Vortex counting from field theory

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collabolation with

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Non-perturbative effect on SUSY vacua

- 4 dim $\mathcal{N} = 2$ theory

Seiberg-Witten prepotential

$$\mathcal{F}(\vec{a}, \Lambda) = \lim_{\epsilon_{1,2} \to 0} \epsilon_1 \epsilon_2 \, \mathcal{Z}_{\mathrm{4d}}(\vec{a}, \Lambda; \epsilon_{1,2})$$

- 2 dim $\mathcal{N} = (2, 2)$ theory

Twisted superpotential

$$\mathcal{W}(\vec{m}, \Lambda) = \lim_{\epsilon \to 0} \epsilon \, \mathcal{Z}_{\mathrm{2d}}(\vec{m}, \Lambda; \epsilon)$$

Non-perturbative partition function

- 4 dim $\mathcal{N} = 2$ $\mathrm{SU}(N)$ gauge theory

Instanton partition function

$$\mathcal{Z}_{\mathrm{inst}}(\vec{a}, \Lambda; \epsilon_{1,2}) = \sum_{k=0}^{\infty} \Lambda^{2Nk} \mathcal{Z}_k(\vec{a}; \epsilon_{1,2})$$

- 2 dim $\mathcal{N} = (2,2)$ $\mathrm{U}(N)$ gauge theory

Vortex partition function

$$\mathcal{Z}_{\mathrm{vortex}}(\vec{m}, \Lambda; \epsilon) = \sum_{k=0}^{\infty} \Lambda^{Nk} \mathcal{Z}_k(\vec{m}; \epsilon)$$

- What's $\mathcal{Z}_k$? $\longrightarrow$ **volume of instanton/vortex moduli space**

- How to describe moduli space?

Vortex moduli space

1. Kähler quotient [Hanany-Tong '03]
2. Direct treatment of BPS eq [Eto-Isozumi-Nitta-Ohashi-Sakai '06]

- Previous results
  [Shadchin '06] [Dimofte-Gukov-Hollands '10] [Yoshida '11]
  [Bonneli-Tanzini-Zhao '11] [Miyake-Ohta-Sakai '11]

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Field theory setup

- $\mathcal{N} = (2,2)$ $\mathrm{U}(N)$ gauge theory with N fundamental chirals

$$\mathcal{L}_{\mathrm{B}} = \mathrm{Tr}\left[-\frac{1}{2g^2}F_{\mu\nu}F^{\mu\nu} - (\mathcal{D}_\mu H)(\mathcal{D}_\mu H)^\dagger - \frac{g^2}{4}(HH^\dagger - v^2\mathbb{1}_N)^2\right]$$

- Potential term

$$HH^\dagger - v^2\mathbb{1}_N = 0 \quad \longrightarrow \quad \langle H\rangle \neq 0$$

Gauge symmetry broken (Higgs phase)

BPS state

- BPS equations

$$(\mathcal{D}_1 + i\mathcal{D}_2)\, H = 0, \qquad F_{12} + \frac{g^2}{2}(v^2 \mathbb{1}_N - HH^\dagger) = 0$$

- Energy saturated

$$T = v^2 \int \mathrm{tr}\, F = 2\pi v^2 \times k$$

k : vortex number

BPS vortex state

Holomorphic gauge : moduli matrix

- BPS configuration

$$H(z,\bar{z}) = S^{-1}(z,\bar{z})H_0(z), \qquad A_z = S^{-1}(z,\bar{z})\partial_z S(z,\bar{z})$$

moduli matrix : $H_0(z)$

- Gauge transformation

$$H_0(z) \longrightarrow V(z)H_0(z), \qquad S(z,\bar{z}) \longrightarrow V(z)S(z,\bar{z})$$

complexified gauge group : $V(z) \in \mathrm{U}(N)_{\mathbb{C}} \cong \mathrm{GL}(N,\mathbb{C})$

Vortex moduli space

Vortex number

$$v^2 \int \mathrm{tr}\, F = -i\frac{v^2}{2} \oint dz\, \partial_z \log\det H_0(z) \longrightarrow 2\pi v^2 \times k$$

where

$$\det H_0(z) = z^k + c_1 z^{k-1} + \cdots c_k = \prod_{i=1}^{k}(z - z_i)$$

Vortex moduli space

$$\mathcal{M}_{N,k} \cong \left\{ H_0(z) \middle| \det H_0(z) = \mathcal{O}(z^k) \right\} / \{V\text{-transformation}\}$$

$$\to (\mathbb{C} \times \mathbb{C}\mathbf{P}^{N-1})^k / \mathfrak{S}_k \qquad \text{(well separated)}$$

Vortex partition function

- How to compute the moduli space volume?

Localization method

1. Find a fixed point in vortex moduli space under isometry
2. Calculate "weight" at that point
3. Read the partition function from the weight

Isometry and fixed points

- Isometry : $\mathrm{U}(1) \times \mathrm{U}(1)^{N-1} \in \mathrm{SO}(2) \times \mathrm{SU}(N)$

$$H_0(z) \quad \longrightarrow \quad H_0(T_\epsilon z) T_{\vec{a}}$$

$$T_\epsilon = e^{i\epsilon}, \qquad T_{\vec{a}} = \mathrm{diag}(e^{ia_1}, \cdots, e^{ia_N})$$

- Fixed point labeled by $\vec{k} = (k_1, \cdots, k_N)$

$$H_0^{\vec{k}}(z) = \mathrm{diag}(z^{k_1}, \cdots, z^{k_N})$$

$$\Downarrow$$

$$H_0^{\vec{k}}(T_\epsilon z) T_{\vec{a}} = V_{\vec{k}}(z) H_0^{\vec{k}}(z) \overset{V\text{-trans}}{\sim} H_0^{\vec{k}}(z)$$

Isometry and fixed points

- Tangent space at the fixed point $\vec{k}$

$$H_0(z) = H_0^{\vec{k}}(z) + \delta H_0(z)$$

$$(\delta H_0(z))_{lm} = \sum_{j=1}^{k_m} c_{lm,j} z^{j-1}$$

$c_{lm,j}$: a moduli space coordinate $\phi_i \quad (i = 1, \cdots, Nk)$

- Isometry on the tangent space

$$c_{lm,j} \longrightarrow e^{ia_{ml}+i\epsilon(-k_l+j-1)} c_{lm,j}$$

Schematically,

$$\phi_i \longrightarrow (\mathcal{T}_{\vec{k}})_{ij} \phi_j$$

diagonalized $\mathcal{T}_{\vec{k}}$ with eigenvalues $e^{ia_{ml}+i\epsilon(-k_l+j-1)}$

- Remark : $\dim_{\mathbb{C}} \mathcal{M}_{N,k} = Nk$

- "weight" at the fixed point $\vec{k}$

$$\chi(T_{\vec{k}}\mathcal{M}) = \mathrm{Tr}\left[\mathcal{T}_{\vec{k}}\right] = \sum_{l,m}^{N}\sum_{j=1}^{k_m} e^{ia_{ml}+i\epsilon(-k_l+j-1)}$$

Vortex partition function

$$\mathcal{Z}_k(\vec{a};\epsilon) = \sum_{\vec{k}\,\text{s.t.}\,|\vec{k}|=k} \mathcal{Z}_{\vec{k}}(\vec{a};\epsilon)$$

$$\mathcal{Z}_{\vec{k}}(\vec{a};\epsilon) = \prod_{l,m}^{N}\prod_{j=1}^{k_m} \frac{1}{a_{ml}+\epsilon(-k_l+j-1)}$$

- Other derivations :
  - surface operator [Dimofte-Gukov-Hollands] [Bonelli-Tanzini-Zhao]
  - S^2 partition function [Benini-Cremonesi] [Doroud-Gomis-Floch-Lee]

Adding anti-fundamental matter

- Additional $\widetilde{N}$ anti-fundamental chiral multiplets
  - No additional bosonic moduli $\widetilde{H}_0 = 0$
  - Additional fermionic zero modes

$$\left(\tilde{\psi}_{0-}\right)_{lm} = \sum_{j=1}^{k_l} \zeta_{lm,j} \bar{z}^{j-1}$$

  - Isometry

$$\zeta_{lm,j} \quad \longrightarrow \quad e^{im_l + ia_m + i\epsilon(j-1)} \zeta_{lm,j}$$

Anti-fundamental matter contribution

$$\mathcal{Z}_{\vec{k}}^{\text{antifund}}(\vec{a}, \vec{m}; \epsilon) = \prod_{l=1}^{\widetilde{N}} \prod_{m=1}^{N} \prod_{j=1}^{k_l-1} (m_l + a_m + \epsilon(j-1))$$

Summary and discussion

- Vortex partition function from moduli matrix method

$$\mathcal{Z}_{\text{vortex}}(\vec{a},\Lambda;\epsilon) = \sum_{\vec{k}} \Lambda^{N|\vec{k}|} \mathcal{Z}_{\vec{k}}(\vec{a};\epsilon)$$

$$\mathcal{Z}_{\vec{k}}(\vec{a};\epsilon) = \prod_{l,m}^{N} \prod_{i=1}^{k_m} \frac{1}{a_{ml} + \epsilon(-k_l + i - 1)}$$

- Adding anti-fundamental, adjoint chiral multiplets

$$\mathcal{Z}_{\vec{k}}^{\text{antifund}}(\vec{a},\vec{m};\epsilon) = \prod_{l=1}^{\widetilde{N}} \prod_{m=1}^{N} \prod_{j=1}^{k_l-1} (m_l + a_m + \epsilon(j-1))$$

$$\mathcal{Z}_{\vec{k}}^{\text{adj}}(\vec{a},\vec{m};\epsilon) = \prod_{l,m}^{N} \prod_{i=1}^{k_m} (a_{ml} + \epsilon(-k_l + i - 1) + M)$$

Summary and discussion

- Orbifold partition function

- Other gauge groups : Kähler quotient not yet known

- Non-perturbative effect on SUSY vacua

$$\mathcal{W}(\vec{m}, \Lambda) = \lim_{\epsilon \to 0} \epsilon \, \mathcal{Z}_{2\mathrm{d}}(\vec{m}, \Lambda; \epsilon)$$

  - 4d/2d, 3d/3d relations, quantized curve...

Next Talk