Topological Supersymmetry in Ω-deformed $\mathcal{N} = 4$ Super Yang-Mills

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1. Introduction

Nonperturbative effect in supersymmetric field theory

$\rightarrow$ BPS soliton and instanton are important.

Contribution to physical quantities

$\sim$ (path-)integral on the moduli space

$\sim$ "volume" of the moduli space $\rightarrow$ difficult to evaluate.

- localization formula

existence of BRST supercharge $\bar{Q}$ ($\bar{Q}^2 = 0$ or gauge transf.)

$\rightarrow$ Integration is localized to the fixed point set of $\bar{Q}$.

(exact WKB or saddle point method)

instanton in 4D $\mathcal{N}=2$ supersymmetric gauge theory

– There exists $\bar{Q}$ (topological twist).

– However, the fixed point set of $\bar{Q}$ is basically U(1) instanton.

still hard to compute for large instanton number

Ω-background deformation [Moore-Nekrasov-Shatashvili]

- $\bar{Q} \to \bar{Q}_\Omega$, $\quad \bar{Q}_\Omega^2 =$ (undeformed) $+$ ($\mathrm{U}(1)^2$ rotation by ϵ_1,ϵ_2)
- $S = \bar{Q}\Xi_0 \to S_\Omega = \bar{Q}_\Omega \Xi = \bar{Q}_\Omega(\Xi_0 + \cdots)$

Ω-background for $\mathcal{N}=2$ theory

$= \underbrace{\text{6D gravitational background}}_{\text{to introduce } \epsilon_1,\ \epsilon_2} + \underbrace{\text{SU(2) R-symmetry Wilson line}}_{\text{to preserve SUSY}}$

Generalization of Ω-background for $\mathcal{N}=4$ super Yang-Mills theory [Ito-H.N.-Saka-Sasaki 2011]

generalized Ω-background
$=$ 10D gravitational background $+$ SU(4) R-symmetry Wilson line

By introducing R-symmetry Wilson line, supersymmetry is preserved for generic ϵ_1 and ϵ_2.

In 10D, SU(4) $=$ SO(6) R-symmetry is the subgroup of local Lorentz symmetry (No R-symmetry in 10D SYM).

$\Rightarrow$ contribution to spin (and Affine) connection ($\sim$ torsion)

$$\omega_{\mathcal{M},NP} \quad \longrightarrow \quad \widehat{\omega}_{\mathcal{M},NP} = \omega_{\mathcal{M},NP} + K_{\mathcal{M},NP}$$

$\Rightarrow$ deformation of parallel (or Killing) spinor equation

$$\nabla_{\mathcal{M}} \zeta = 0 \quad \longrightarrow \quad \widehat{\nabla}_{\mathcal{M}} \zeta = \left(\nabla_{\mathcal{M}} + \frac{1}{4} K_{\mathcal{M},NP} \Gamma^{NP} \right) \zeta = 0.$$

Classification of parallel spinors and supersymmetry in the theory

advantage of 10D formalism with torsion

- Calculation is easier than 4D. Geometrical meaning is clear.

disadvantage

- Construction of off-shell SUSY is not easy in general.

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2. 10D SYM in curved background with torsion

- curved background with torsion   ($\mathcal{M}$: curved, M: tangent)

$$e_{\mathcal{M}}^{M}: \text{ vielbein} \qquad T_{\mathcal{M}\mathcal{N}}{}^{P}: \text{ torsion}$$

$$\widehat{\omega}_{\mathcal{M},NP}: \text{ spin connection} \qquad \widehat{\Gamma}_{\mathcal{M}\mathcal{N}}{}^{\mathcal{P}}: \text{ Affine connection}$$

decomposition of connections: (vielbein part)+(torsion part)

$$\widehat{\omega}_{\mathcal{M},NP} = \omega_{\mathcal{M},NP} + K_{\mathcal{M},NP}, \quad \widehat{\Gamma}_{\mathcal{M}\mathcal{N}}{}^{\mathcal{P}} = \Gamma_{\mathcal{M}\mathcal{N}}{}^{\mathcal{P}} + K_{\mathcal{M},}{}^{\mathcal{P}}{}_{\mathcal{N}},$$

contorsion

$$K_{M,NP} = -\frac{1}{2}(T_{MN,P} - T_{NP,M} + T_{PM,N}).$$

- 10D SYM in curved background with torsion

$$\widehat{\mathcal{L}} = \frac{1}{\kappa g^2}\mathrm{Tr}\left[-\frac{1}{4}e\left(F_{\mathcal{M}\mathcal{N}} - T_{\mathcal{M}\mathcal{N}}{}^{\mathcal{P}}A_{\mathcal{P}}\right)^2 - \frac{i}{2}e\,\bar{\Psi}\Gamma^{\mathcal{M}}\widehat{\nabla}^{(G)}_{\mathcal{M}}\Psi\right],$$

$$\begin{aligned}\widehat{\nabla}^{(G)}_{\mathcal{M}}\Psi &= \widehat{\nabla}_{\mathcal{M}}\Psi + i[A_{\mathcal{M}}, \Psi] \\ &= D_{\mathcal{M}}\Psi + \frac{1}{4}(\omega_{\mathcal{M},NP} + K_{\mathcal{M},NP})\Gamma^{NP}\Psi.\end{aligned}$$

Explicit dependence on $A_{\mathcal{M}}$ (mass term of $A_{\mathcal{M}}$)

$\rightarrow$ No (10D) gauge invariance unless torsion $= 0$

$\rightarrow$ No (10D) SUSY invariance unless torsion $= 0$

supersymmetry after dimensional reduction?

3. SUSY in dimensionally reduced theory

dimensional reduction (to 4D) and gauge invariance

$$x^{\mathcal{M}} = (x^\mu,\, x^{\mathcal{A}}), \qquad \partial_{\mathcal{A}} = 0,$$

$$A_{\mathcal{M}} = (A_\mu,\, \varphi_{\mathcal{A}}), \qquad \Psi = (\Lambda^A_\alpha,\, \bar{\Lambda}_{\dot{\alpha} A}),$$

vielbein and torsion: independent of $x^{\mathcal{A}}$.

α, $\dot{\alpha}$: $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$ indices, $A = 1, \ldots, 4$: **4** repr. of $\mathrm{SO}(6)$

- gauge invariance in dimensionally reduced theory

$$-T_{\mathcal{M}\mathcal{N}}{}^{\mathcal{P}} A_{\mathcal{P}} = -T_{\mathcal{M}\mathcal{N}}{}^{\mu} A_\mu - T_{\mathcal{M}\mathcal{N}}{}^{\mathcal{A}} \varphi_{\mathcal{A}}$$

$$\Longrightarrow \; T_{\mathcal{M}\mathcal{N}}{}^{\mu} = 0.$$

• SUSY invariance of dimensionally reduced theory

SUSY transformation in 10D notation (ansatz)

$$\delta A_{\mathcal{M}} = i e^{M}_{\mathcal{M}} \bar{\zeta} \Gamma_M \Psi,$$
$$\delta \Psi = -\frac{1}{2} e^{\mathcal{M}}_{M} e^{\mathcal{N}}_{N} \left(F_{\mathcal{M}\mathcal{N}} - T_{\mathcal{M}\mathcal{N}}{}^{\mathcal{P}} A_{\mathcal{P}} \right) \Gamma^{MN} \zeta.$$

SUSY variation of $\widehat{\mathcal{L}}$ (should be total dertivative)

$$\delta \widehat{\mathcal{L}} = \frac{1}{\kappa g^2} \mathrm{Tr} \Big[+ \frac{i}{2} e \bar{\Psi} \Gamma^{\mathcal{M}} \Gamma^{\mathcal{N}\mathcal{P}} \widehat{F}_{\mathcal{N}\mathcal{P}} (\widehat{\nabla}_{\mathcal{M}} \zeta) + \frac{i}{2} e \bar{\Psi} \Gamma^{\mathcal{M}\mathcal{N}\mathcal{P}} \zeta (\widehat{\nabla}^{(G)}_{\mathcal{M}} \widehat{F}_{\mathcal{N}\mathcal{P}}) - \frac{i}{4} e \widehat{\nabla}^{(G)}_{\mathcal{M}} (\bar{\Psi} \Gamma^{\mathcal{N}\mathcal{P}} \Gamma^{\mathcal{M}} \zeta \widehat{F}_{\mathcal{N}\mathcal{P}}) \Big].$$

first term

$$\widehat{\nabla}_{\mathcal{M}}\zeta = 0 \ \ : \ \ \text{parallel spinor condition}$$

second term

$$\widehat{\nabla}^{(G)}_{[\mathcal{M}}\widehat{F}_{\mathcal{N}\mathcal{P}]} = -(\partial_\mu A_{[\mathcal{M}})T_{\mathcal{N}\mathcal{P}]}{}^{\mu} - (\partial_{\mathcal{A}} A_{[\mathcal{M}})T_{\mathcal{N}\mathcal{P}]}{}^{\mathcal{A}}$$
$$- (\partial_{[\mathcal{M}}T_{\mathcal{N}\mathcal{P}]}{}^{\mathcal{R}} + T_{[\mathcal{M}\mathcal{N}}{}^{\mathcal{Q}}T_{\mathcal{P}]\mathcal{Q}}{}^{\mathcal{R}})A_{\mathcal{R}}$$

$$\Rightarrow \partial_{[\mathcal{M}}T_{\mathcal{N}\mathcal{P}]}{}^{\mathcal{R}} + T_{[\mathcal{M}\mathcal{N}}{}^{\mathcal{Q}}T_{\mathcal{P}]\mathcal{Q}}{}^{\mathcal{R}} = 0.$$

third term

$$e\,\widehat{\nabla}_{\mathcal{M}}V^{\mathcal{M}} = \partial_{\mathcal{M}}(e\,V^{\mathcal{M}}) - e\,T_{\mathcal{M}\mathcal{N}}{}^{\mathcal{N}}V^{\mathcal{M}}$$

$$\Rightarrow T_{\mathcal{M}\mathcal{N}}{}^{\mathcal{N}} = 0.$$

4. Ω-background and SUSY condition

Ω-background for $\mathcal{N} = 2$ theory
$=$ 6D gravitational background $+$ SU(2) R-symmetry Wilson line

generalized Ω-background
$=$ 10D gravitational background $+$ $\underbrace{\text{SU(4) R-symmetry Wilson line}}_{\text{torsion}}$

• 10D metric for generalized Ω-background (on $\mathbb{R}^4 \times \mathbb{T}^6$)

$$ds^2_{\mathrm{10D}} = (dx^m + \Omega^m_a dx^a)^2 + dx^a dx^a, \quad \Omega^m_a = \Omega^m{}_{na} x^n.$$

Here $\Omega_{mna} = -\Omega_{nma}$ are constant and commute with each other.
(m, n, . . . : flat 4D, a, b, . . . : flat 6D)

$$\text{commuting } \Omega_{mna} \quad \longrightarrow \quad \Omega_{mna} = \frac{1}{2\sqrt{2}} \begin{pmatrix} 0 & \epsilon_{1a} & 0 & 0 \\ -\epsilon_{1a} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\epsilon_{2a} \\ 0 & 0 & \epsilon_{2a} & 0 \end{pmatrix}.$$

spin connection from Ω-background metric

$$\omega_{a,mn} = \Omega_{mna}, \quad \text{(other components)} = 0.$$

contorsion $\to$ R-symmetry Wilson line

$$\Longrightarrow K_{a,bc} : \text{ constant,} \quad \text{(other components)} = 0.$$

gauge invariance condition

$$(K_{a,b}{}^{c} - K_{b,a}{}^{c})\Omega_{mnc} = 0 \quad (\text{from } T_{\mathcal{M}\mathcal{N}}{}^{\mu} = 0)$$

4D action

$$S_\Omega = \int d^4x \, \frac{1}{g^2} \mathrm{Tr}\Bigg[\frac{1}{4}F_{mn}F^{mn} + \frac{1}{2}(D_m\varphi_a - F_{mn}\Omega_a^n)^2 + \frac{1}{4}\Big[[\varphi_a,\varphi_b] + i\Omega_a^m D_m\varphi_b - i\Omega_b^m D_m\varphi_a - i\Omega_a^m\Omega_b^n F_{mn} - i\big(K_{a,b}{}^c - K_{b,a}{}^c\big)\varphi_c\Big]^2 + \text{(fermions)}\Bigg].$$

gauge invariance and SUSY condition

$$\begin{aligned}(K_{a,b}{}^c - K_{b,a}{}^c)\Omega_{mnc} &= 0,\\ \widehat{\nabla}_{\mathcal{M}}\,\zeta &= 0,\\ T_{[ab}{}^d T_{c]d}{}^e &= 0,\\ T_{ab}{}^b &= 0.\end{aligned}$$

• parallel spinor condition and topological twist

$$\widehat{\nabla}_{\mathcal{A}}\zeta = \frac{1}{4}\delta^{a}_{\mathcal{A}}(\Omega_{mna}\Gamma^{mn} + K_{a,bc}\Gamma^{bc})\zeta = 0.$$

$\Longrightarrow$ 4D and 6D rotation should be canceled.

$$\omega_{a,mn} = \Omega_{mna} \in \mathsf{U}(1)_L \times \mathsf{U}(1)_R \subset \mathsf{SU}(2)_L \times \mathsf{SU}(2)_R.$$

Reducing $\mathsf{SO}(6)$

$$SO(6) \supset SO(4)' \times SO(2) = SU(2)_{L'} \times SU(2)_{R'} \times U(1)_E$$

We can restrict contorsion s.t.

$$K_{a,bc} \to K_{a,\hat{b}\hat{c}} \in \mathsf{U}(1)_{L'} \times \mathsf{U}(1)_{R'} \subset \mathsf{SU}(2)_{L'} \times \mathsf{SU}(2)_{R'}.$$

Here $a = (\hat{a},\, a')$, $\hat{a} = 5, \ldots, 8$, $a' = 9, 10$.

	$U(1)_L$	$U(1)_R$	$U(1)_{L'}$	$U(1)_{R'}$
$\zeta_\alpha^{A'}$	$\pm\frac{1}{2}$	0	0	$\pm\frac{1}{2}$
$\zeta_\alpha^{\hat{A}}$	$\pm\frac{1}{2}$	0	$\pm\frac{1}{2}$	0
$\bar{\zeta}_{A'}^{\dot{\alpha}}$	0	$\pm\frac{1}{2}$	0	$\pm\frac{1}{2}$
$\bar{\zeta}_{\hat{A}}^{\dot{\alpha}}$	0	$\pm\frac{1}{2}$	$\pm\frac{1}{2}$	0

cancellation of U(1) charges

$\rightarrow$ identification of SU(2)'s

$\rightarrow$ topological twist

SUSY condition can be solved for each topological twist.
(We assume generic $\Omega_{mna'} = \Omega_{mn}, \bar{\Omega}_{mn}$.)

- half twist $\quad$ ($\mathrm{SU}(2)_R \sim \mathrm{SU}(2)_{R'}$) $\quad \Rightarrow \quad \mathcal{N} = 2^*$ theory

  parameters : $\Omega_{mna'}$, $\quad K^+_{a'} \sim M, \bar{M}$ $\quad$ ($\mathcal{N} = 2^*$ mass)

  one scalar and one tensor supercharges are preserved.

- Vafa-Witten twist $\quad$ ($\mathrm{SU}(2)_R \sim \mathrm{diag}(\mathrm{SU}(2)_{L'} \times \mathrm{SU}(2)_{R'})$)

  parameters : $\Omega_{mna'}$, $\Omega_{mn\hat{5}}$, $\Omega_{mn\hat{6}}$

  two scalar and two tensor supercharges are preserved.

- Marcus twist $\quad$ ($\mathrm{SU}(2)_L \sim \mathrm{SU}(2)_{L'}$, $\mathrm{SU}(2)_R \sim \mathrm{SU}(2)_{R'}$)

  parameters : $\Omega_{mna'}$ $\quad$ (special case of half twist)

  two scalar and two tensor supercharges are preserved.

5. Summary

summary

1. Correspondence between torsion and R-symmetry Wilson line in generalized Ω-background
2. SUSY condition can be solved for each topological twist and supercharges are obtained (on shell).
3. In Nekrasov-Shatashvili limit ($\epsilon_{1a} = 0$ or $\epsilon_{2a} = 0$), Some of supersymmetry are restored. (skip)

future work

- off-shell BRST superchage
  (done for half and VW twists.)

- extension to more complicated backgrounds
  [Festuccia-Seiberg], [Dumitrescu-Festuccia-Seiberg], [Hama-Hosomichi], etc.

- embedding to superstring/SUGRA
  R-R 3-form (in instanton (D(-1)) effective action)
  [Ito-H.N.-Sasaki], [Ito-H.N.-Saka-Sasaki]

  (cf. [Hellerman-Orlando-Reffert], [Reffert], [Nakayama-Ooguri])