G/G Gauged WZW Higgs system, Bethe ansatz for Q-boson and a deformed Verlinde formula

立教大学　奥田諭史

吉田豊氏 (KEK) との共同研究

Introduction

・Moore-Nekrasov-Shatashvili が topological Yang-Mills-Higgs model の分配関数に局所化を適用させせると、局所化する配位が non-linear Shcrodinger 模型の Bethe 方程式になることを指摘した。('97)

・Gerasimov-Shatashvili が YMH 模型の分配関数が、non-linear Schrodinger 模型の波動関数のノルムで書けることを見つけた。(08')

・Gerasimov-Shatashvili が G/G Gauged WZW Higgs 模型に局所化を適用すると、spin-S XXZ 模型のスピン無限大極限の Bethe 方程式になることを指摘した。

・さらに、Gerasimov-Shatashvili は全ての TFT に、同様の可積分構造が現れると予想した。

目的

・ G/G GWZW (+Higgs) model の分配関数は波動関数のノルムで書くことができることを示す。また、Gauged WZW model でなぜ波動関数のノルムで書けるのかを説明する。

G/G Gauged WZW Model and The Phase Model

Plan

1,Partition function of G/G Gauged WZW model

2,q-boson model and Phase model

3,G/G Gauged WZW model and Phase model

4,WZW model (Theory of representation of Affine Lie algebra) and Phase model

The G/G Gauged WZW Model on the genus h Riemann surface Σ_h

ref) *M.Blau and G.Thompson* ('93)
A.Gerasimov ('93)

Matter contents

- $g(z, \bar{z})$: scalar field ($\Sigma_h \to G$, 0-form ,Grassman even)
- A : gauge field (a connection on a principal G-bundle over Σ_h,Grasmann even)
- λ : fermion (adjoint rep of G, 1-form, Grassmann odd)

Partition function: $Z_{\rm GWZW}(\Sigma_h) = \int_{\Sigma_h} \mathcal{D}g\mathcal{D}A\mathcal{D}\lambda \, e^{ikS_{\rm GWZW}}$

where

$$S_{G/G}(g, A, \lambda)_{GWZW} = S_{WZW}(g) - \frac{1}{2\pi}\int_{\Sigma_h} d^2z {\rm Tr}(A_z g^{-1}\bar{\partial}_{\bar{z}} g + g\partial_z g^{-1} A_{\bar{z}} + gA_z g^{-1}A_{\bar{z}} - A_z A_{\bar{z}}) + \frac{1}{2\pi}\int_{\Sigma_h} {\rm Tr}(\lambda \wedge \lambda)$$

$$S_{WZW}(g) = -\frac{1}{4\pi}\int_{\Sigma_h} d^2z {\rm Tr}(g^{-1}\partial_z g \cdot g^{-1}\partial_{\bar{z}} g) \underbrace{-i\frac{1}{12\pi}\int_B d^3y \epsilon^{ijk} {\rm Tr} g^{-1}\partial_i g \cdot g^{-1}\partial_j g \cdot g^{-1}\partial_k g}_{\text{WZ-term}}$$

· The G/G Gauged WZW model is a topological field theory.

Partition function of G/G Gauged WZW model $=$

$\sharp$ of conformal block of level k G WZW model on a genus h Riemann surface:

$$Z_{\mathrm{GWZW}}(\Sigma_h) = \mathrm{dim} V_{h,k} = \sum_R |\mathcal{S}_{0,R}|^{-2(h-1)}$$

The Partition Function of the G/G Gauged WZW Model on Σ_h

$G = U(N)$

- Gauge fixing: $g \in U(N) \to$ Maximal Torus

$$g(z,\bar{z}) = \exp\left(2\pi i \sum_{j=1}^{N} \phi_j(z,\bar{z}) H^j\right) \quad \text{where } H^j \text{ are Cartan subalgebra of } u(N)$$

- Effective abeliannized GWZW action

$$Z_{\text{GWZW}}(\Sigma_h) = \sum_{(\ell_1,\cdots,\ell_N)\in\mathbb{Z}_N} \int \prod_{j=1}^{N} \mathcal{D}\phi_i \mathcal{D}A_j \prod_{\substack{a,b=1\\ a\neq b}}^{N} \left(1 - e^{2\pi i(\phi_a-\phi_b)}\right)^{1-h+\ell_a-\ell_b} e^{2\pi i k \sum_{j=1}^{N} \int_{\Sigma_h} \phi_j F^{(j)}}$$

where U(1)-charge of the backgound gauge field $\ell_j = c_1(\mathcal{L}_j) = \dfrac{1}{2\pi}\displaystyle\int_{\Sigma_h} F^{(j)}$

$\longrightarrow$

$$\begin{aligned} Z^{U(N)}_{\text{GWZW}}(\Sigma_h) = &\quad (k+N)^{Nh} \sum_{(n_1,\cdots,n_N)\in\mathbb{Z}_N} \int \prod_{j=1}^{N} d\phi_j \left\{ \frac{\prod_{\substack{a,b=1\\ a\neq b}}^{N} \left(e^{2\pi i\phi_a} - e^{2\pi i\phi_b}\right)}{\prod_{a=1}^{N} e^{2\pi i(N-1)\phi_a}} \right\}^{1-h} \prod_{j=1}^{N} \delta\left((N+k)\phi_j - \sum_{\ell=1}^{N} \phi_\ell + \frac{N-1}{2} - n_j\right) \\ = &\quad (k+N)^{kh} \sum_{(n_1,\cdots,n_k)\in\mathbb{Z}_k} \int \prod_{j=1}^{k} d\phi_j \left\{ \frac{\prod_{\substack{a,b=1\\ a\neq b}}^{k} \left(e^{2\pi i\phi_a} - e^{2\pi i\phi_b}\right)}{\prod_{a=1}^{k} e^{2\pi i(k-1)\phi_a}} \right\}^{1-h} \prod_{j=1}^{k} \delta\left((N+k)\phi_j - \sum_{\ell=1}^{k} \phi_\ell + \frac{k-1}{2} - n_j\right) \end{aligned}$$

The q-boson Model and Phase Model

$ref)N.M.Bogoliubov, A.G.Izergin, N.A.Kitanine('98)$

The q-boson model (the phase model) is the quantm integrable field theory on the lattice.

the Hamiltonian

$$H = -\frac{1}{2}\sum_{i=1}^{N}(\beta_i\beta_{i+1}^\dagger + \beta_{i+1}\beta_i^\dagger)$$

the q-boson (q-oscillator) algebra $\{q^{\pm\hat{N}}, \beta, \beta^\dagger\}$,

$$q^{\hat{N}}q^{-\hat{N}} = q^{-\hat{N}}q^{\hat{N}} = 1, \quad q^{\hat{N}}\beta = \beta q^{\hat{N}-1}, \quad q^{\hat{N}}\beta^\dagger = \beta^\dagger q^{\hat{N}+1},$$

$$\beta\beta^\dagger - \beta^\dagger\beta = (1-q^2)q^{2\hat{N}}, \quad \beta\beta^\dagger - q^2\beta^\dagger\beta = 1 - q^2,$$

where $q^{\pm N}$ denote generators and $q^{\pm pN+x}$ is shorthand for $(q^{\pm N})^p q^x$.

- t serve a coupling constant. $(q = e^t)$

Comment

q-boson algebra $\leftrightarrow$ free boson algebra

$$\beta_j = \sqrt{\frac{1-q^{2(\hat{N}_j+1)}}{1+\hat{N}_j}}a_j, \quad \beta_j^\dagger = a_j^\dagger\sqrt{\frac{1-q^{2(\hat{N}_j+1)}}{1+\hat{N}_j}}$$

free boson algebra

$$[\hat{N}, a] = -a, \quad [\hat{N}, a^\dagger] = a^\dagger, \quad [a, a^\dagger] = 1$$

Various limits

- $q \to 1$ Limit: q-boson model $\longrightarrow$ free boson model
- Continuum Limit: q-boson model $\longrightarrow$ non-linear Schrodinger model

"strong coupling" limit

- $q \to 0$ Limit: q-boson model $\longrightarrow$ phase model

$$H = -\frac{1}{2}\sum_{i=1}^{N}(\varphi_i \varphi_{i+1}^\dagger + \varphi_{i+1}\varphi_i^\dagger)$$

$$q \to 0 : \qquad \beta_j \to \varphi_j = \frac{1}{\sqrt{1+\hat{N}_j}} a_j, \quad \beta_j^\dagger \to \varphi_j^\dagger = a_j^\dagger \frac{1}{\sqrt{1+\hat{N}_j}}$$

The Algebraic Bethe Ansatz Of The Phase Model

Fock space of the phase model $(0 \equiv N,\ \text{Ex. } |0\rangle_0 \equiv |0\rangle_N)$

$$|0\rangle := \bigotimes_{i=0}^{N-1} |0\rangle_i, \quad \varphi_i|0\rangle = 0, \quad \varphi_i^\dagger|0\rangle = |0\rangle_0 \otimes |0\rangle_1 \otimes \cdots \otimes (\varphi^\dagger|0\rangle)_i \otimes \cdots \otimes |0\rangle_{N-1}$$

Algebraic Bethe Ansatz

- L-matrix: $L_i(\mu) = \begin{pmatrix} 1 & \mu\varphi_i^\dagger \\ \varphi_i & \mu \cdot 1 \end{pmatrix}$ where $\mu \in \mathbb{C}$ is a spectral parameter.

- L matrix satisfy the Yang-Baxter eq: $R(\mu,\nu)(L(\mu) \otimes L(\nu)) = (L(\nu) \otimes L(\mu))R(\mu,\nu)$

with R-matrix is $R(\mu,\nu) = \begin{pmatrix} \frac{\mu}{\mu-\nu} & 0 & 0 & 0 \\ 0 & \frac{\nu}{\mu-\nu} & 1 & 0 \\ 0 & 0 & \frac{\mu}{\mu-\nu} & 0 \\ 0 & 0 & 0 & \frac{\mu}{\mu-\nu} \end{pmatrix}$.

- Monodoromy matrix: $T(\mu) = \begin{pmatrix} A(\mu) & B(\mu) \\ C(\mu) & D(\mu) \end{pmatrix} := L_N(\mu)L_{N-1}(\mu)\cdots L_1(\mu)$

- Transfer matrix: $\tau(\mu) = \mathrm{tr}T(\mu) = A(\mu) + D(\mu)$

- $B(\mu) =$ creation operator, $C(\mu) =$ annihilation operator

- Yang-Baxter eq $\longrightarrow R(\mu,\nu)(T(\mu)\otimes T(\nu)) = (T(\nu)\otimes T(\mu))R(\mu,\nu)$

$$\longrightarrow \begin{cases} (\mu-\nu)A(\mu)B(\nu) &= \nu(-B(\nu)A(\mu)+B(\mu)A(\nu)) \\ (\mu-\nu)D(\mu)B(\nu) &= \mu B(\nu)D(\mu)-\nu B(\mu)D(\nu) \\ (\mu-\nu)C(\mu)B(\nu) &= \nu(A(\nu)D(\mu)-A(\mu)D(\nu)) \end{cases}$$

- Suppose that $|\psi(\{\lambda\}_k)\rangle := \prod_{j=1}^{k} B(\lambda_j)|0\rangle$ is the eigenstate of the transfer matrix:

$$\tau(\mu)\prod_{j=1}^{k} B(\lambda_j)|0\rangle = \Lambda(\mu,\{\lambda\})\prod_{j=1}^{k} B(\lambda_j)|0\rangle$$

$$\longrightarrow \text{ Bethe eqs:}\quad (-1)^{k-1}\cdot\lambda_j^{-(N+k)}\cdot\prod_{a=1}^{k}\lambda_a = 1$$

- Bethe norm: $\langle 0|\prod_{j=1}^{k} C(\lambda_j)\prod_{j=1}^{k} B(\lambda_j)|0\rangle = \dfrac{\prod_{b=1}^{k}\lambda_b^{k-1}}{\prod_{\substack{a,b=1\\ a\neq b}}^{k}(\lambda_a-\lambda_b)} N(N+k)^{k-1}$

where $\{\lambda\}$ satisfy the Bethe eqs.

The G/G Gauged WZW Model and The Phase Model

$G = U(N)$

$$Z_{\mathrm{GWZW}}(\Sigma_h) = \left(\frac{N}{k+N}\right)^{1-h} \sum_{\{Solution\}} \left\{ \frac{1}{(k+N)^{k-1}N} \frac{\prod_{\substack{a,b=1\\a\neq b}}^{k}\left(e^{2\pi i\phi_a} - e^{2\pi i\phi_b}\right)}{\prod_{a=1}^{k} e^{2\pi i(k-1)\phi_a}} \right\}^{1-h}$$

$$\text{with constraints } (N+k)\phi_j - \sum_{\ell=1}^{k}\phi_\ell + \frac{k-1}{2} - n_j = 0 \quad \text{for } j = 1, \cdots, k$$

N : Site number $\equiv$ Rank of Guage group U(N)

k : Particle number $\equiv$ Level of $\hat{\mathfrak{su}}(N)_k$

$$\lambda_j = e^{2\pi i\phi_j}, \qquad \text{Bethe eqs} \longrightarrow (N+k)\phi_j - \sum_{\ell=1}^{k}\phi_\ell + \frac{k-1}{2} - n_j = 0$$

$$Z_{\mathrm{GWZW}}^{U(N)}(\Sigma_h) = \underbrace{\left(\frac{N}{k+N}\right)^{1-h}}_{U(1)\text{ part}} \sum_{\{\text{Bethe roots}\}} \left(\langle 0| \prod_{j=1}^{k} C(\lambda_j) \prod_{j=1}^{k} B(\lambda_j)|0\rangle \right)^{1-h}$$

$G = SU(N)$

$$Z_{\mathrm{GWZW}}^{SU(N)}(\Sigma_h) = \sum_{\{\text{Bethe roots}\}} \left(\langle 0| \prod_{j=1}^{k} C(\lambda_j) \prod_{j=1}^{k} B(\lambda_j)|0\rangle \right)^{1-h}$$

Fock space of the phase model and affine weights of affine Lie algebra

$ref)C.Korff\ ,\ C.Stroppel('10)$

Fock space of the phase model $(N \equiv 0)$

$$|0\rangle := \bigotimes_{i=0}^{N-1} |0\rangle_i, \quad \varphi_i|0\rangle = 0, \quad \varphi_i^\dagger|0\rangle = |0\rangle_0 \otimes \cdots \otimes (\varphi^\dagger|0\rangle)_i \otimes \cdots \otimes |0\rangle_{N-1} =: |0\rangle_0 \otimes \cdots \otimes |1\rangle_i \otimes \cdots \otimes |0\rangle_{N-1}$$

$$\prod_{i=0}^{N-1} \left(\varphi_i^\dagger\right)^{\lambda_i} |0\rangle = |\lambda_0\rangle_0 \otimes |\lambda_1\rangle_1 \otimes \cdots \otimes |\lambda_{N-1}\rangle_{N-1} =: |\hat{\lambda}\rangle$$

- total particle number: $k = \sum_{i=0}^{N-1} \lambda_i$

Affine dominant integral weight of level k affine Lie algebra $\widehat{\mathfrak{su}}(N)_k$

$$P_k^+ = \left\{ \hat{\lambda} = \sum_{i=0}^{N-1} \lambda_i \hat{\omega}_i \;\middle|\; \sum_{i=0}^{N-1} \lambda_i = k\,,\ \lambda_i \in \mathbb{Z}_{\geq 0} \right\}$$

where $\hat{\omega}_i$ are affine fundamental weights and λ_i are Dinkin labels.

Identify a Fock space on the i-th site $|0\rangle_i$ and an affine fundamental weight $\hat{\omega}_i$: $|0\rangle_i \equiv \hat{\omega}_i$

$$\longrightarrow\ \varphi_i^\dagger :\ P_k^+ \to P_{k+1}^+, \quad \varphi_i^\dagger \hat{\lambda} = \hat{\lambda} + \hat{\omega}_i \quad ; \quad \varphi_i :\ P_k^+ \to P_{k-1}^+, \quad \varphi_i \hat{\lambda} = \begin{cases} \hat{\lambda} - \hat{\omega}_i, & \text{if } \hat{\lambda} - \hat{\omega}_i \in P_{k-1}^+ \\ 0 \quad , & \text{otherwise} \end{cases}$$

$$B(x_1^{-1})B(x_2^{-1})\cdots B(x_k^{-1})|0\rangle = \sum_{\hat{\lambda}\in P_k^+} s_{\hat{\lambda}^t}(x_1^{-1}, x_2^{-1}, \cdots, x_k^{-1})\hat{\lambda}$$

where $s_{\hat{\lambda}^t}(x_1, x_2, \cdots, x_k) =$ Schur polynomial

- Suppose that $\{x\}_{i=1}^k$ satisfy the Bethe equations of the phase model.
- $\mathrm{Sol}(N,k) := \left\{x \in \mathbb{R}^k | x \text{ solve Bethe eqs and the } x_i \text{ are pairwise distinct}\right\}$

$$\mathfrak{P}_{\leq N-1,k} \cong \widehat{\mathrm{Sol}(N,k)},$$

$$\sigma \mapsto x_\sigma := z^{\frac{1}{N}} \zeta^{\frac{|\sigma|}{N}} \left(\zeta^{I_1}, \ldots, \zeta^{I_k}\right)$$

$$\text{where } \zeta = \exp\frac{2\pi\iota}{k+N} \text{ and } I = I(\sigma^t) := \left(\tfrac{k+1}{2} + \sigma_k^t - k, \ldots, \tfrac{k+1}{2} + \sigma_1^t - 1\right)$$

- Character: $\chi_\mu(x^{-1}) = \langle 0|C(x_1^{-1})C(x_2^{-1})\cdots C(x_k^{-1})|\hat{\mu}\rangle$ for $\hat{\mu} \in P_k^+$
- Modular S-matrix: $\mathcal{S}_{\hat{\mu},\hat{\sigma}} = \mathcal{S}_{0,\hat{\sigma}}\chi_\mu(x^{-1})$, and $\mathcal{S}_{0\hat{\sigma}} = \left(\langle 0|\prod_{j=1}^k C(x_j^{-1})\prod_{j=1}^k B(x_j^{-1})|0\rangle\right)^{-1}$

Summary and Discussion

- G/G Gauged WZW model の分配関数を phase model のベーテノルムで表わすことができることを示した。
- G/G Gauged WZW model の $\mathrm{Tr}g(z,\bar{z})$ の相関関数を phase model のの言葉で表わすことができることを示した。
- また、ベーテノルムで表わすことができる数学的な理由は phase 模型を affine Lie algebra の表現論で記述することができること。(逆も可)
- G/G Gauged WZW Higgs model の分配関数を q-boson model のベーテノルムで表わすことができることを示した。
- ここで現れたタイプの可積分性は、全てのゲージ群のランク、レベルを持つ G/G Gauged WZW model を同時に考えることにより現れる。しかし、物理的な理解は得られなかった