Wave functions and correlation functions for GKP strings from integrability

Shota Komatsu
(University of Tokyo, Komaba)

based on work with Yoichi Kazama
arXiv:1205.6060 [hep-th]
and work in progress

@ YITP workshop,
"Field theory and String theory"
23.7.2012

Classical wave functions in AdS for strings from integrability

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Theme:

Boundary terms for the string world-sheet action in AdS.

Boundary terms and wave functions

Wave functions from integrability

Applications and Prospects

Boundary terms and wave functions

AdS_5/CFT_4 correspondence:

$\mathcal{N} = 4$ SU(N_c) super Yang-Mills

4d gauge theory

=

superstring on $AdS_5 \times S^5$

10d string theory

Relation between two theories

$\mathcal{N}=4$ SU(N_c) super Yang-Mills

superstring on $AdS_5 \times S^5$

$$\lambda = g_{\mathrm{YM}}^2 N_c \quad \longleftrightarrow \quad T = \sqrt{\lambda}$$

't Hooft coupling constant — string tension

$$N_c \quad \longleftrightarrow \quad g_s \sim \frac{1}{N_c}$$

color — string loop effect

- Strong coupling limit → Classical string
- Large N limit → No string loop

A well-known example

½-BPS circular Wilson loop

$$\frac{1}{N}\mathrm{tr}\mathcal{P}\exp\left(\oint(iA_\mu\dot{x}^\mu+\phi_i\dot{y}^i)ds\right)$$

$$(\dot{x}^2=\dot{y}^2)$$

on $S^1 \subset S^4$

$\longleftrightarrow$

Minimal surface in AdS

$\lambda \gg 1,\ N \to \infty$

Calculation in the gauge theory

$$W := \frac{1}{N}\mathrm{tr}\mathcal{P}\exp\left(\oint (iA_\mu \dot{x}^\mu + \phi_i \dot{y}^i)ds\right)$$

Sum of ladder diagrams
[Erickson, Semenoff, Zarembo]

Localization of the path integral
[Pestun]

Exact result

$$\langle W \rangle \sim e^{\sqrt{\lambda}} \quad (\lambda \gg 1,\, N \to \infty)$$

Calculation in the string theory

$(\lambda \gg 1,\ N \to \infty)$

Roughly speaking,

$$\langle W \rangle \sim e^{-\sqrt{\lambda} A}$$

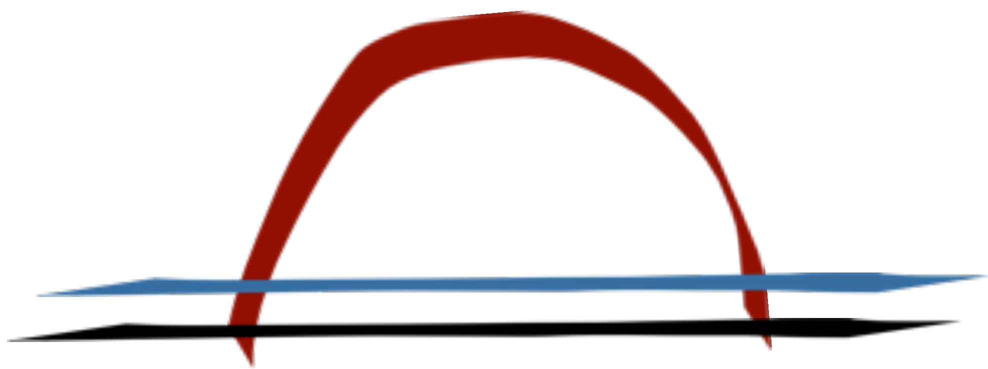
A : Minimal area of a string attached to the loop

However, the area is divergent since the world-sheet reaches the boundary, where the metric diverges.

$$A \to \infty$$

To obtain the correct answer, we need to

i) Regularize the divergence by introducing a cut-off.



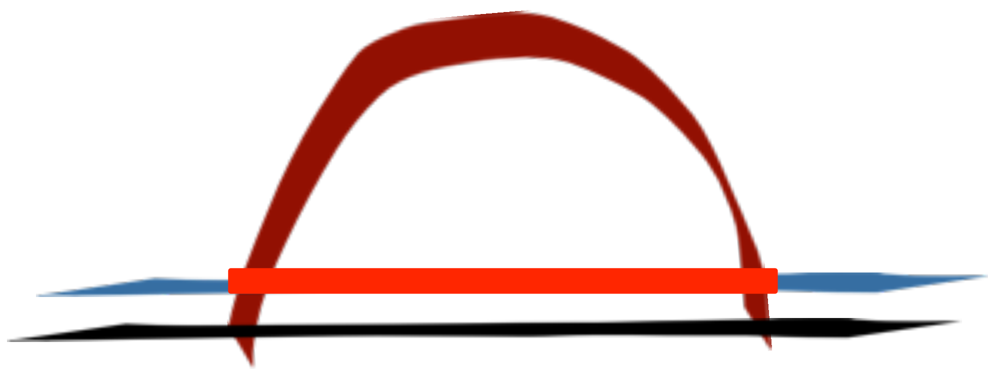
$$A_{\mathrm{reg}} = \frac{R}{\epsilon} - 1 + O(\epsilon)$$

R : radius of the Wilson loop

To obtain the correct answer, we need to

ii) Add a boundary term.

[Drukker, Gross, Ooguri]



$$A_{\rm bdy} = \frac{1}{2\pi}\oint d\sigma\, Y_i\, \dot{y}^i = -\frac{R}{\epsilon}$$

$$\rightarrow \langle W \rangle \sim e^{-\sqrt{\lambda}(A_{\rm reg}+A_{\rm bdy})} = e^{\sqrt{\lambda}}$$

Remark:

This boundary term has a definite physical meaning.

$$A_{\rm bdy} = \frac{1}{2\pi} \oint d\sigma \, Y_i \, \dot{y}^i$$

- Dirichlet → Neumann.
- Necessary to realize supersymmetry (1/2-BPS).

Lesson:

Boundary terms of a classical string are important in AdS/CFT

Correlation functions

- Boundary terms are important also for holographic calculations of non-BPS correlation functions.

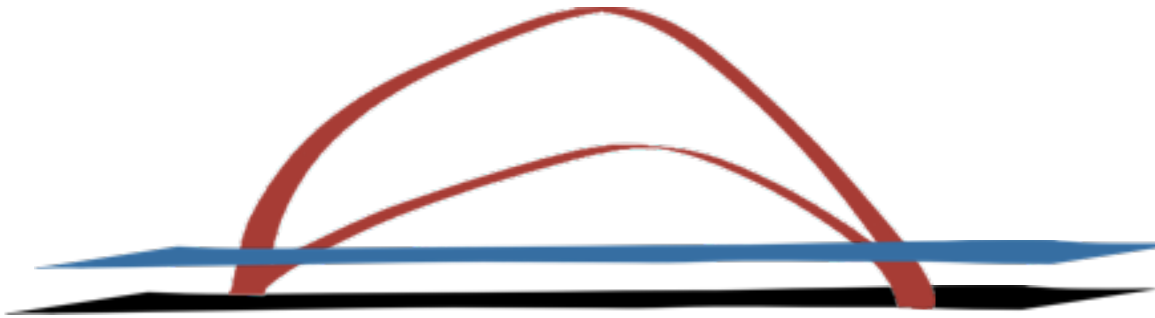
$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\rangle$$

$\mathcal{O}_i$: non-BPS operator with large charge

$\lambda \gg 1,\ N \to \infty$

$\text{Area} \to \infty$

Necessary not only to cancel the divergence but also to reproduce the correct space-time dependence.



Without boundary terms

$$e^{-\sqrt{\lambda}A_{\mathrm{reg}}} \sim \left(\frac{\epsilon}{x_1 - x_2}\right)^{\tilde{\Delta}}$$

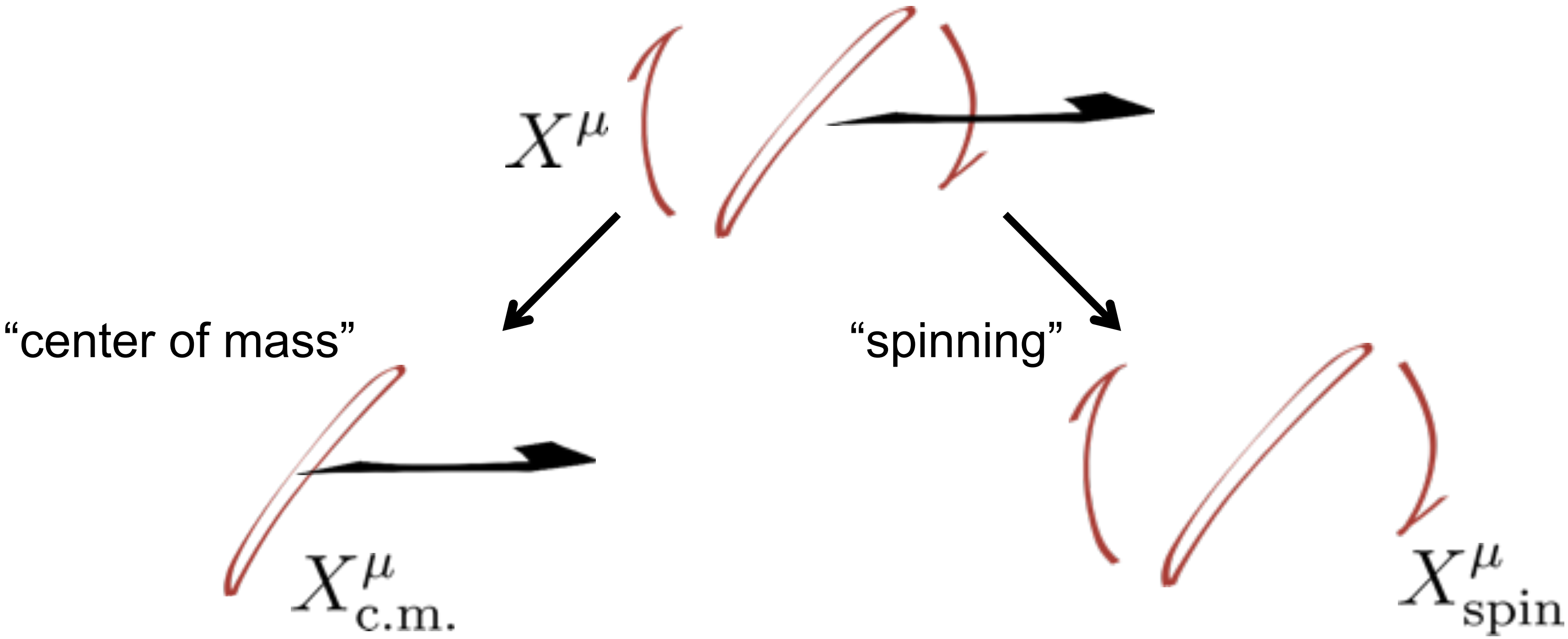
Prediction from the gauge theory: $\left(\frac{1}{x_1 - x_2}\right)^{\Delta}$

$$\tilde{\Delta} \neq \Delta$$

Previous approach

[Tsuji], [Janik-Surowka-Wereszczynski] (cf. [Asano-Sekino-Yoneya])

Decompose the motion of the string into the "center of mass" motion and the "spinning" motion.



$$X^\mu = X^\mu_{\rm c.m.} + X^\mu_{\rm spin}$$

Previous approach

[Tsuji], [Janik-Surowka-Wereszczynski] (cf. [Asano-Sekino-Yoneya])

Perform the Legendre transformation only for the "spinning" degrees of freedom X_{spin}

$$S_{\text{string}} \to S_{\text{string}} - \int d\tau d\sigma \, \Pi_{\text{spin}} \partial_\tau X_{\text{spin}}$$

$(\Pi_{\text{spin}} : \text{ conjugate momenta for } X_{\text{spin}})$

$$\to \quad \sim \left(\frac{\epsilon}{x_1 - x_2} \right)^{\Delta}$$

"Dirichlet → Neumann" for the spinning motion

Problems in the previous approach

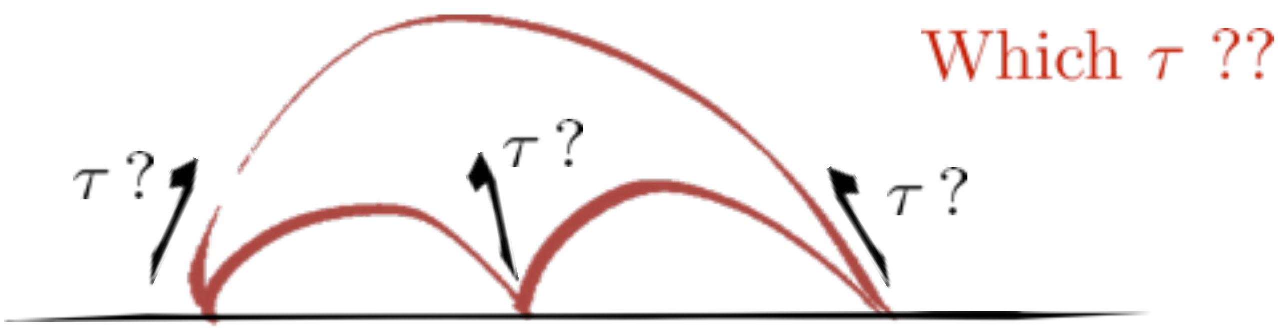
- Separation into the “center of mass” and the “spinning” is ambiguous.
  (The center of mass motion of a string in AdS is inherently coupled to the spinning motion)

  → Ambiguity in the final answer.

- It can be applied only to two point functions.

$$\int d\tau d\sigma\, \Pi_{\rm spin} \partial_\tau X_{\rm spin}$$

For multi-point function, there is no globally well-defined “time” on the world-sheet.



Our work:

Determine the boundary terms
from first principles.

- No ambiguity.
- Applicable to multi-point functions.
- Based on integrability.

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Determine the boundary terms
from first principles.

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What is the origin of the boundary terms?

→ Vertex operators.

- Correlation functions in the gauge theory are (believed to be) dual to the path integral on the worldsheet with insertions of appropriate vertex operators.

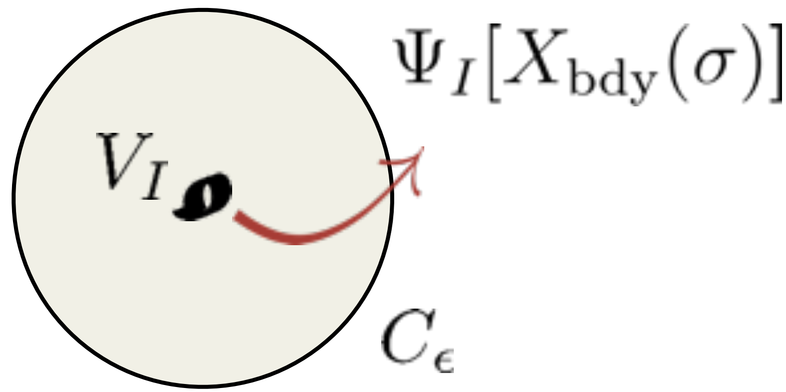
$$\langle \mathcal{O}_1(x_1)\,\mathcal{O}_2(x_2)\,\mathcal{O}_3(x_3)\rangle_{\text{gauge theory}}$$
$$= \frac{1}{\text{Möbius}}\int\prod_i d^2z_i\,\langle V_1\left[X^\mu(z_1)\right]V_2\left[X^\mu(z_2)\right]V_3\left[X^\mu(z_3)\right]\rangle_{\text{worldsheet}}$$

$$\mathcal{O}(x^\mu) \longleftrightarrow V[X^\mu(z);\,x^\mu]$$

gauge　　　　　　　　　　　　string

GKP-Witten for stringy modes

- The state-operator correspondence maps the vertex operators to the wave functions.



- In the classical limit, the wave functions provide the boundary terms for the worldsheet action.

$$\Psi_I[X^*_{\rm bdy}(\sigma)]$$

X^* : saddle-point classical solution

Therefore,

we need to know wave functions to determine correct boundary terms.

Wave functions from integrability

- In the classical limit,

$$\Psi[X] \sim e^{-\frac{1}{\hbar}W}$$

W : a solution to the Hamilton-Jacobi eq.

Difficult to solve...

- In terms of action-angle variables, the Hamilton-Jacobi eq. can be easily solved.

$$\Psi[\theta_I] = e^{i\sum_I J_I \theta_I}$$

J_I : action-variable
θ_I : angle-variable

- Fortunately, the powerful integrability-based method to construct such variables is known:

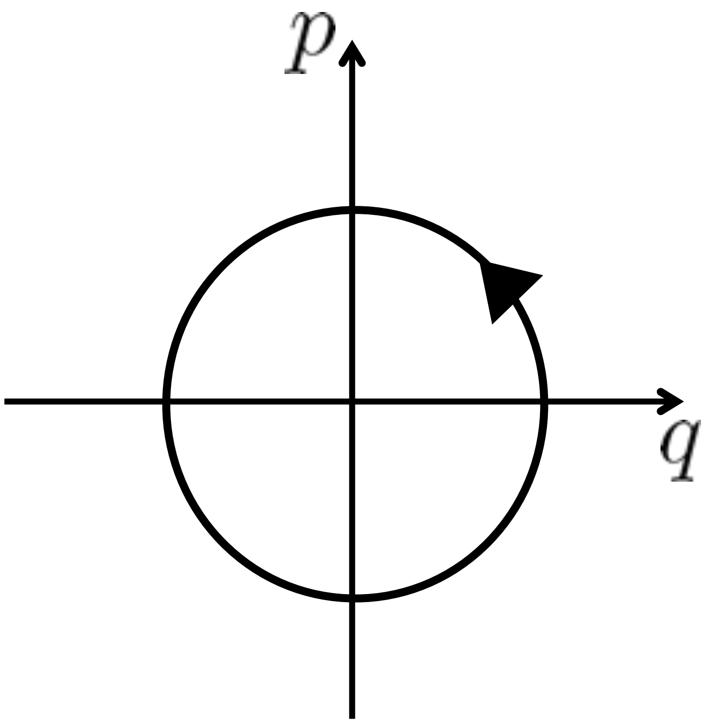
Sklyanin's magic recipe

Magic recipe

Consider a harmonic oscillator.

$$\ddot{q} = -q$$

Action-angle variables can be easily obtained.



$$(p = \dot{q})$$

$$J = \frac{1}{2\pi} \oint p dq$$

$$\theta = \frac{\partial}{\partial J} \int p dq$$

Reformulation

Eq. of motion is equivalent to the following eq.

$$\frac{d\Omega(x)}{dt} + [i\sigma_3\,, \Omega(x)] = 0$$

"Monodromy matrix"

$$\Omega(x) := p\sigma_1 + q\sigma_2 + ix\sigma_3 = \begin{pmatrix} ix\,, & p - iq \\ p + iq\,, & -ix \end{pmatrix}$$

x : spectral parameter (independent of time)

Spectral curve (independent of time):

$$\det\left(y - \Omega(x)\right) = 0$$

$$\Longleftrightarrow \; y^2 + x^2 = p^2 + q^2 (= 2E)$$

A pair of canonical variables (q,p) appears as a pole of the normalized eigenvector of Ω(x).

$$\Omega(x) \cdot \vec{\psi} = y(x)\vec{\psi}$$

Normalization condition:

$$\vec{n} \cdot \vec{\psi} = 1 \qquad \vec{n}\text{: arbitrary constant vector}$$

e.g.

For $\vec{n} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$,

$$\vec{\psi} \propto \left(q^2 - x^2 - ipq + ix\sqrt{p^2 + q^2 - x^2}\right)^{-1/2}$$

→ A pole at $x = q$

$$(x_{\text{pole}}\,, y(x_{\text{pole}})) = (q, p)$$

In this formulation, action-angle variables can be constructed as follows.

$$J = \frac{1}{2\pi} \oint y(x) dx$$

period integral
on the spectral curve

$$\theta = \int^{x_{\mathrm{pole}}} \frac{\partial y(x)}{\partial J} dx$$

motion of the pole
on the spectral curve

- Both are characterized by
  the spectral curve and the motion of the pole.

- Generalizable to a string on AdS.

In summary,

Conserved charges
from the spectral curve.

Canonical variables
from poles of the eigenvector.

Generalization to string on AdS

Consider,  $\subset \mathrm{AdS}_3$

$$\mathrm{AdS}_3 : \ X_{-1}^2 + X_0^2 - X_1^2 - X_4^2 = -1$$

Eq. of motion: $$\partial\bar{\partial}X^\mu + (\partial X^\nu \bar{\partial} X_\nu) X^\mu = 0$$

$$z = \tau + i\sigma$$

Generalization to string on AdS

Consider,  $\subset \mathrm{AdS}_3$

$$\mathrm{AdS}_3 : \; X_{-1}^2 + X_0^2 - X_1^2 - X_4^2 = -1$$

$$\texttt{Factor}[\partial\bar{\partial}X^{\mu} + (\partial X^{\nu}\bar{\partial}X_{\nu})X^{\mu} = 0]$$

Generalization to string on AdS

Consider,  $\subset \mathrm{AdS}_3$

$$\mathrm{AdS}_3 : \; X_{-1}^2 + X_0^2 - X_1^2 - X_4^2 = -1$$

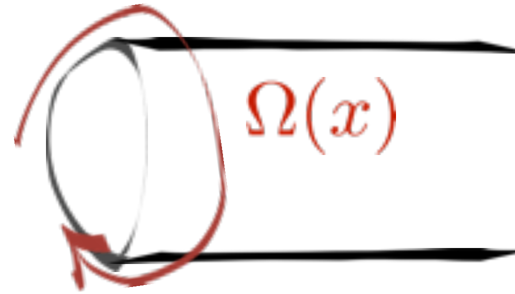
$$\left[\partial + \frac{J_z}{1-x}, \bar{\partial} + \frac{J_{\bar{z}}}{1+x}\right] = 0$$

$J : \; 2 \times 2$ matrix $\qquad x : \;$ arbitrary parameter

$$J_z = g^{-1}\partial_z g, \quad g = \begin{pmatrix} X_{-1} + X_4 & X_0 + X_1 \\ -X_0 + X_1 & X_{-1} - X_4 \end{pmatrix}$$

- Monodromy matrix:

$$\Omega(x) := \mathcal{P}\exp\left(\oint \frac{J_z}{1-x}dz + \frac{J_{\bar z}}{1+x}d\bar z\right)$$



- Normalized eigenvector:

$$\Omega(x)\vec{\psi}_{\rm norm}(x) = e^{ip(x)}\vec{\psi}_{\rm norm}(x) \qquad \vec{n}\cdot\vec{\psi}_{\rm norm}(x) = 1$$

- Poles of the eigenvector:

$$\vec{\psi}_{\rm norm}(x_i) \to \infty$$

In general, infinitely many.

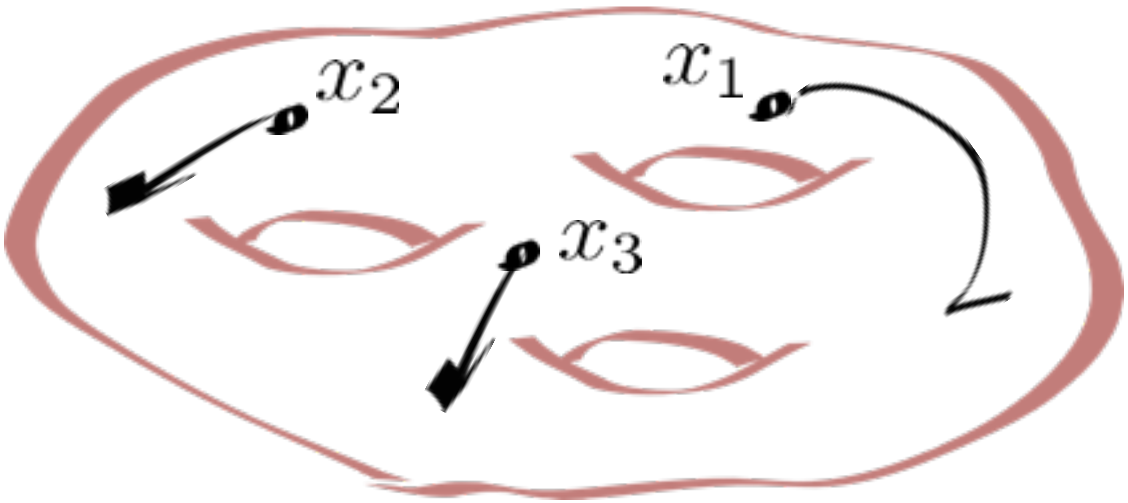
- A complete set of canonical variables:

$$\{z(x_i)\,,p(x_j)\} = \delta_{ij} \quad z(x) := x + \frac{1}{x}\ :\ \text{Zhukovski map}$$

$$\{z(x_i)\,,z(x_j)\} = \{p(x_i)\,,p(x_j)\} = 0$$

Spectral curve: $\det\,(y-\Omega(x))=0$

$$\Longleftrightarrow\quad (y-e^{ip(x)})(y-e^{-ip(x)})=0$$



$(x,y)=(x_i\,,e^{ip(x_i)})$

Action variables: $S_i=\oint_{a_i} p(x)dz(x)$

(Filling fraction)

Integrals along various cycles on the curve

Angle variables: $\phi_i=\sum_j\int^{x_j}\omega_i$

(Abel-Jacobi map)

ω_i: normalized holomorphic 1-form

Relation to the gauge theory

gauge | **string**

$$\mathcal{O}(x^\mu) \qquad \Psi[\phi_j] = e^{i\sum_j S_j \phi_j}$$

Charges: $\Delta\,, S\,, \cdots$ ⟷ Spectral curve

x^μ ⟷ $\vec{n}$

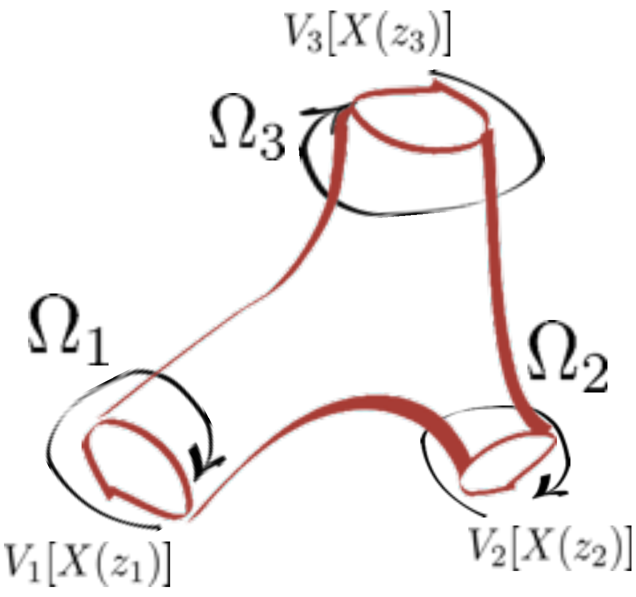
$(\vec{n}\cdot\vec{\psi}_{\mathrm{norm}}(x) = 1)$

Applications and Prospects

Three point functions

Combining with other integrability-based techniques, we can calculate three point functions holographically.

[Kazama-Komatsu ’10, ‘11], [work in progress]



$$\frac{C_{IJK}}{|x_{12}|^{\Delta_I+\Delta_J-\Delta_K}|x_{23}|^{\Delta_J+\Delta_K-\Delta_I}|x_{31}|^{\Delta_K+\Delta_I-\Delta_J}}$$

For GKP strings…

$$\begin{aligned}\ln C_{IJK} &= -\frac{\pi}{12} + \pi \left[-\kappa_1 K(\kappa_1) - \kappa_2 K(\kappa_2) - \kappa_3 K(\kappa_3) \right. \\ &+ \frac{\kappa_1+\kappa_2+\kappa_3}{2} K\left(\frac{\kappa_1+\kappa_2+\kappa_3}{2}\right) \\ &+ \left|\frac{-\kappa_1+\kappa_2+\kappa_3}{2}\right| K\left(\left|\frac{-\kappa_1+\kappa_2+\kappa_3}{2}\right|\right) \\ &+ \left|\frac{\kappa_1-\kappa_2+\kappa_3}{2}\right| K\left(\left|\frac{\kappa_1-\kappa_2+\kappa_3}{2}\right|\right) \\ &\left. + \left|\frac{\kappa_1+\kappa_2-\kappa_3}{2}\right| K\left(\left|\frac{\kappa_1+\kappa_2-\kappa_3}{2}\right|\right) \right] + \cdots\end{aligned}$$

$$K(x) = -\frac{1}{\pi} \int_{-\infty}^{\infty} d\theta \, e^{-\theta} \log\left(1 - e^{-4\pi x \cosh\theta}\right)$$

Three point functions

Combining with other integrability-based techniques, we can calculate three point functions holographically.

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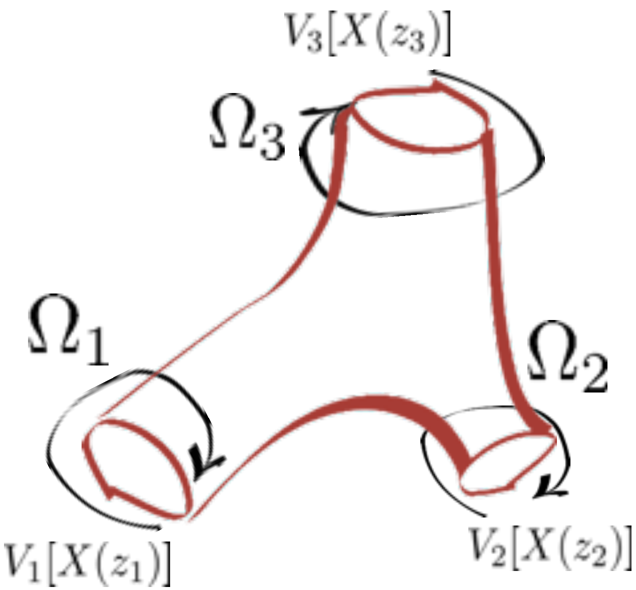

$$\frac{C_{IJK}}{|x_{12}|^{\Delta_I+\Delta_J-\Delta_K}|x_{23}|^{\Delta_J+\Delta_K-\Delta_I}|x_{31}|^{\Delta_K+\Delta_I-\Delta_J}}$$

Prospects

- Four point functions.
- Magic recipe for the spin-chain?
- Semi-classical calculation.
- Other backgrounds.

Thank you for listening