ABJM行列模型の最近の進展

森山翔文（Nagoya/KMI）

JHEP [arXiv:1106.4631]: with H.Fuji and S.Hirano

[arXiv:1207.4283]: with Y.Hatsuda and K.Okuyama

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Previously in Fuji-Hirano-M

Perturbative Terms of ABJM Matrix Model

$N_1 = N_2 = N$

$$Z(N) = \frac{1}{N_1! N_2!} \int \prod_{i=1}^{N_1} \frac{d\mu_i}{2\pi} \prod_{k=1}^{N_2} \frac{d\nu_k}{2\pi} e^{-(\sum \mu_i^2 - \sum \nu_k^2)/2g_s}$$

$$\times \frac{\prod_{i<j} \left(2\sinh\frac{\mu_i - \mu_j}{2}\right)^2 \prod_{k<l} \left(2\sinh\frac{\nu_k - \nu_l}{2}\right)^2}{\prod_{i,k} \left(2\cosh\frac{\mu_i - \nu_k}{2}\right)^2}$$

$g_s = 2\pi i/k$

in 't Hooft Expansion Sum Up To ･･･

Previously in Fuji-Hirano-M

$$Z(N) = \mathrm{Ai}\left(\left[\frac{\pi k^2}{\sqrt{2}}\right]^{2/3}\lambda_{\mathrm{ren}}\right)$$

(Up To Constant Maps & Instanton Effects)

cf: [Marino-Putrov, Honda et al]

- Airy Function

$$\mathrm{Ai}\left(z\right) = \frac{1}{2\pi i}\int_C \exp\left(-zt + \frac{1}{3}t^3\right)$$

- Renormalization of 't Hooft coupling $\lambda = N/k$

$$\lambda_{\mathrm{ren}}^{-1} = \frac{(\lambda - 1/24)^{-1}}{1 + (1/24)k^{-2}(\lambda - 1/24)^{-1}}$$

Today: Exact Results

Among Others (For k=1) (← *Mathematica8*)

$Z(1) = 1/4$

$Z(2) = 1/16\pi$

$Z(3) = (\pi-3)/2^6\pi$

$Z(4) = (-\pi^2+10)/2^{10}\pi^2$

$Z(5) = (-9\pi^2+20\pi+26)/2^{12}\pi^2$

$Z(6) = (36\pi^3-121\pi^2+78)/2^{14}3^2\pi^3$

$Z(7) = (-75\pi^3+193\pi^2+174\pi-126)/2^{16}3\pi^3$

$Z(8) = (1053\pi^4-2016\pi^3-4148\pi^2+876)/2^{21}3^2\pi^4$

$Z(9) = (5517\pi^4-13480\pi^3-15348\pi^2+8880\pi+4140)/2^{23}3^2\pi^4$

Other Related Work

[Hanada-Honda-Honma-Nishimura-Shiba-Yoshida] 1202

(Numerical Studies)

··· ··· $Z(20)$ = ···

(cf. Honda's Talk)

[Putrov-Yamazaki] 1207

··· ··· $Z(19)$ = ···

(A Beautiful Formula with Bernoulli Polynomial)

Other Related Work

[Hanada-Honda-Honma-Nishimura-Shiba-Yoshida] 1202

(Numerical Studies)

$\cdots \quad \cdots \quad Z(20) = \cdots$

(cf. Honda's Talk)

[Putrov-Yamazaki] 1207

$\cdots \quad \cdots \quad Z(19) = \cdots$

(A Beautiful Formula with Bernoulli Polynomial)

"Are You Using *Mathematica*16?" (2012/07/20)

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Motivation① M2-brane

ABJM

N=6 Chern-Simons Theory (N_1,N_2,k)

⇕

$(N_1+N_2)/2$ M2 with (N_1-N_2) Fractional M2 on C^4/Z_k

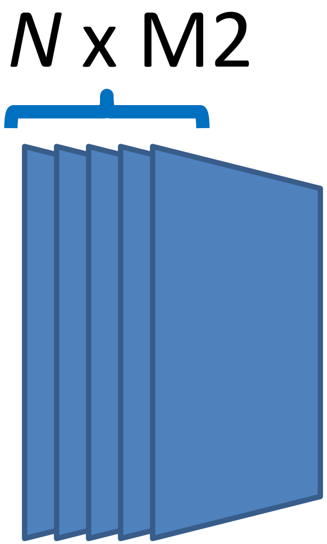
Special Case

No Fractional Branes: $N_1=N_2=N$

Flat Space: $k=1$

Motivation① M2-brane

Partition Function of M2 WorldVolume Theory



Z = Airy(N) ≈ Exp $-N^{3/2}$

DOF $N^{3/2}$ Reproduced

Motivation② From Gaussian To ABJM

ABJM

$$Z = \frac{1}{N_1!N_2!}\int \prod_{i=1}^{N_1}\frac{d\mu_i}{2\pi}\prod_{k=1}^{N_2}\frac{d\nu_k}{2\pi}\, e^{-(\sum \mu_i^2 - \sum \nu_k^2)/2g_s} \times \frac{\prod_{i<j}\left(2\sinh\frac{\mu_i-\mu_j}{2}\right)^2 \prod_{k<l}\left(2\sinh\frac{\nu_k-\nu_l}{2}\right)^2}{\prod_{i,k}\left(2\cosh\frac{\mu_i-\nu_k}{2}\right)^2}$$

CS

$$Z = \frac{1}{N!}\int \prod_{i=1}^{N}\frac{d\mu_i}{2\pi}\, e^{-\sum \mu_i^2/2g_s} \times \prod_{i<j}\left(2\sinh\frac{\mu_i-\mu_j}{2}\right)^2$$

Super

$$Z = \frac{1}{N_1!N_2!}\int \prod_{i=1}^{N_1}\frac{d\mu_i}{2\pi}\prod_{k=1}^{N_2}\frac{d\nu_k}{2\pi}\, e^{-(\sum \mu_i^2 - \sum \nu_k^2)/2g_s} \times \frac{\prod_{i<j}(\mu_i-\mu_j)^2 \prod_{k<l}(\nu_k-\nu_l)^2}{\prod_{i,k}(\mu_i+\nu_k)^2}$$

CS q-deform

Gauss

$$Z = \frac{1}{N!}\int \prod_{i=1}^{N}\frac{d\mu_i}{2\pi}\, e^{-\sum \mu_i^2/2g_s} \times \prod_{i<j}(\mu_i-\mu_j)^2$$

Superalg

Message from Airy① Hidden Structure?

- String Theory (Dual Resonance Model)

  Veneziano Amplitude

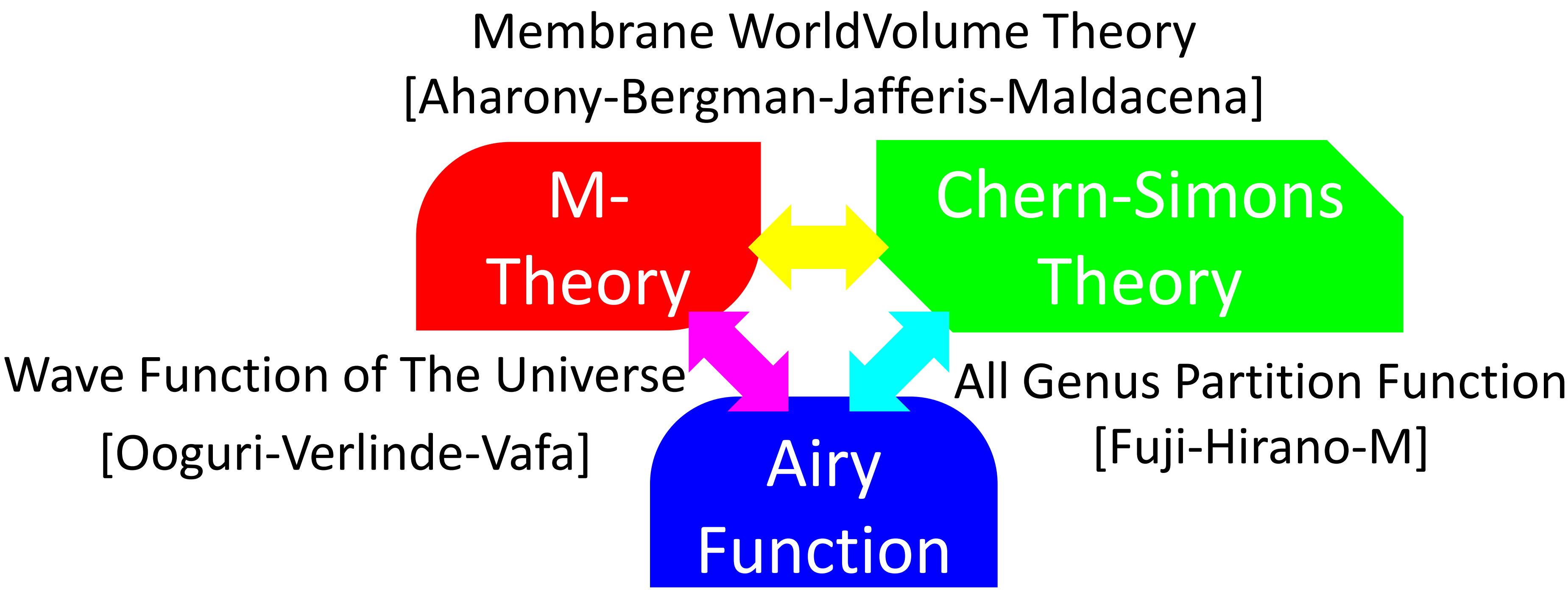
  ⇒ String Conformal Symmetry [Virasoro, Nambu]

- Membrane Theory

  Free Energy as Airy Function

  ⇒ Hidden Structure for Membrane?

Message from Airy② Trinity?



Today: Exact Result

Why Exact?

- Academic Interest
- Non-Perturbative Effects
- Hidden Structure?

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1. Motivation
2. **Fermi Gas**
3. Exact Results
4. Discussion

ABJM Matrix Model

$N_1 = N_2 = N$

$$Z(N) = \frac{1}{N_1!N_2!}\int \prod_{i=1}^{N_1}\frac{d\mu_i}{2\pi}\prod_{k=1}^{N_2}\frac{d\nu_k}{2\pi}\, e^{-(\sum \mu_i^2 - \sum \nu_k^2)/2g_s}$$

$$\times \frac{\prod_{i<j}\left(2\sinh\frac{\mu_i-\mu_j}{2}\right)^2 \prod_{k<l}\left(2\sinh\frac{\nu_k-\nu_l}{2}\right)^2}{\prod_{i,k}\left(2\cosh\frac{\mu_i-\nu_k}{2}\right)^2}$$

ABJM Matrix Model

$N_1=N_2=N$

$$Z(N) = \frac{1}{N_1!N_2!}\int\prod_{i=1}^{N_1}\frac{d\mu_i}{2\pi}\prod_{k=1}^{N_2}\frac{d\nu_k}{2\pi}\,e^{-(\sum\mu_i^2-\sum\nu_k^2)/2g_s}$$

$$\times\frac{\prod_{i<j}\left(2\sinh\frac{\mu_i-\mu_j}{2}\right)^2\prod_{k<l}\left(2\sinh\frac{\nu_k-\nu_l}{2}\right)^2}{\prod_{i,k}\left(2\cosh\frac{\mu_i-\nu_k}{2}\right)^2}$$

=

$[\prod \sinh \prod \sinh / \prod \cosh]^2$

Hidden Structure as Fermi Gas

1. Cauchy Determinant

$$\prod \sinh \prod \sinh / \prod \cosh = \mathrm{Det}\, \cosh^{-1}$$
$$= \sum_{\sigma} (-1)^{\sigma} \prod_i [2 \cosh (\mu_i - \nu_{\sigma(i)})/2]^{-1}$$

2. Trivialization of One Permutation

$$\sum_{\sigma,\tau} (-1)^{\sigma+\tau} \prod_i [2 \cosh(\mu_i - \nu_{\sigma(i)})/2]^{-1} [2 \cosh(\mu_i - \nu_{\tau(i)})/2]^{-1}$$
$$= N! \sum_{\sigma} (-1)^{\sigma} \prod_i [2 \cosh(\mu_i - \nu_i)/2]^{-1} [2 \cosh(\mu_i - \nu_{\sigma(i)})/2]^{-1}$$

3. Fourier Transform

$$(\mu,\nu) \Rightarrow (p,q)$$

4. (μ,ν) Gaussian Integration

Hidden Structure as Fermi Gas

After Some Calculation, ... [Marino-Putrov]

- Non-Interacting Fermi Gas

$$Z(N) = (N!)^{-1} \sum_\sigma (-1)^\sigma \int \prod_i dq_i \langle q_i | \boldsymbol{\rho} | q_{\sigma(i)} \rangle$$

Density Matrix $\boldsymbol{\rho} = \mathbf{e}^{-H}$

$$\boldsymbol{\rho} = [2 \cosh q/2]^{-1/2} [2 \cosh p/2]^{-1} [2 \cosh q/2]^{-1/2}$$

- Statistical Mechanics Approach

$N^{3/2}$ & Airy Easily(!) Reproduced

Contents

Quantum Mechnics

- Density Matrix $\boldsymbol{\rho} = \mathbf{e}^{-H}$

  $\rho = [2 \cosh q/2]^{-1/2} \, [2 \cosh p/2]^{-1} \, [2 \cosh q/2]^{-1/2}$
- tr ρ^n = ??
- Eigenvalues ??

Beyond Perturbation Theory !!

Difficulty

$$\mathrm{tr}\, \boldsymbol{\rho}^n = \iiint \cdots \iiint (2\pi)^{-2n} \prod \mathrm{d}q_j \mathrm{d}p_j$$

$$[2 \cosh q_1/2]^{-1} \langle q_1 | p_1 \rangle [2 \cosh p_1/2]^{-1} \langle p_1 | q_2 \rangle$$

$$\cdots [2 \cosh q_n/2]^{-1} \langle q_n | p_n \rangle [2 \cosh p_n/2]^{-1} \langle p_n | q_1 \rangle$$

Integration Over n-Dimensional Phase Space

$$\Downarrow$$

Reduce the Dimensions?

First Trial

$$[p,q] = i\hbar \quad \& \quad e^X e^Y = e^{[X,Y]} e^Y e^X$$

- Implication?

$$[2 \cosh q/2]\,[2 \cosh p/2] = i\,[2 \sinh p/2]\,[2 \sinh q/2]$$

$$[2 \cosh p/2]\,[2 \cosh q/2] = -i\,[2 \sinh q/2]\,[2 \sinh p/2]$$

- Reciprocal?

$$C_p\, C_q = -i\, S_q\, S_p \;?$$

$$C_q\, C_p = i\, S_p\, S_q \;?$$

$$C_x := [2 \cosh x/2]^{-1} \qquad S_x := [2 \sinh x/2]^{-1}$$

However, Anomalous

- Study of Matrix Element

$$\langle q_1 |\, C_p\, C_q + i\, S_q\, S_p \,| q_2 \rangle = [S_{q_1}][C_{q_2} \,\mathrm{Sinh}\, q_2/2]$$

$$\langle q_1 |\, C_q\, C_p - i\, S_p\, S_q \,| q_2 \rangle = [C_{q_1} \,\mathrm{Sinh}\, q_1/2][S_{q_2}]$$

- Anomalous Terms, in terms of Projection

$$C_p\, C_q = -i\, S_q\, \Pi\, S_p \qquad C_q\, C_p = i\, S_p\, \Pi^\dagger\, S_q$$

$$\Pi = 1 - |p_{=0}\rangle\langle q_{=0}| \qquad \Pi^\dagger = 1 - |q_{=0}\rangle\langle p_{=0}|$$

Important Implication

$$C_p\, C_q = -\mathrm{i}\, S_q\, \Pi\, S_p \qquad C_q\, C_p = \mathrm{i}\, S_p\, \Pi^\dagger\, S_q$$

- Then

$$\mathrm{tr}\,(\Pi\, C_q\, C_p)^n = \mathrm{i}^n\, \mathrm{tr}\,(\Pi\, S_p\, \Pi^\dagger\, S_q)^n = \mathrm{i}^{2n}\, \mathrm{tr}\,(C_p\, C_q\, \Pi^\dagger)^n$$

- Note! $\mathrm{tr}\,(\Pi\, C_q\, C_p)^n = \mathrm{tr}\,(C_p\, C_q\, \Pi^\dagger)^n$: **Real**

$$\mathbf{tr}\,(\Pi\, C_q\, C_p)^{\mathrm{odd}} = 0$$

- Hence ($\Pi = 1 - |p_{=0}\rangle\langle q_{=0}|$)

$$\mathbf{tr}\,(C_q\, C_p)^{\mathrm{odd}} = \sum_\#\, \prod_\#\, \langle q_{=0}|\, (C_q\, C_p)^\#\, |p_{=0}\rangle^\#$$

Formula

- Definition

$$\Xi(z) = \det(1 + z\,\boldsymbol{\rho}) = \det(1 + z\,C_q\,C_p)$$

$$\boldsymbol{G}(z) = \boldsymbol{g}_n\,z^n \qquad \boldsymbol{g}_n = \langle q_{=0}|\,(C_q\,C_p)^n\,|p_{=0}\rangle$$

- Relation ONLY for z^{odd}

$$\log \Xi(z) = \log \boldsymbol{G}(z)$$

- Correctly Speaking

$$(\log \Xi(z) - \log \Xi(-z))/2 = (\log \boldsymbol{G}(z) - \log \boldsymbol{G}(-z))/2$$

Explicitly

$$\mathrm{tr}\,\boldsymbol{\rho}^1/1 = \boldsymbol{g}_1$$

$$\mathrm{tr}\,\boldsymbol{\rho}^3/3 = \boldsymbol{g}_3 - \boldsymbol{g}_2\,\boldsymbol{g}_1 + \boldsymbol{g}_1{}^3/3$$

$$\mathrm{tr}\,\boldsymbol{\rho}^5/5 = \boldsymbol{g}_5 - \boldsymbol{g}_4\,\boldsymbol{g}_1 - \boldsymbol{g}_3\,\boldsymbol{g}_2 + \boldsymbol{g}_3\,\boldsymbol{g}_1{}^2 + \boldsymbol{g}_2{}^2\,\boldsymbol{g}_1 - \boldsymbol{g}_2\,\boldsymbol{g}_1{}^3 + \boldsymbol{g}_1{}^5/5$$

- Computation of $\mathrm{tr}\,\boldsymbol{\rho}^{\mathrm{odd}}$

Without Product of Density Matrix $\boldsymbol{\rho}(q_1,q_2)$

Even Power

- Relation ONLY for z^{odd}

$$\log \Xi(z) = \log G(z)$$

- How About z^{even} of $\log G(z)$?

A Novel Identity

$$\det(1 + z\,\rho_-)/\det(1 - z\,\rho_+) = G(z)$$

- Explicitly

$(\mathrm{tr}\,\rho_+^2 - \mathrm{tr}\,\rho_-^2)/2 = g_2 - g_1^2/2$

$(\mathrm{tr}\,\rho_+^4 - \mathrm{tr}\,\rho_-^4)/4 = g_4 - g_3\,g_1 - g_2^2/2 + g_2\,g_1^2 - g_1^4/4$

$(\mathrm{tr}\,\rho_+^6 - \mathrm{tr}\,\rho_-^6)/6 = g_6 - g_5\,g_1 - g_4\,g_2 + g_4\,g_1^2 - g_3^2/2 + 2\,g_3\,g_2\,g_1 - g_3\,g_1^3 + g_2^3/3 - 3\,g_2^2\,g_1^2/2 + g_2\,g_1^4 - g_1^6/6$

ρ_+ & ρ_-

- Z_2 Symmetry

$$\rho(-q_1,-q_2) = \rho(q_1,q_2)$$

- Preserving Wave Function Parity

$$\psi_+(q) = +\ \psi_+(-q)$$

$$\psi_-(q) = -\ \psi_-(-q)$$

- Decomposition

$$\rho_+(q_1,q_2) = (\ \rho(q_1,q_2) + \rho(q_1,-q_2)\ )/2$$

$$\rho_-(q_1,q_2) = (\ \rho(q_1,q_2) - \rho(q_1,-q_2)\ )/2$$

Explicit Values

$\mathrm{tr}\, \rho_+^1 = \sqrt{2}/8$ $\mathrm{tr}\, \rho_-^1 = (2-\sqrt{2})/8$

$\mathrm{tr}\, \rho_+^2 = 1/16\pi$ $\mathrm{tr}\, \rho_-^2 = (\pi-3)/16\pi$

$\mathrm{tr}\, \rho_+^3 = (3-2\sqrt{2})/64$ $\mathrm{tr}\, \rho_-^3 = ((2\sqrt{2}+1)\pi-12)/64\pi$

$\mathrm{tr}\, \rho_+^4 = (\pi^2-8)/512\pi^2$ $\mathrm{tr}\, \rho_-^4 = (11\pi^2-32\pi-8)/512\pi^2$

$\mathrm{tr}\, \rho_+^5 = ((5-8\sqrt{2})\pi+20)/2^{10}\pi$

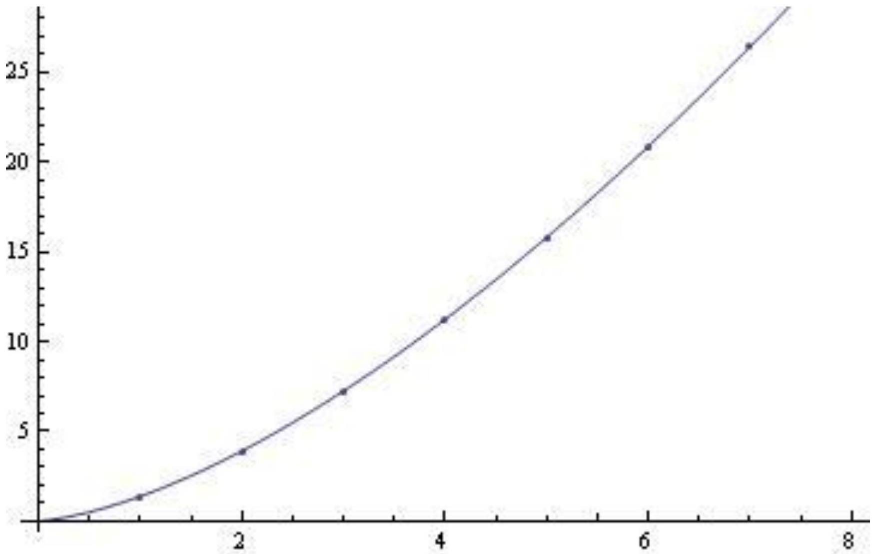
$\mathrm{tr}\, \rho_-^5 = ((8\sqrt{2}-9)\pi^2-20\pi+40)/2^{10}\pi^2$

$\mathrm{tr}\, \rho_+^6 = (9\pi-28)/2^{12}3\pi$ $\mathrm{tr}\, \rho_-^6 = (-11\pi^2+4\pi+96)/2^{12}\pi^2$

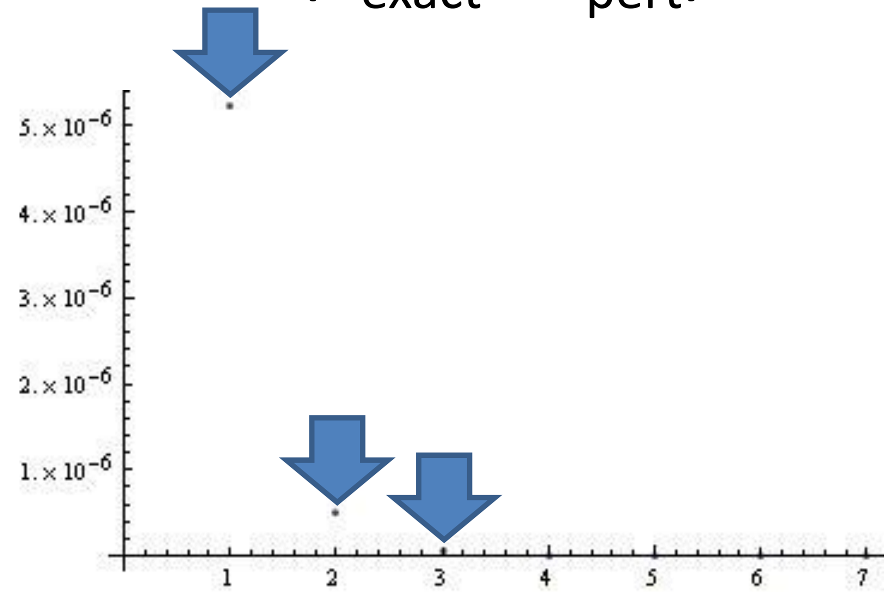
...

Exact vs Perturbation

F_{exact} & F_{pert}



$-(F_{\text{exact}} - F_{\text{pert}})$



$(F_{\text{pert}} = F_{\text{Airy}} + F_{\text{ConstMap}})$

[・・・, Marino-Putrov, Honda et al]

Instanton Effects

- Worldsheet Instanton

$$\exp(-2\pi \sqrt{2N/k})$$

- Membrane Instanton (D2 wrapping RP^3 in CP^3)

$$\exp(-\pi \sqrt{2kN})$$

Case k=1

No RP^3 To Wrap $\Rightarrow$ $\exp(-2\pi \sqrt{2N})$

Content

Discussion

- tr ρ^n = ??   Eigenvalues ??
- Similar to SFT Anomaly:

  Associativity Anomaly, Twist Anomaly, ...
- Techniques from String Field Theory ??
- Trick: Reducing Matrices to Vectors [Tracy-Widom]
- Polynomial in $1/\pi$ with Rational Coefficient

  $\Rightarrow$ Counting ?? Hidden Structure ??
- Relation Between Two Parities tr $\rho_\pm{}^n$

Thank you for your attention!