Supersymmetry, chiral symmetry and the generalized BRS transformation in lattice formulations of 4D $\mathcal{N} = 1$ SYM

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- H.S., arXiv:1202.2598 [hep-lat], Nucl. Phys. B861 (2012) 290–320.

Target: 4D $\mathcal{N} = 1$ SYM

- Simplest 4D SUSY gauge theory ($\because$ no scalar field)

$$S = \int d^4x \left[\frac{1}{2}\,\mathrm{tr}\,(F_{\mu\nu}F_{\mu\nu}) + \mathrm{tr}\,(\bar{\psi}D\!\!\!/\psi)\right], \qquad \bar{\psi} = \psi^T(-C^{-1})$$

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- Chiral $U(1)_A$ symmetry

$$\delta_\theta\psi = i\theta\gamma_5\psi, \qquad \delta_\theta\bar{\psi} = i\theta\bar{\psi}\gamma_5$$

is an R symmetry

$$[\delta_\theta, \delta_\xi] = \delta_{(\xi \to -i\theta\gamma_5\xi)}$$

Expected non-perturbative physics (for $G = SU(N_c)$)

- No spontaneous SUSY breaking: Witten index $\mathrm{Tr}(-1)^F = N_c \neq 0$
- Chiral symmetry breaking

$$U(1)_A \xrightarrow{\text{anomaly \& instanton}} \mathbb{Z}_{2N_c} \xrightarrow{\langle \mathrm{tr}(\bar{\psi}\psi)\rangle \neq 0} \mathbb{Z}_2, \qquad \text{(domain wall)}$$

- Lowest-lying SUSY multiplet (Veneziano–Yankielowicz (1982))

$$\text{gluino-glue } \sigma_{\mu\nu}\,\mathrm{tr}(\psi F_{\mu\nu}) \Leftrightarrow \text{adjoint-}\eta'\ \mathrm{tr}(\bar{\psi}\gamma_5\psi),\ \text{adjoint-}f_0\ \mathrm{tr}(\bar{\psi}\psi)$$

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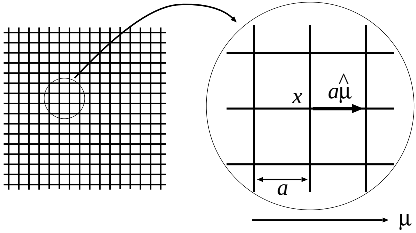
- Non-perturbative study by the lattice regularization?



Lattice formulation?

- A possible lattice action:

$$S_{\text{gluon}} = \sum_x \sum_{\mu,\nu} \left(-\frac{1}{g^2}\right) \operatorname{Re}\operatorname{tr}\left[U_\mu(x)U_\nu(x+a\hat{\mu})U_\mu^\dagger(x+a\hat{\nu})U_\nu^\dagger(x)\right],$$

$$S_{\text{gluino}} = a^4 \sum_x \operatorname{tr}\left(\bar{\psi}(x)\left\{\frac{1}{2}\sum_\mu \left[\gamma_\mu(\nabla_\mu + \nabla_\mu^*) - ra\nabla_\mu^*\nabla_\mu\right]\right\}\psi(x)\right)$$

  - Link variables

$$U_\mu(x) = e^{iagA_\mu(x)}$$

  - Covariant differences

$$\nabla_\mu\psi(x) \equiv \frac{1}{a}\left[U_\mu(x)\psi(x+a\hat{\mu})U_\mu^\dagger(x) - \psi(x)\right],$$

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- These $O(a)$ effects become $O(1)$ through $O(1/a)$ radiative corrections!

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- We want to understand this symmetry restoration in terms of Ward–Takahashi (WT) relation. . .

Lattice WT relation for $U(1)_A$

- (Localized) $U(1)_A$ transformation

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$$X_A(x) = (1-\mathcal{Z}_A)\partial_\mu^*\,\mathrm{tr}\left[\bar{\psi}(x)\gamma_\mu\gamma_5\psi(x)\right] - \mathcal{Z}_{F\tilde{F}}[F\tilde{F}]^L(x) - \frac{1}{a}\mathcal{Z}_P\,\mathrm{tr}\left[\bar{\psi}(x)\gamma_5\psi(x)\right] + \cdots$$

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(Necessary) condition for the symmetry restoration

- Thus, for SUSY and $U(1)_A$ WT identities without breaking to be restored,

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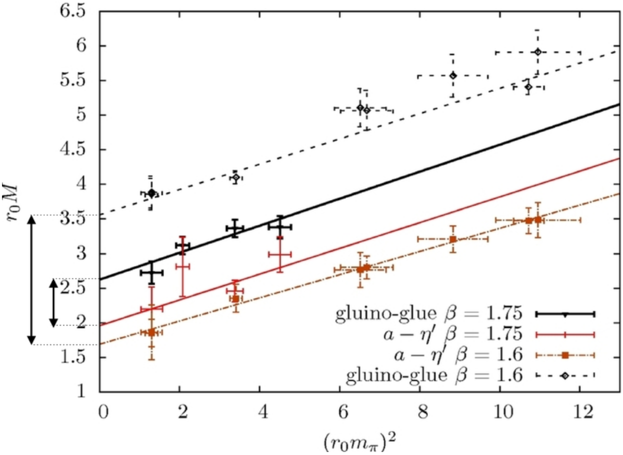
- This must be important from the perspective of recent numerical simulations (DESY–Münster, Giedt et al. (USA), Endres (RIKEN), Kim et al. (JLQCD))

Wilson fermion, tree-level Symanzik improved gauge action, $G = SU(2)$ (Bergner–Münster–Sandbrink–Özugurel–Montvay (2011))

- $32^3 \times 64$, $a = 0.114 r_0$,



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the gauge anomaly

$$\mathcal{A} \equiv s_0 \Gamma, \qquad \Gamma\text{: effective action},$$

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- This BRS-like transformation should include also translation and gauge transformations

$$[\delta_\xi, \delta_{\xi'}] = -t_\mu \partial_\mu + \mathcal{G}_{t_\mu A_\mu}, \qquad t_\mu = \bar\xi \gamma_\mu \xi' - \bar\xi' \gamma_\mu \xi$$

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sA_\mu &\equiv D_\mu c + \bar{\xi}\gamma_\mu\psi - it_\nu\partial_\nu A_\mu, \\
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  are constant and possess opposite statistics as the corresponding transformation parameters
- Then one finds

$$s^2\Phi = 0$$

for all variables Φ, except ψ on which,

$$s^2\psi = \gamma_5\xi\bar{\xi}\gamma_5 D\!\!\!\!/\,\psi \propto \text{(eq. of motion of } \psi\text{; on-shell nilpotency)}$$

Basic line of the proof

- In continuum theory, the formal invariance implies the Slavnov–Taylor (ST) identity or the Zinn-Justin equation for the effective action,

$$\mathcal{S}(\Gamma) = 0$$

where

$$\begin{aligned}\mathcal{S}(F) \equiv \int d^4x \left[\frac{\delta F}{\delta K^a_{A_\mu}(x)} \frac{\delta F}{\delta A^a_\mu(x)} + \frac{\delta F}{\delta \bar{K}^a_\psi(x)} \frac{\delta F}{\delta \psi^a(x)} + \frac{\delta F}{\delta K^a_c(x)} \frac{\delta F}{\delta c^a(x)} \right] \\ + \int d^4x \left[s\bar{c}^a(x) \frac{\delta F}{\delta \bar{c}^a(x)} + sB^a(x) \frac{\delta F}{\delta B^a(x)} \right] \\ + s\xi \frac{\partial F}{\partial \xi} + st_\mu \frac{\partial F}{\partial t_\mu} + s\theta \frac{\partial F}{\partial \theta} + \cdots\end{aligned}$$

Basic line of the proof

- We can define a lattice analogue of the generalized BRS transformation s but s is not nilpotent by $O(a)$ (of course!)

$$s^2 A_\mu = O(a), \qquad s^2\psi = \gamma_5\xi\bar{\xi}\gamma_5 D\psi + O(a), \qquad s^2 c = O(a),$$

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$$\mathcal{S}(\Gamma) = \left\langle a^4 \sum_x \left[\bar{\xi} X_S(x) + i\theta X_A(x)\right] + \bar{c}\cdot\mathcal{B}_{\bar{c}} + K'\cdot\mathcal{B}_{K'} + t\cdot\mathcal{B}_t \right\rangle_{J,K,\xi,t,\theta,u,v}$$

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- Here, $X_S(x)$ and $X_A(x)$ are $O(a)$ symmetry breaking terms

$$\delta_\xi\left(S_{\mathrm{gluon}} + S_{\mathrm{gluino}}\right) = a^4 \sum_x \bar{\xi} X_S(x),$$

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- The crucial ingredient is the "linearized" $\mathcal{S}(F)$, defined by

$$\mathcal{D}(F) \equiv a^4 \sum_x \left[\frac{\delta F}{\delta A_\mu^a(x)} \frac{\delta}{\delta K_{A_\mu}^a(x)} + \frac{\delta F}{\delta K_{A_\mu}^a(x)} \frac{\delta}{\delta A_\mu^a(x)} \right.$$

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Very final step of the proof

- The expectation value survives only through radiative corrections, thus $O(\hbar^n)$ with $n \geq 1$. Taking $O(\hbar^n)$ terms of both sides,

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