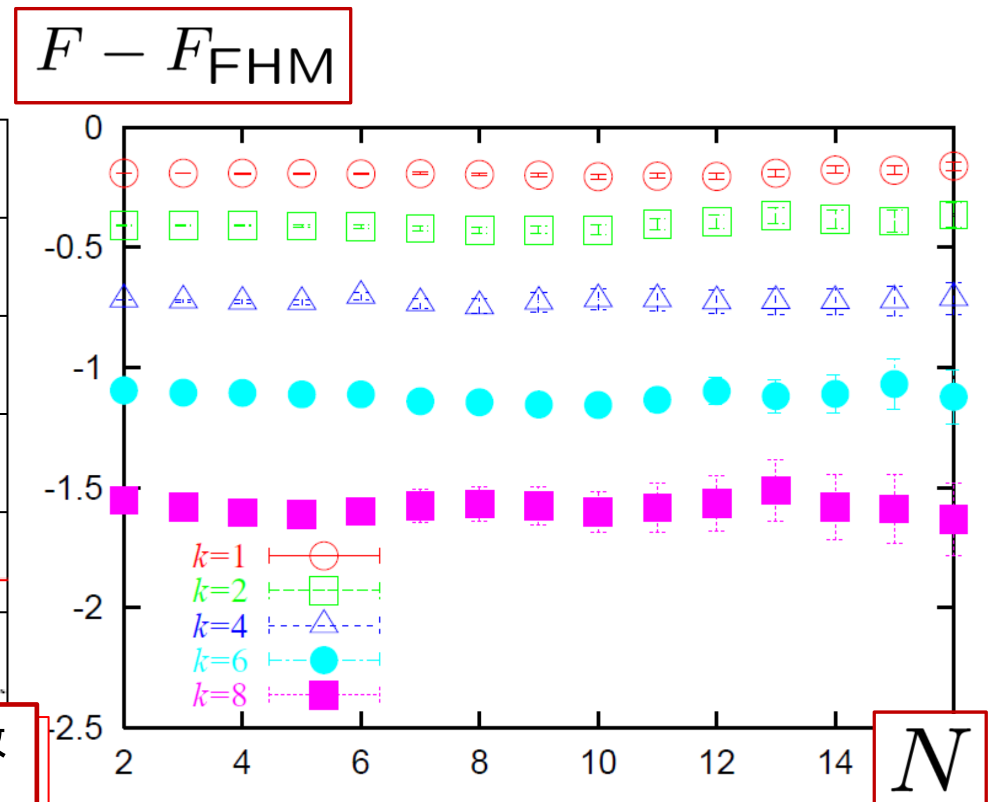

強結合でFHMとシミュレーション結果はほぼ一致
→差をとってより精密に比較

strong ⟷ weak

Fuji-Hirano-Moriyamaとの比較

$$F_{\mathrm{FHM}} = \log\left[\frac{1}{\sqrt{2}}\left(\frac{4\pi^2 N}{\lambda}\right)^{1/3} \mathrm{Ai}\left[\left(\frac{\pi}{\sqrt{2}}\left(\frac{N}{\lambda}\right)^2\right)^{2/3}\left(\lambda - \frac{1}{24} - \frac{\lambda^2}{3N^2}\right)\right]\right] \quad \text{up to } \mathrm{O}(e^{-2\pi\sqrt{2\lambda}})$$

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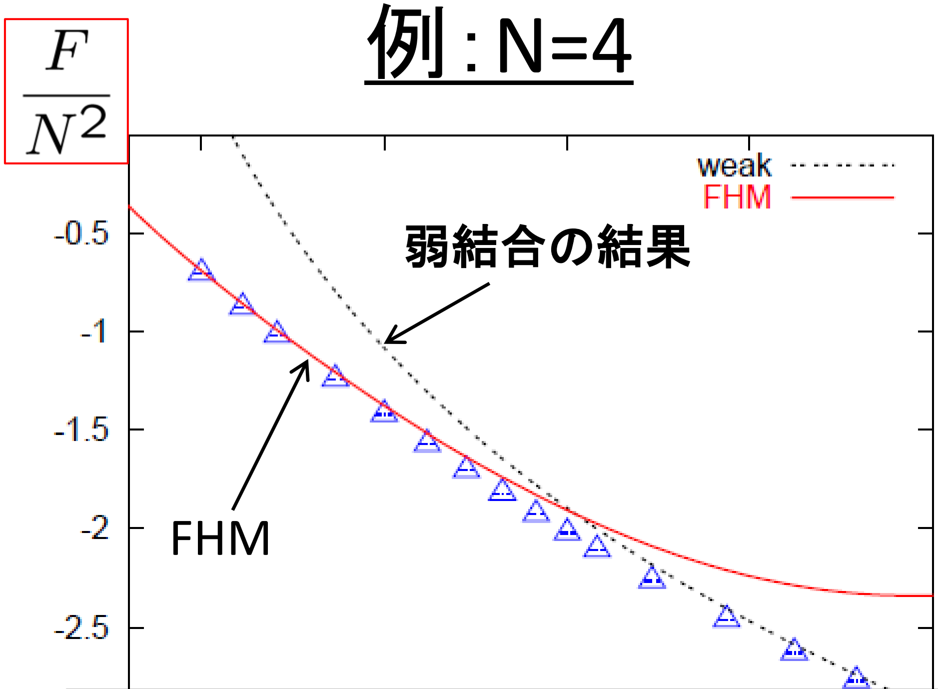
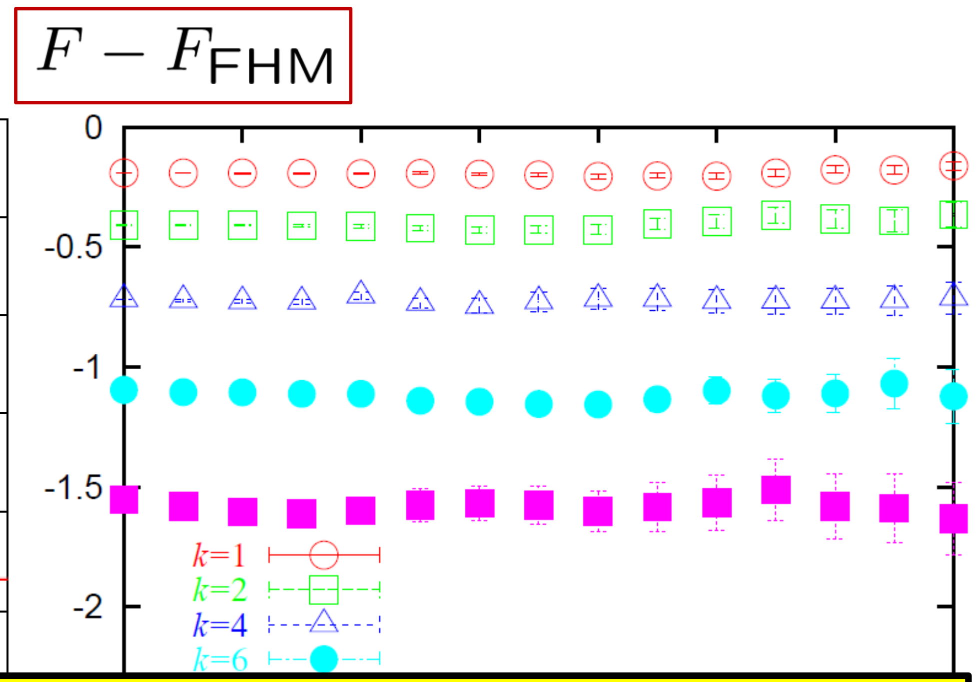
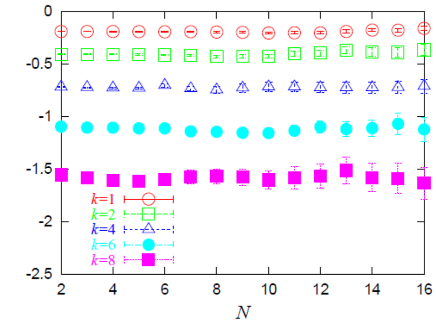




Nに独立で、kのみに依存するずれ
→instanton的振る舞い(～exp dumped)とは異なる

ずれの起源



ずれをkの関数としてフィットすると、

$$F - F_{\mathsf{FHM}} \simeq -0.60103\frac{k^2}{4\pi^2} - \frac{1}{6}\log k - 0.25558$$

実はABJM行列模型は他の模型と解析接続でつながる（と考えられている）:

[Marino-Putrov '10]

$$Z_{ABJM} = \frac{1}{N!^2}\int \frac{d^N\mu}{(2\pi)^N}\frac{d^N\nu}{(2\pi)^N}\frac{\prod_{i<j}\left[2\sinh\left(\frac{\mu_i-\mu_j}{2}\right)\right]^2\left[2\sinh\left(\frac{\nu_i-\nu_j}{2}\right)\right]^2}{\prod_{i,j}\left[2\cosh\left(\frac{\mu_i-\nu_j}{2}\right)\right]^2}\exp\left[\frac{ik}{4\pi}\sum_{i=1}^{N}(\mu_i^2-\nu_i^2)\right]$$

解析接続: $N_1 \to N,\ \ N_2 \to -N$

[Cf. Yost '91, Dijkgraaf-Vafa '03]

Lens space L(2,1)=S^3/Z_2 行列模型:

$$Z_{L(2,1)}(N_1,N_2) = \frac{1}{N_1!N_2!}\int \frac{d^{N_1}\mu}{(2\pi)^{N_1}}\frac{d^{N_2}\nu}{(2\pi)^{N_2}}\prod_{i<j}\left[2\sinh\left(\frac{\mu_i-\mu_j}{2}\right)\right]^2\prod_{a<b}\left[2\sinh\left(\frac{\nu_a-\nu_b}{2}\right)\right]^2\prod_{i,b}\left[2\cosh\left(\frac{\mu_i-\nu_b}{2}\right)\right]^2 e^{-\frac{ik}{4\pi}\left(\sum_i\mu_i^2+\sum_a\nu_a^2\right)}$$

[Aganagic-Klemm-Marino-Vafa '02]

正準量子化

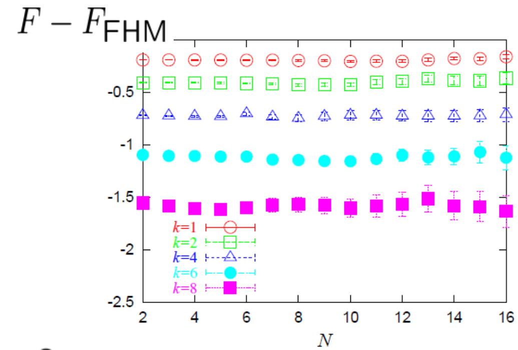
Chern – Simons theory on S^3/Z_2 ⟺ The topological A – model on local $F_0 = P^1 \times P^1$

``constant map''と呼ばれる寄与が知られている:

[Bershadsky-Cecotti-Ooguri-Vafa '93, Faber-Pandharipande '98]

$$F_{\mathsf{const}} = -\frac{\zeta(3)}{8\pi^2}k^2 - \frac{1}{6}\log k + \frac{1}{6}\log\frac{\pi}{2} + 2\zeta'(-1) + \sum_{g=2}^{\infty}\frac{4^g B_{2g}B_{2g-2}}{4g(2g-2)(2g-2)!}\left(\frac{2\pi i}{k}\right)^{2g-2}$$

(続)ずれの起源

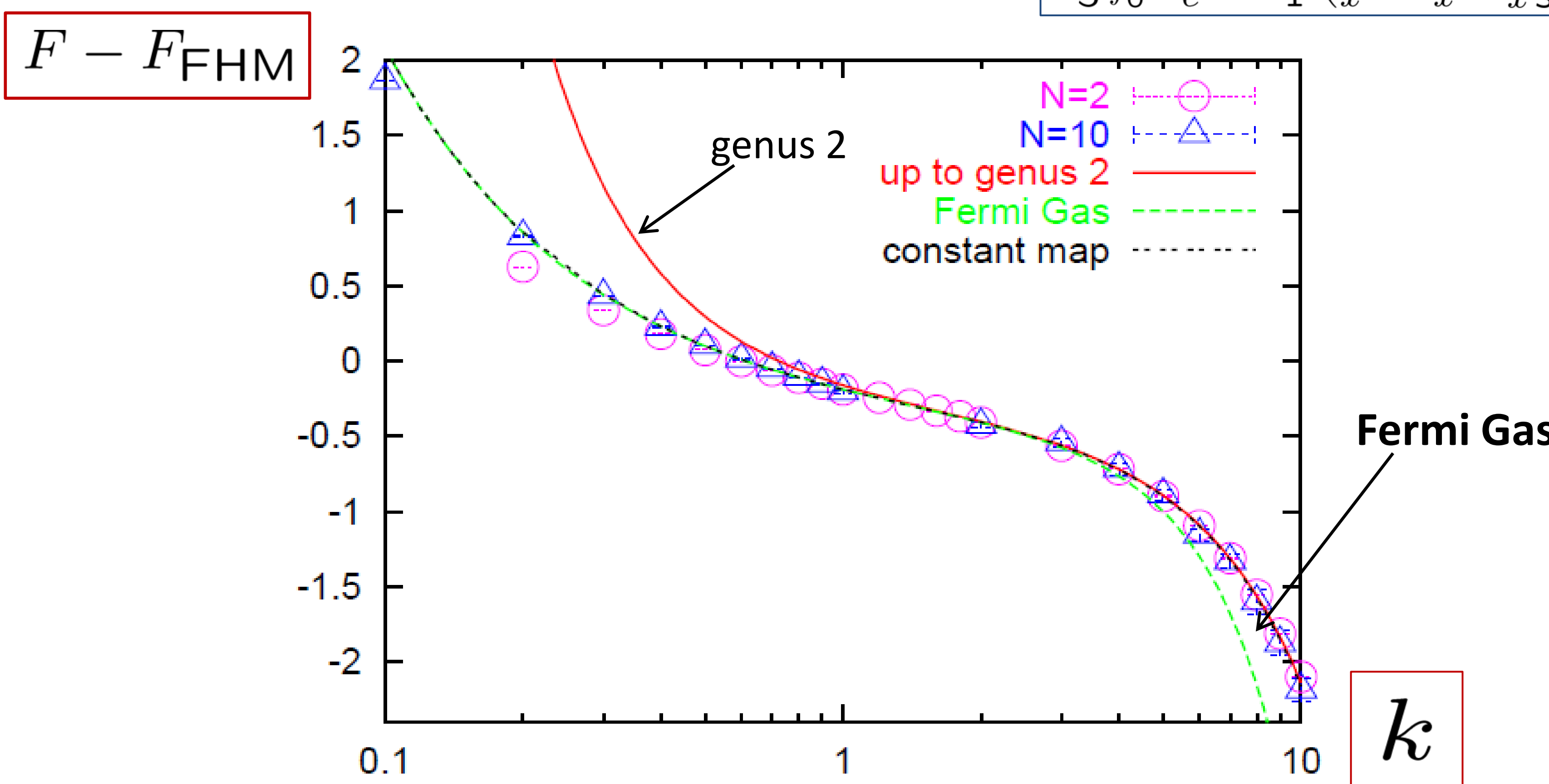


ずれ: $F - F_{\rm FHM} \simeq -0.60103\dfrac{k^2}{4\pi^2} - \dfrac{1}{6}\log k - 0.25558$

topological stringで``constant map''と呼ばれる寄与で説明できる:

$$F_{\rm const} = -\frac{\zeta(3)}{8\pi^2}k^2 - \frac{1}{6}\log k + \frac{1}{6}\log\frac{\pi}{2} + 2\zeta'(-1) + \underline{\sum_{g=2}^{\infty}\frac{4^g B_{2g}B_{2g-2}}{4g(2g-2)(2g-2)!}\left(\frac{2\pi i}{k}\right)^{2g-2}}$$

発散するがボレル和がとれる: $\boxed{-\dfrac{1}{3}\displaystyle\int_0^{\infty}\frac{1}{e^{kx}-1}\left(\frac{3}{x^3}-\frac{1}{x}-\frac{3}{x\sinh^2 x}\right)}$



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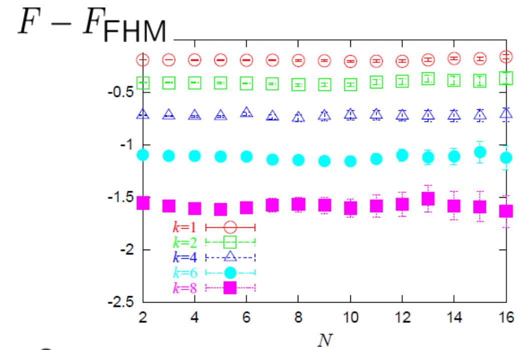

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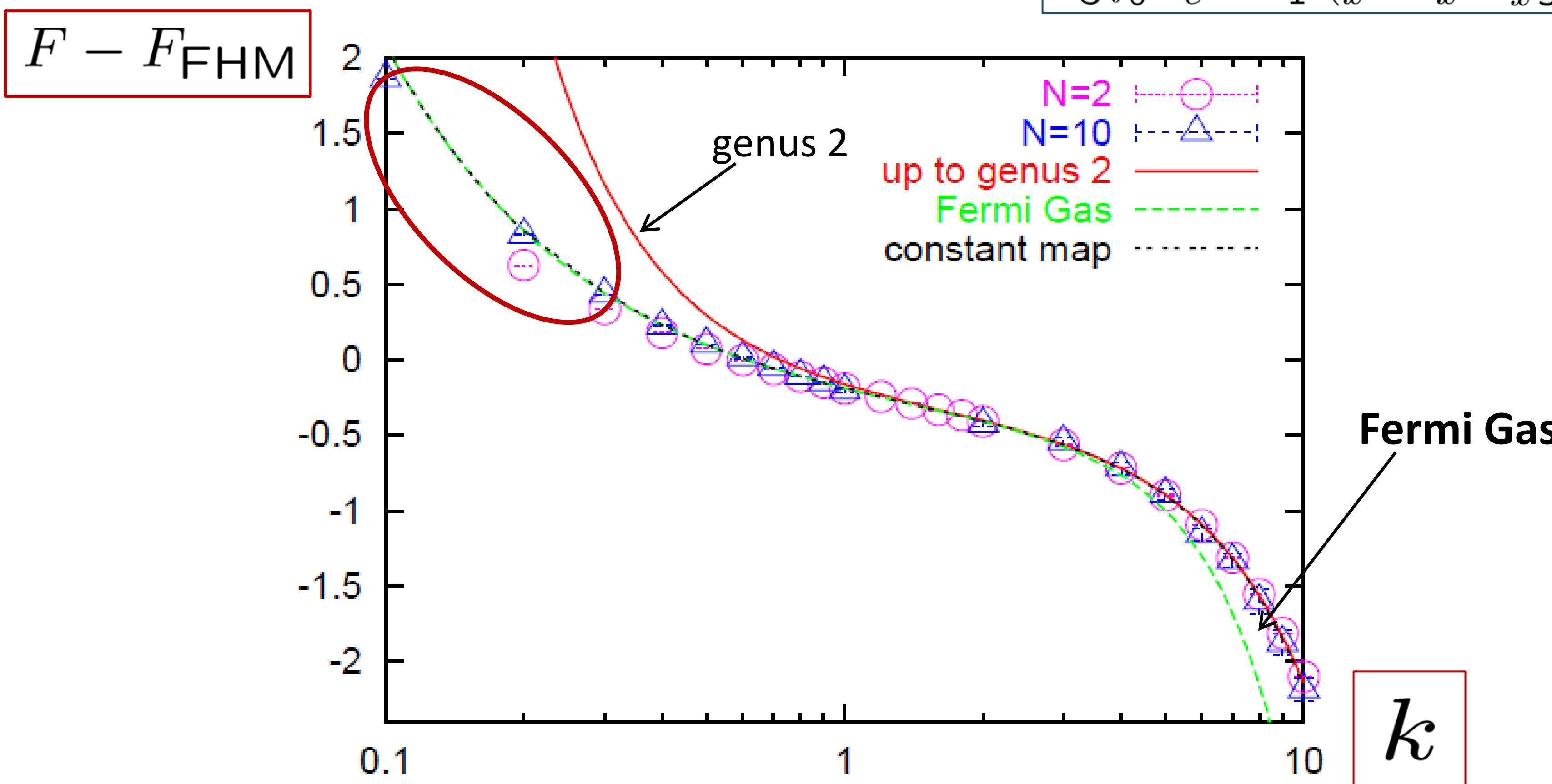
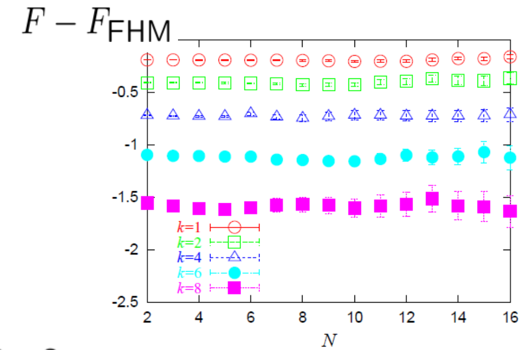
$$F_{\rm const} = -\frac{\zeta(3)}{8\pi^2}k^2 - \frac{1}{6}\log k + \frac{1}{6}\log\frac{\pi}{2} + 2\zeta'(-1) + \sum_{g=2}^{\infty}\frac{4^g B_{2g}B_{2g-2}}{4g(2g-2)(2g-2)!}\left(\frac{2\pi i}{k}\right)^{2g-2}$$

発散するがボレル和がとれる: $\boxed{-\frac{1}{3}\int_0^\infty \frac{1}{e^{kx}-1}\left(\frac{3}{x^3} - \frac{1}{x} - \frac{3}{x\sinh^2 x}\right)}$



（続）ずれの起源



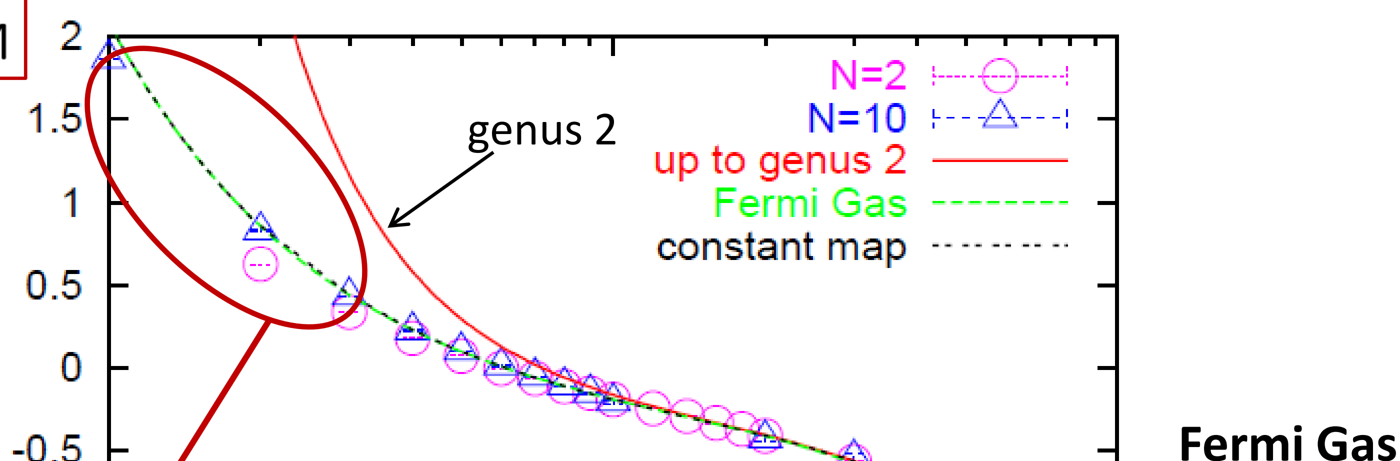
ずれ:　$F - F_{\mathrm{FHM}} \simeq -0.60103\dfrac{k^2}{4\pi^2} - \dfrac{1}{6}\log k - 0.25558$

topological stringで``constant map''と呼ばれる寄与で説明できる:

$$F_{\mathrm{const}} = -\frac{\zeta(3)}{8\pi^2}k^2 - \frac{1}{6}\log k + \frac{1}{6}\log\frac{\pi}{2} + 2\zeta'(-1) + \underline{\sum_{g=2}^{\infty}\frac{4^g B_{2g}B_{2g-2}}{4g(2g-2)(2g-2)!}\left(\frac{2\pi i}{k}\right)^{2g-2}}$$

発散するがボレル和がとれる: $\boxed{-\dfrac{1}{3}\displaystyle\int_0^\infty \frac{1}{e^{kx}-1}\left(\frac{3}{x^3} - \frac{1}{x} - \frac{3}{x\sinh^2 x}\right)}$

$\boxed{F - F_{\mathrm{FHM}}}$



Constant mapのボレル和が
Fermi Gas（small k）の結果まで再現！！
→関係を解析的に理解できないか？

Fermi Gas from Constant map

Constant mapの寄与：

$$F_{\rm const}=-\frac{\zeta(3)}{8\pi^2}k^2-\frac{1}{6}\log k+\frac{1}{6}\log\frac{\pi}{2}+2\zeta'(-1)+\sum_{g=2}^{\infty}\frac{4^g B_{2g}B_{2g-2}}{4g(2g-2)(2g-2)!}\left(\frac{2\pi i}{k}\right)^{2g-2}$$

ボレル和

$$F_{\rm const}=-\frac{\zeta(3)}{8\pi^2}k^2-\frac{1}{6}\log k+\frac{1}{6}\log\frac{\pi}{2}+2\zeta'(-1)-\frac{1}{3}\int_0^\infty \frac{1}{e^{kx}-1}\left(\frac{3}{x^3}-\frac{1}{x}-\frac{3}{x\sinh^2 x}\right)$$

k=0周りで展開

全てのkで正しい表式と考えられる

$$\begin{aligned}F_{\rm const}&=-\frac{1}{2}\log 2+\frac{2\zeta(3)}{\pi^2 k}+\sum_{n=1}^{\infty}(-1)^n\frac{B_{2n}B_{2n-2}}{(2n)!}\pi^{2n-2}k^{2n-1}\\&=-\frac{1}{2}\log 2+\frac{2\zeta(3)}{\pi^2 k}-\frac{k}{12}-\frac{\pi^2k^3}{4320}+\frac{\pi^4k^5}{907200}+\cdots,\end{aligned}$$

All order form？

Fermi Gasの計算結果と一致！

→Fermi Gasの結果はk=0周りの漸近級数

まとめ

S^3上のU (N) × U(N) ABJM理論において、
（全ての対称性を厳密に保ちつつ）自由エネルギーをモンテカルロ計算

- Fuji-Hirano-Moriyamaからのinstanton由来でないズレが constant mapで説明できる
- Constant mapのall genus足し上げは漸近級数だがボレル和可能
- Up to instanton的効果での全領域の自由エネルギーの表式:

$$F = \log\left[\frac{1}{\sqrt{2}}\left(4\pi^2 k\right)^{1/3}\mathrm{Ai}\left[\left(\frac{\pi k^2}{\sqrt{2}}\right)^{2/3}\left(\frac{N}{k}-\frac{1}{24}-\frac{1}{3k^2}\right)\right]\right] + F_{\mathrm{const}} + O\left(e^{-2\pi\sqrt{2N/k}}, e^{-\pi\sqrt{2kN}}\right)$$

～instanton的効果

ただし、

$$F_{\mathrm{const}} = -\frac{\zeta(3)}{8\pi^2}k^2 - \frac{1}{6}\log k + \frac{1}{6}\log\frac{\pi}{2} + 2\zeta'(-1) - \frac{1}{3}\int_0^\infty \frac{1}{e^{kx}-1}\left(\frac{3}{x^3}-\frac{1}{x}-\frac{3}{x\sinh^2 x}\right)$$

- Fermi Gasのall order形を予言:

$$A(k) = \frac{2\zeta(3)}{\pi^2 k} + \sum_{n=1}^{\infty}\frac{(-1)^n B_{2n}B_{2n-2}}{(2n)!}\pi^{2n-2}k^{2n-1}$$

疑問：形式的にはうまくいった。物理的意味は？

Type IIA superstring on $AdS_4 \times CP^3$

M − theory on $AdS_4 \times S^7/Z_k$

？？

AdS/CFT対応

局所化

ABJM理論

ABJM行列模型

フーリエ変換等
（k=1のとき、mirror）

解析接続：
$N_1 \to N,$
$N_2 \to -N$

シミュレーションした行列模型

Lens space L(2,1)=S^3/Z_2 行列模型

フェルミガス

Membrane Instantonのエネルギー ＝ Constant map

The topological A − model on local $F_0 = P^1 \times P^1$

展望

- 他の物理量の計算

  Ex.) BPS Wilson loop [Hanada-M.H.-Honma-Nishimura-Shiba-Yoshida, work in progress]

- 他のゲージ群での計算

  ―本当にU(N)×U(N)で良いのか？

- 他の空間上での計算

  Ex.) Squashed orbifolded sphere [M.H.-Imamura-Yokoyama, work in progress]

- 他の理論での計算

  ー局所化で行列模型に落とした後に解かれていない理論は無数にある

- ミラー対称性の幅広い検証

  ー値が合うか合わないかを調べればよいので、数値計算と相性が良い

完

Appendix

Note on Level shift

[Kao-Lee-Lee '95]

$$\mathcal{L} = k\mathcal{L}_{\text{CS}} + \frac{1}{g_{\text{YM}}^2}\mathcal{L}_{\text{YM}}$$

topological mass $\sim g_{\text{YM}}^2$

Integrate out all fields except the gauge field

At 1-loop level,

$$\mathcal{N} = 0 \text{ SUSY} \quad : \quad \delta k = N$$

$$\mathcal{N} = 1 \text{ SUSY} \quad : \quad \delta k = N/2$$

$$\mathcal{N} = 2 \text{ SUSY} \quad : \quad \delta k = 0$$

$$\mathcal{N} = 3 \text{ SUSY} \quad : \quad \delta k = 0$$

How to calculate the free energy

Problem: Monte Carlo can calculate only expectation value

$$\boxed{Z(N,k)=\frac{1}{2^N N!}\int\frac{d^N x}{(2\pi k)^N}\frac{\prod_{i<j}\tanh^2\left(\frac{x_i-x_j}{2k}\right)}{\prod_i 2\cosh\left(\frac{x_i}{2}\right)}=\frac{1}{2^N N!}\int\frac{d^N x}{(2\pi k)^N}e^{-S(N,k)}}$$

We regard the partition function as an expectation value under another ensemble:

$$\begin{aligned}Z(N_1+N_2,k)&=Z(N_1,k)Z(N_2,k)\frac{Z(N_1+N_2,k)}{Z(N_1,k)Z(N_2,k)}\\&=\frac{N_1!N_2!}{(N_1+N_2)!}Z(N_1,k)Z(N_2,k)\frac{\int d^{N_1+N_2}x\ e^{-S(N_1+N_2,k)}}{\int d^{N_1+N_2}x\ e^{-S(N_1,k)-S(N_2,k)}}\\&=\frac{N_1!N_2!}{(N_1+N_2)!}Z(N_1,k)Z(N_2,k)\left\langle e^{-S(N_1+N_2,k)+S(N_1,k)+S(N_2,k)}\right\rangle_{N_1,N_2}\\&=\frac{N_1!N_2!}{(N_1+N_2)!}Z(N_1,k)Z(N_2,k)\left\langle\prod_{i=1}^{N_1}\prod_{J=N_1+1}^{N}\tanh^2\left(\frac{x_i-x_J}{2k}\right)\right\rangle_{N_1,N_2}\end{aligned}$$

VEV under the action:

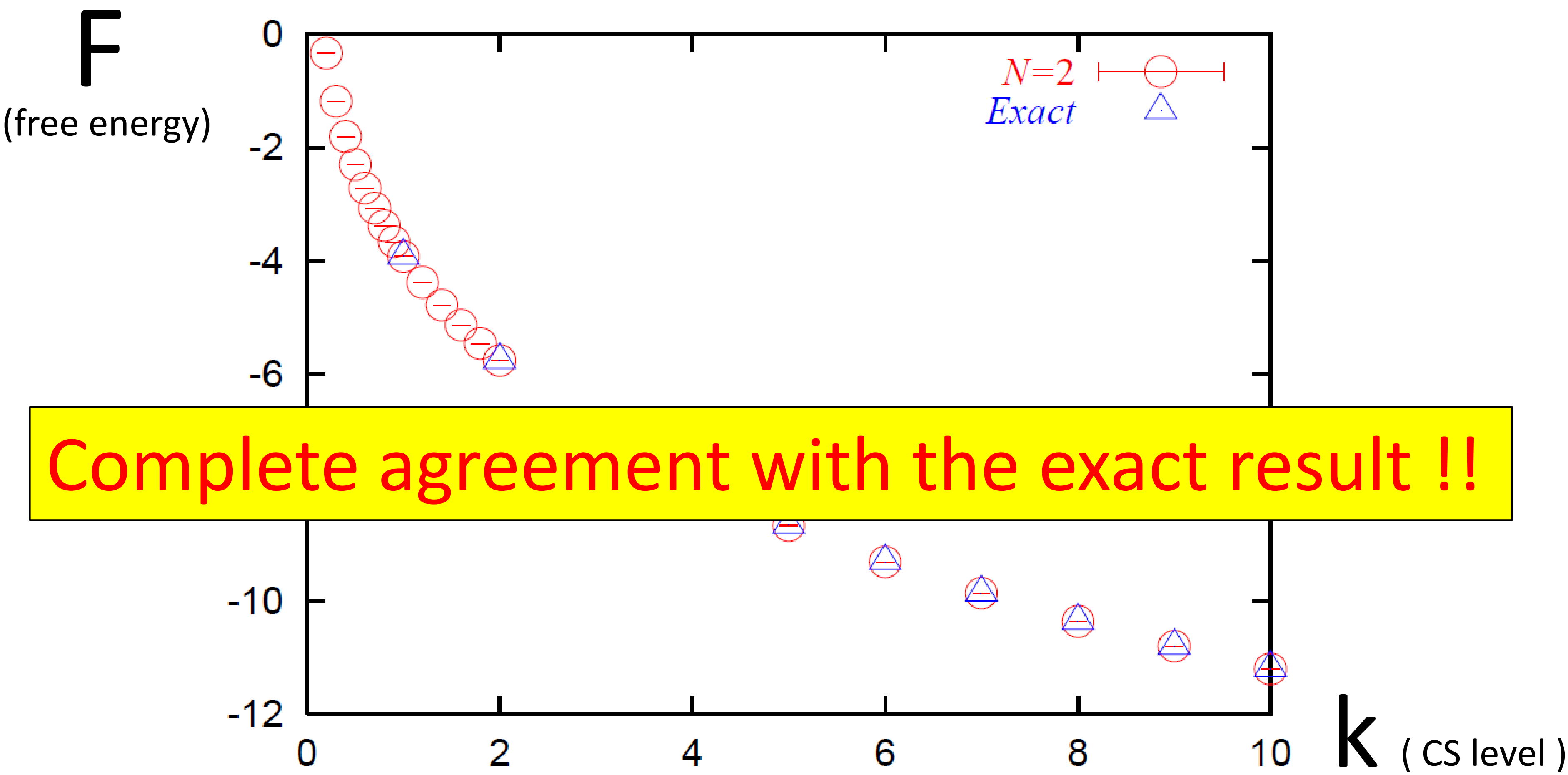
$$S(N_1,k)+S(N_2,k)$$

Note: $Z(1,k)=\frac{1}{2}\int\frac{dx}{2\pi k}\frac{1}{2\cosh\left(\frac{x}{2}\right)}=\frac{1}{4k}$

Warming up: Free energy for N=2

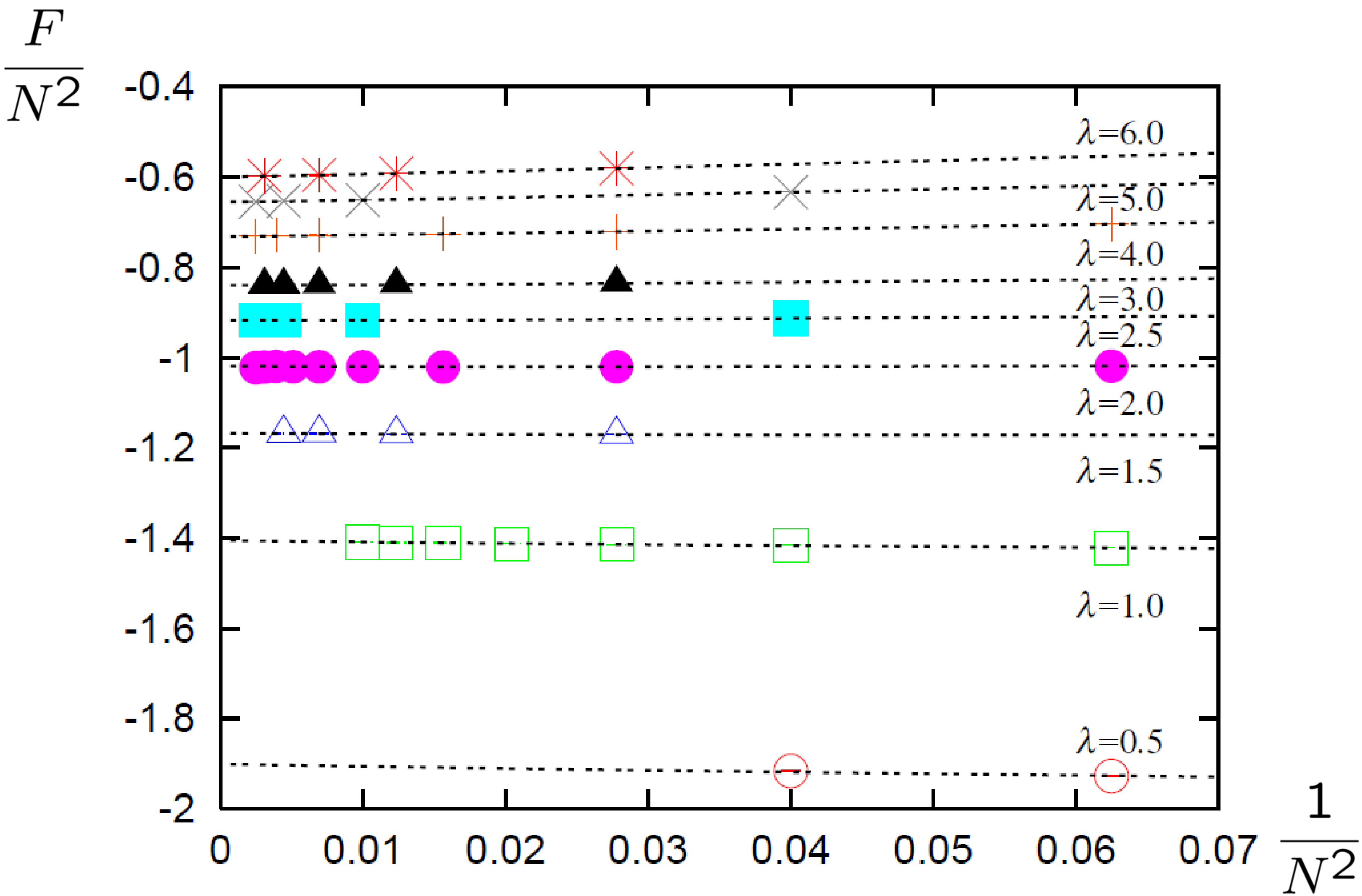
There is the exact result for N=2: [Okuyama '11]

$$
\begin{cases}
Z_{ABJM} = \dfrac{1}{16}\left[\dfrac{1}{k}\displaystyle\sum_{s=1}^{k-1}(-1)^{s-1}\left(\dfrac{1}{2}-\dfrac{s}{k}\right)\tan^2\dfrac{\pi s}{k} + \dfrac{(-1)^{\frac{k-1}{2}}}{\pi}\right] & \text{for odd k} \\
Z_{ABJM} = \dfrac{1}{16}\left[\dfrac{1}{k}\displaystyle\sum_{s=1}^{k-1}(-1)^{s-1}\left(\dfrac{1}{2}-\dfrac{s}{k}\right)^2\tan^2\dfrac{\pi s}{k}\right] & \text{for even k}
\end{cases}
$$



Complete agreement with the exact result !!

Taking planar limit



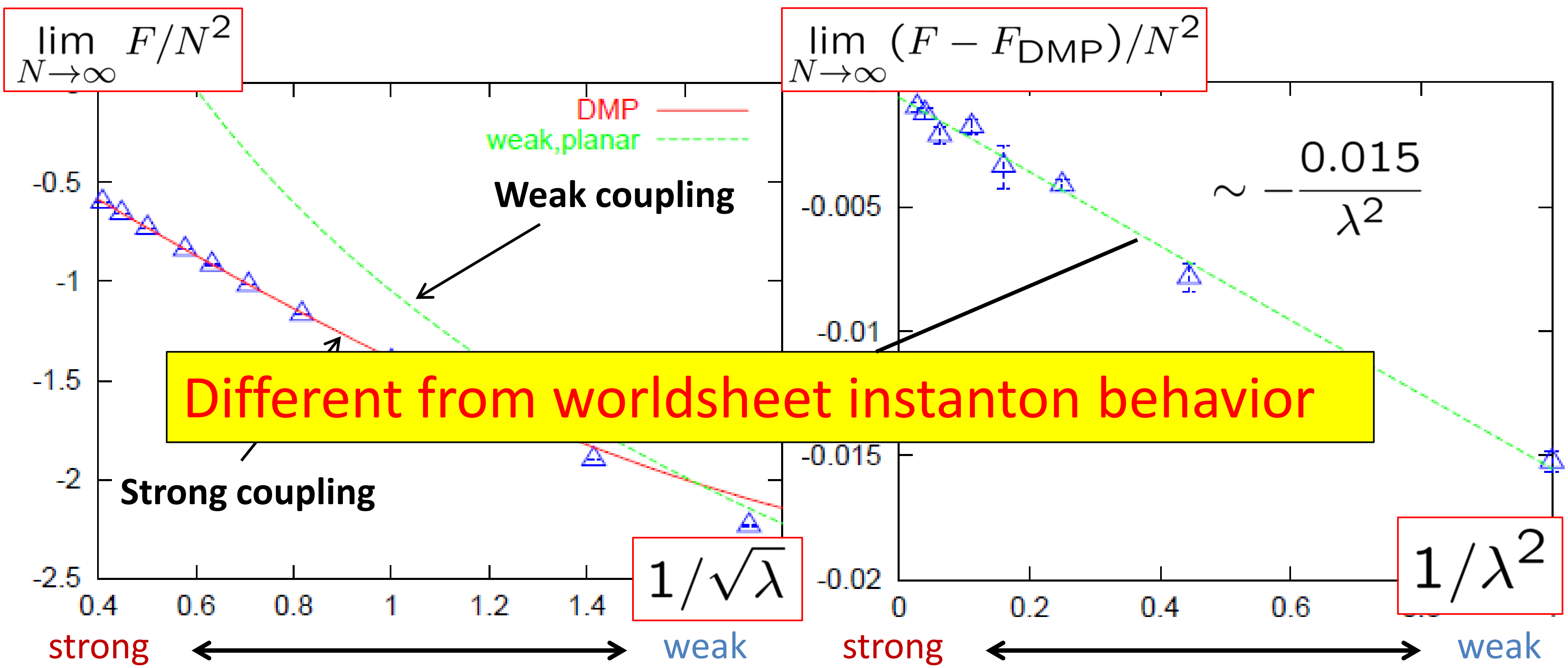
Result for Planar limit

$\left(\lambda = \frac{N}{k} = \text{fixed},\ \ \mathrm{N} \to \infty\right)$

[Drukker-Marino-Putrov '10]

- Weak couling : $F_{\text{weak,planar}} = N^2 \left(\log 2\pi\lambda - \frac{3}{2} - 2\log 2 \right)$ up to $\mathrm{O}(\lambda^2)$
- Strong coupling : $F_{\text{DMP}} = -\frac{\pi\sqrt{2}}{3} \frac{(\lambda - 1/24)^{3/2}}{\lambda^2} N^2$ up to $\mathrm{O}(e^{-2\pi\sqrt{2\lambda}})$

Worldsheet instanton



Different from worldsheet instanton behavior

Origin of Discrepancy for the Planar limit (without MC)

$$Z_{ABJM}=\frac{1}{N!^2}\int\frac{d^N\mu}{(2\pi)^N}\frac{d^N\nu}{(2\pi)^N}\frac{\prod_{i<j}\left[2\sinh\left(\frac{\mu_i-\mu_j}{2}\right)\right]^2\left[2\sinh\left(\frac{\nu_i-\nu_j}{2}\right)\right]^2}{\prod_{i,j}\left[2\cosh\left(\frac{\mu_i-\nu_j}{2}\right)\right]^2}\exp\left[\frac{ik}{4\pi}\sum_{i=1}^{N}(\mu_i^2-\nu_i^2)\right]$$

[Marino-Putrov '10]

Analytic continuation: $N_1\to N,\ \ N_2\to -N$

[Cf. Yost '91, Dijkgraaf-Vafa '03]

Lens space L(2,1)=S^3/Z_2 matrix model:

$$Z_{L(2,1)}(N_1,N_2)=\frac{1}{N_1!N_2!}\int\frac{d^{N_1}\mu}{(2\pi)^{N_1}}\frac{d^{N_2}\nu}{(2\pi)^{N_2}}\prod_{i<j}\left[2\sinh\left(\frac{\mu_i-\mu_j}{2}\right)\right]^2\prod_{a<b}\left[2\sinh\left(\frac{\nu_a-\nu_b}{2}\right)\right]^2\prod_{i,b}\left[2\cosh\left(\frac{\mu_i-\nu_b}{2}\right)\right]^2 e^{-\frac{ik}{4\pi}\left(\sum_i\mu_i^2+\sum_a\nu_a^2\right)}$$

※This is dual to the topological A-model on local $F_0=P^1\times P^1$

[Aganagic-Klemm-Marino-Vafa '02]

Genus expansion:

$$F(g_s,\lambda)=\sum_{g=0}^{\infty}F_g(\lambda)g_s^{2g-2}$$

$$=-\frac{N^2}{(2\pi\lambda)^2}F_0(\lambda)+F_1(\lambda)-\frac{(2\pi\lambda)^2}{N^2}F_2(\lambda)+\cdots$$

$$\boxed{g_s=\frac{2\pi i}{k}}$$

(Cont'd))Origin of Discrepancy for the Planar limit (without MC)

[Drukker-Marino-Putrov '10]

The "derivative" of planar free energy is exactly found as

$$\partial_\lambda F_0(\lambda) = \frac{\kappa}{4} G^{2,3}_{3,3}\left(\left. \begin{matrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & -\frac{1}{2} \end{matrix} \right| -\frac{\kappa^2}{16} \right) + \frac{i\pi^2\kappa}{2} {}_3F_2\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}; 1, \frac{3}{2}; -\frac{\kappa^2}{16} \right) \qquad \lambda(\kappa) = \frac{\kappa}{8\pi} {}_3F_2\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}; 1, \frac{3}{2}; -\frac{\kappa^2}{16} \right)$$

We impose the boundary condition: $F_0(0) = 0$ (Cf. $F_{\text{weak,planar}} = N^2 \left(\log 2\pi\lambda - \frac{3}{2} - 2\log 2 \right)$)

By using asymptotic behavior,

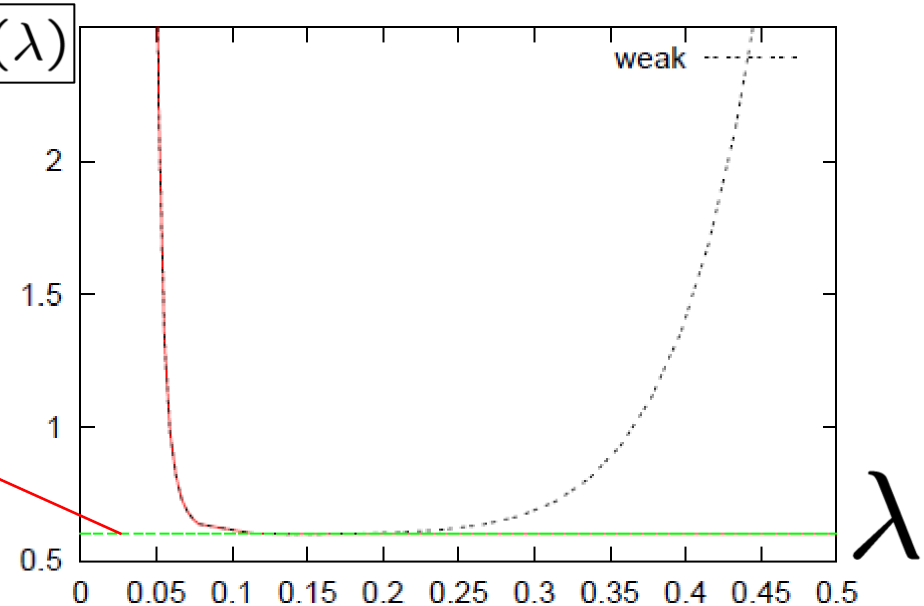
$$F_0(\lambda) = c_0 + \hat{F}_0(\lambda) + \mathrm{O}\left(e^{-8\pi\sqrt{2\hat{\lambda}}} \right) \qquad \left(\hat{\lambda} = \lambda - \frac{1}{24} \right)$$

$$\hat{F}_0(\lambda) = \frac{4\pi^3\sqrt{2}}{3} \hat{\lambda}^{3/2} - e^{-2\pi\sqrt{2\hat{\lambda}}} + e^{-4\pi\sqrt{2\hat{\lambda}}} \left(\frac{9}{8} + \frac{1}{\pi\sqrt{2\hat{\lambda}}} \right) - e^{-6\pi\sqrt{2\hat{\lambda}}} \left(\frac{82}{27} + \frac{9\sqrt{2}}{4\pi\sqrt{\hat{\lambda}}} + \frac{1}{\pi^2\hat{\lambda}} + \frac{\sqrt{2}}{12\pi^3\hat{\lambda}^{3/2}} \right)$$

Necessary for satisfying b.c. , taken as 0 for previous works

$\boxed{F_0(\lambda) - \hat{F}_0(\lambda)}$

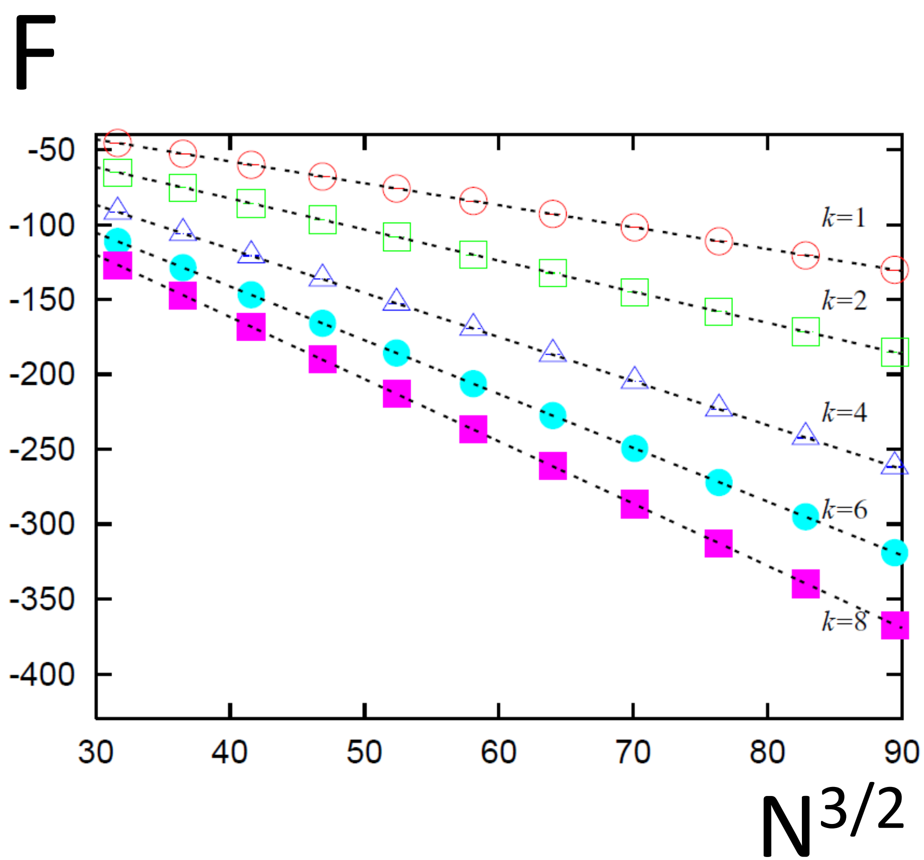
$$c_0 \simeq 0.60103$$

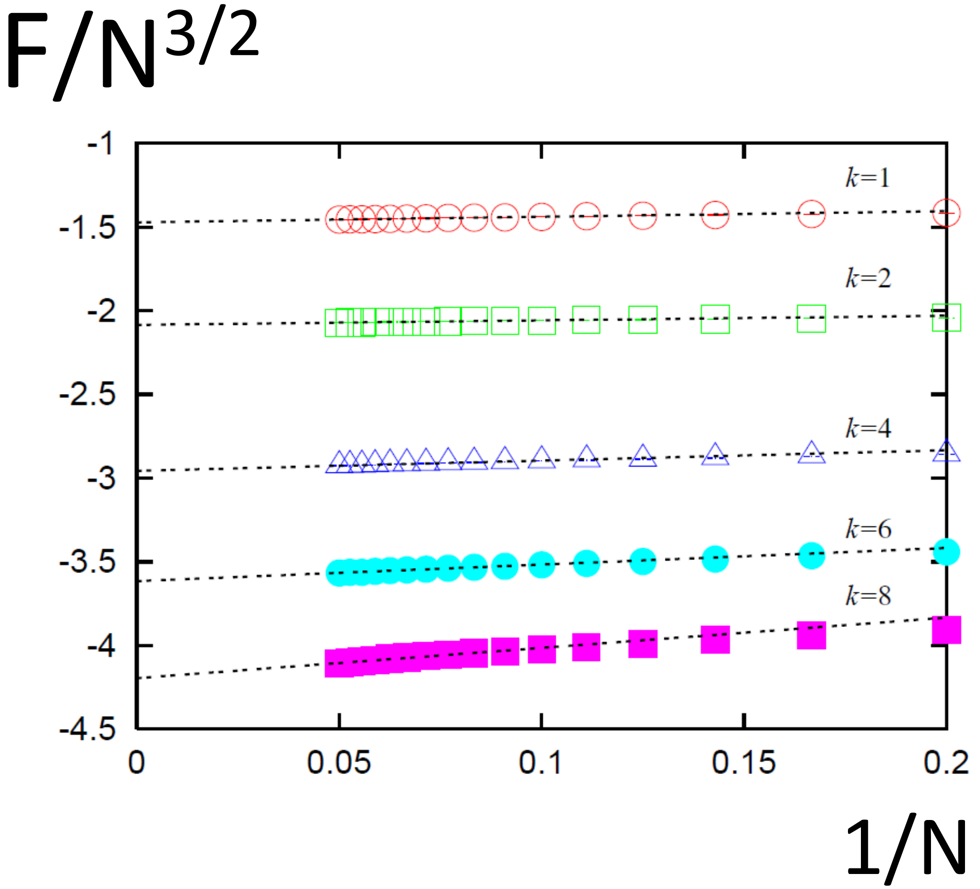


3/2 power low in 11d SUGRA limit

[Drukker-Marino-Putrov '10, Herzog-Klebanov-Pufu-Tesileanu '10]

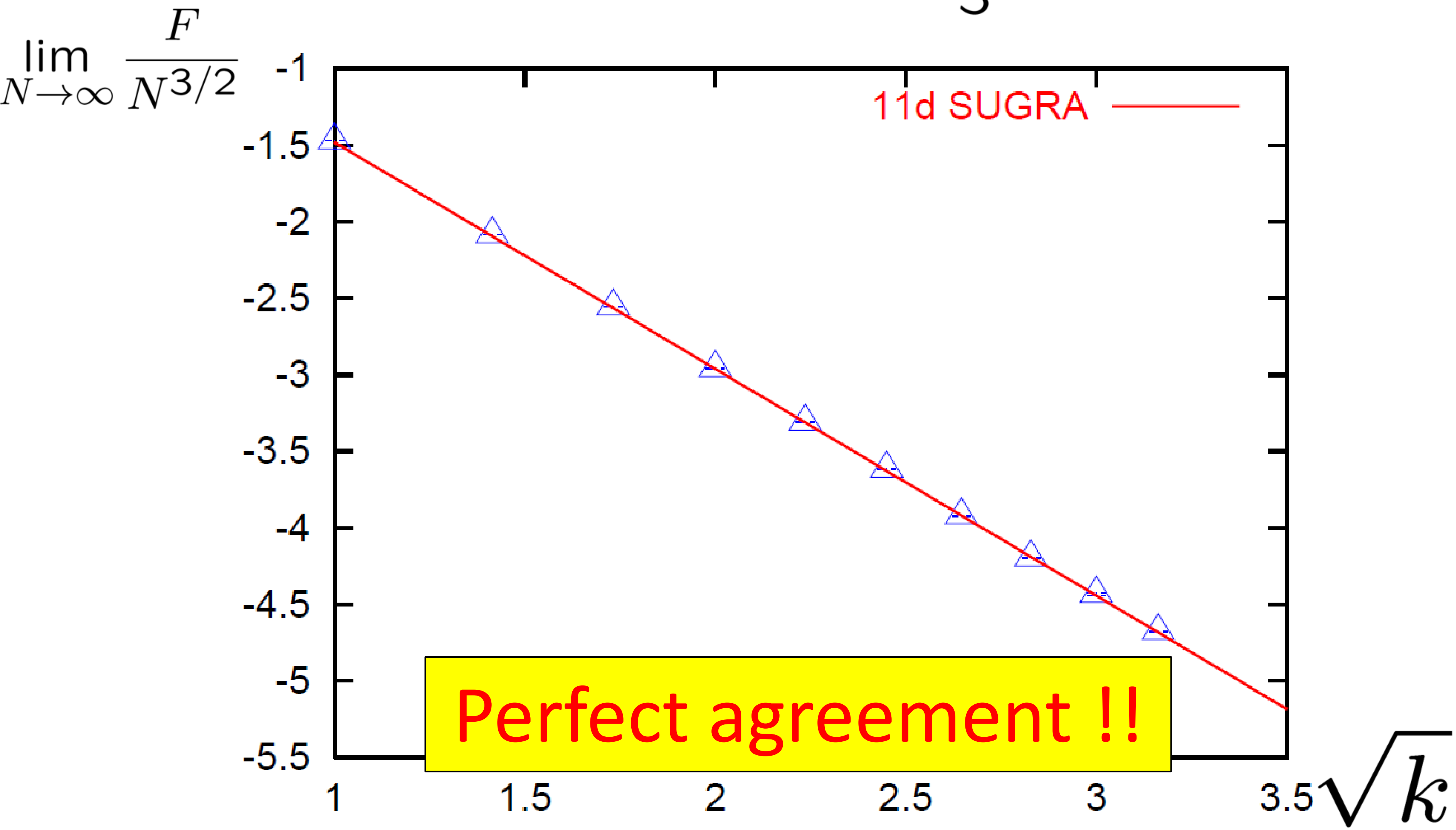
11d classical SUGRA: $F = -\frac{\pi\sqrt{2}}{3}\sqrt{k}N^{\frac{3}{2}}$



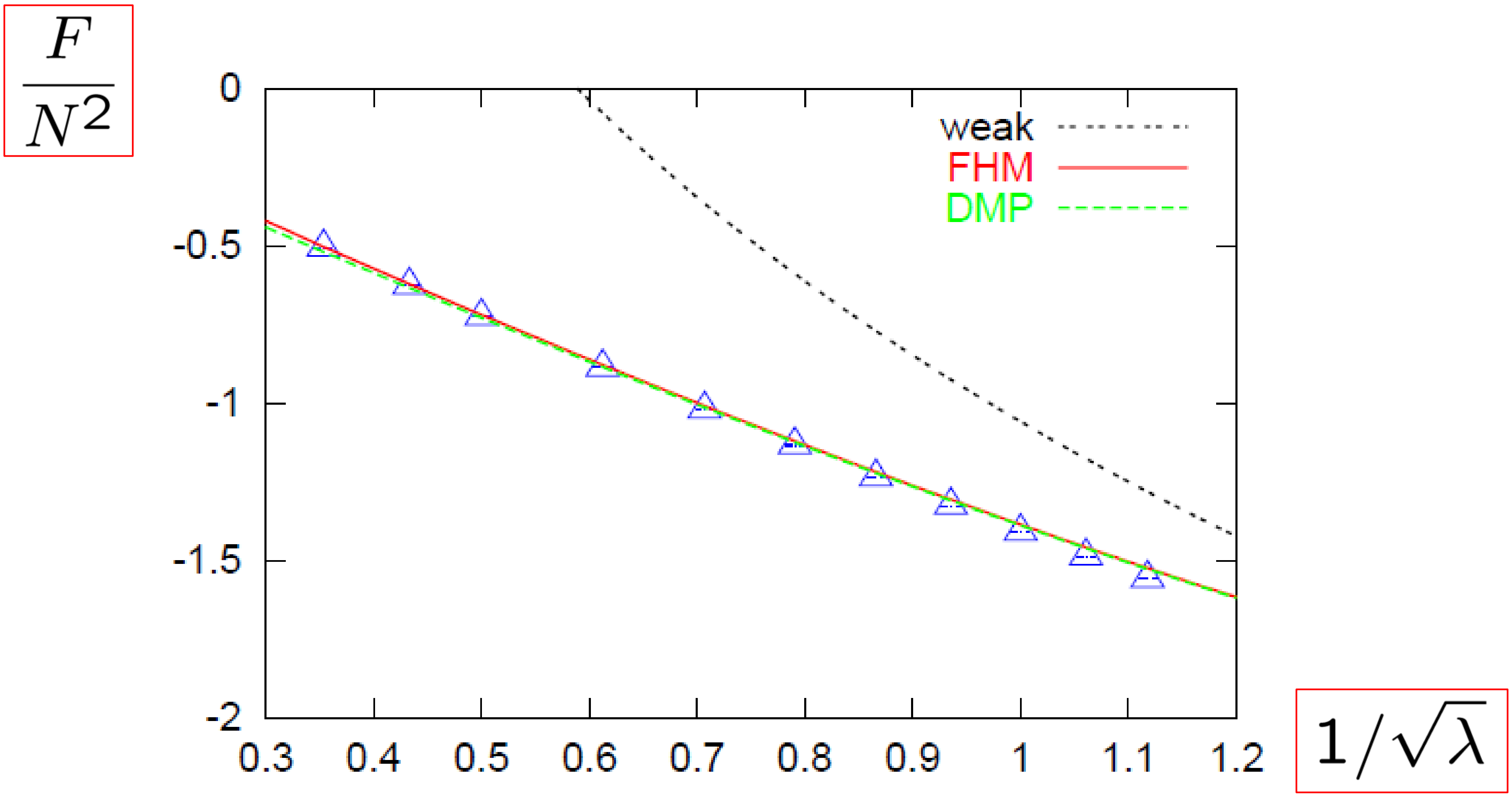


(Cont'd) 3/2 power low in 11d SUGRA limit

11d classical SUGRA: $F = -\frac{\pi\sqrt{2}}{3}\sqrt{k}N^{\frac{3}{2}}$



N=8



$\frac{F}{N^2}$
0
-0.5
-1
-1.5
-2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
1.1
1.2
weak
FHM
DMP
$1/\sqrt{\lambda}$

Higher genus

$$F^{(1)}_{\mathrm{modular}}(\lambda) = -\log \eta(\tau)$$

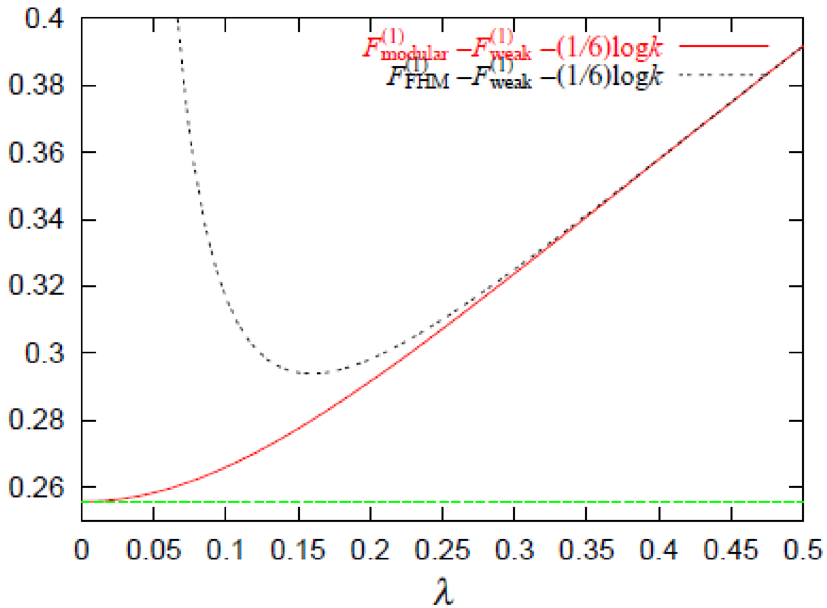
$$F^{(2)}_{\mathrm{modular}}(\lambda) = \frac{1}{432\vartheta_2^4\vartheta_4^8}\left(-\frac{5}{3}E_2^3 + 3\vartheta_2^4 E_2^2 - 2E_4 E_2\right) + \frac{16\vartheta_2^{12} + 15\vartheta_2^8\vartheta_4^4 + 21\vartheta_2^4\vartheta_4^8 + 2\vartheta_4^{12}}{12960\vartheta_2^4\vartheta_4^8},$$

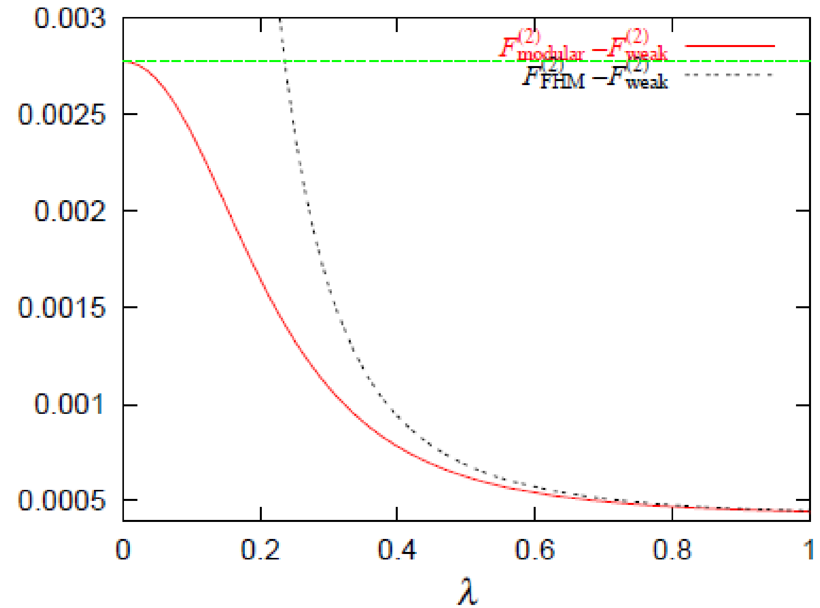
$$F^{(1)}_{\mathrm{weak}}(\lambda) = -\frac{1}{6}(\log\lambda + \log k) + 2\zeta'(-1)$$

$$F^{(2)}_{\mathrm{weak}}(\lambda) = -\frac{B_4}{16\pi^2\lambda^2}.$$

$F^{(1)}_{\mathrm{modular}} - \left(F^{(1)}_{\mathrm{weak}} + \frac{1}{6}\log k\right)$:

$F^{(2)}_{\mathrm{modular}} - F^{(2)}_{\mathrm{weak}}$





$$\Delta F^{(1)} = -\frac{1}{6}\log k - c_1, \qquad \Delta F^{(2)} = -c_2, \qquad c_1 \simeq 0.25558, \qquad c_2 \simeq 0.0027777$$

ABJ(M) matrix model and Lens space matrix model

Hermitian supermatrix: $\Phi = \begin{pmatrix} A & \Psi \\ \Psi^\dagger & C \end{pmatrix}$

$$\left(\begin{array}{c} A(C) : N_1 \times N_1 (N_2 \times N_2) \text{“Bosonic” Hermitian matrix} \\ \Psi : N_1 \times N_2 \text{Complex “Fermionic” matrix} \end{array} \right)$$

Supermatrix model

[Alvarez-Gaume '91, Yost '91,]

$$Z_s(N_1|N_2) = \int \mathcal{D}\Phi \ e^{-\frac{1}{g_s}\mathrm{Str}V(\Phi)}$$

Diagonalizing $A = \mathrm{diag}(\mu_i),\ C = \mathrm{diag}(\nu_i)$,

$$Z_s(N_1|N_2) = \int d\mu d\nu \frac{\prod_{i<j}(\mu_i - \mu_j)^2 \prod_{a<b}(\nu_a - \nu_b)^2}{\prod_{i,a}(\mu_i - \nu_a)^2} \times e^{-\frac{1}{g_s}(\sum_i V(\mu_i) - \sum_a V(\nu_a))}$$

Bosonic matrix model

$$Z_b(N_1|N_2) = \int \mathcal{D}A\mathcal{D}C e^{-\frac{1}{g_s}(\mathrm{tr}V(A)+\mathrm{tr}V(C))}$$

Diagonalizing $A = \mathrm{diag}(\mu_i),\ C = \mathrm{diag}(\nu_i)$,

$$Z_b(N_1|N_2) = \int d\mu d\nu \prod_{i<j}(\mu_i - \mu_j)^2 \prod_{a<b}(\nu_a - \nu_b)^2 \prod_{i,a}(\mu_i - \nu_a)^2 \times e^{-\frac{1}{g_s}(\sum_i V(\mu_i) + \sum_a V(\nu_a))}$$

[Diagramatic proof: Dijkgraaf-Vafa '03 , Dijkgraaf-Gukov-Kazakov-Vafa '03]

$$Z_s(N_1|N_2) = Z_b(N_1|-N_2)$$

(Cont'd)ABJ(M) matrix model and Lens space matrix model

[Marino-Putrov '10]

ABJ(M) matrix model:

$$Z_{\mathrm{ABJ}}(N_1,N_2)=\frac{1}{N_1!N_2!}\int\frac{d^{N_1}\mu}{(2\pi)^{N_1}}\frac{d^{N_2}\nu}{(2\pi)^{N_2}}\frac{\prod_{i<j}\left[2\sinh\left(\frac{\mu_i-\mu_j}{2}\right)\right]^2\prod_{a<b}\left[2\sinh\left(\frac{\nu_a-\nu_b}{2}\right)\right]^2}{\prod_{i,b}\left[2\cosh\left(\frac{\mu_i-\nu_b}{2}\right)\right]^2}e^{-\frac{ik}{4\pi}\left(\sum_i\mu_i^2-\sum_a\nu_a^2\right)}$$

Lens space L(2,1)=S^3/Z_2 matrix model:

$$Z_{L(2,1)}(N_1,N_2)=\frac{1}{N_1!N_2!}\int\frac{d^{N_1}\mu}{(2\pi)^{N_1}}\frac{d^{N_2}\nu}{(2\pi)^{N_2}}\prod_{i<j}\left[2\sinh\left(\frac{\mu_i-\mu_j}{2}\right)\right]^2\prod_{a<b}\left[2\sinh\left(\frac{\nu_a-\nu_b}{2}\right)\right]^2\prod_{i,b}\left[2\cosh\left(\frac{\mu_i-\nu_b}{2}\right)\right]^2e^{-\frac{ik}{4\pi}\left(\sum_i\mu_i^2+\sum_a\nu_a^2\right)}$$

$$\boxed{Z_{\mathrm{ABJ}}(N_1,N_2)=Z_{L(2,1)}(N_1,-N_2)}$$

Lens space matrix model and topological string

[Aganagic-Klemm-Marino-Vafa '02]

Chern – Simons theory on S^3/Z_2

[Witten '92]

Low energy effective theory
of D – branes wrapping S^3/Z_2
in the topological A – model on $T^*(S^3/Z_2)$

Canonical
quantization

Lens space S^3/Z_2 Matrix Model

Large N transition
[Cf. Gopakumar-Vafa '98]

The topological A – model
on local $F_0 = P^1 \times P^1$

Mirror symmetry

The topological B – model
onthe mirror manifold