ABJ Wilson loop and Seiberg Duality

Department of Physics Nagoya University Nagoya 464-8602, Japan Keita Nii
E-mail: nii@th.phys.nagoya-u.ac.jp

We study the supersymmetric Wilson loop in a 3-dimensional $\mathcal{N} = 6$ supersymmetric $U(N_1)_k \times U(N_2)_{-k}$ Chern-Simons Matter gauge theory, which is known as ABJ theory, and its Seiberg dual. Using the Lens space matrix model, we obtain the useful expressions for the supersymmetric Wilson loops in ABJ theory. The result is given in the form of a $\mathsf{min}(N_1, N_2)$-dimensional integral. This result contains not only perturbative effects but also non-perturbative ones. By using these expressions we find the correct mapping of the BPS Wilson loops under the Seiberg duality.

Recently the localization technique is used to derive the exact results[1]. This enable us to obtain the matrix model representation for the partition function or the BPS Wilson loops. In the case of ABJ theory which is a 3-dimensional $\mathcal{N} = 6$ supersymmetric $U(N_1)_k \times U(N_2)_{-k}$ Chern-Simons Matter gauge theory[2], the 1/6 and 1/2 BPS Wilson loop become as follows[3].

$$W_{I,p}^{1/6\,\mathsf{BPS}}(N_1, N_2, k) = \langle \textstyle\sum_{i=1}^{N_1} \mathrm{e}^{p\mu_i} \rangle, \quad W_{II,p}^{1/6\,\mathsf{BPS}}(N_1, N_2, k) = \langle \textstyle\sum_{a=1}^{N_2} \mathrm{e}^{p\nu_a} \rangle \tag{1}$$

$$W_p^{1/2\,\mathsf{BPS}}(N_1, N_2, k) \equiv W_{I,p}^{1/6\,\mathsf{BPS}} - (-1)^p W_{II,p}^{1/6\,\mathsf{BPS}} \tag{2}$$

where $g_s = 2\pi i/k$ and k is the Chern-Simons coupling and p is the winding number of Wilson loops and the bracket means the following integral

$$\langle \cdots \rangle \quad \equiv \int \prod_{j=1}^{N_1} \frac{d\mu_j}{2\pi} \prod_{a=1}^{N_2} \frac{d\nu_a}{2\pi} \frac{\prod_{j<k}\left(2\sinh\frac{\mu_j-\mu_k}{2}\right)^2 \prod_{a<b}\left(2\sinh\frac{\nu_a-\nu_b}{2}\right)^2}{\prod_{j,a}\left(2\cosh\frac{\mu_j-\nu_a}{2}\right)^2} \tag{3}$$

$$\times \exp\left[-\frac{1}{2g_s}\left(\textstyle\sum_j \mu_j^2 - \sum_a \nu_a^2\right)\right](\cdots) \tag{4}$$

These integrals are very tedious due to the denominators. Alternatively we can calculate easily the Lens space matrix model. This matrix model is known to be analytically continued to the above ABJ matrix model under $N_2 \to -N_2$[4, 5].

$$\begin{aligned} W_{\mathsf{lens}}^{1/6\,\mathsf{BPS}}(N_1, N_2, k) = \sum_i \int \prod_{j=1}^{N_1} \frac{d\mu_j}{2\pi} \prod_{a=1}^{N_2} \frac{d\nu_a}{2\pi} \prod_{j<k}\left(2\sinh\frac{\mu_j-\mu_k}{2}\right)^2 \prod_{a<b}\left(2\sinh\frac{\nu_a-\nu_b}{2}\right)^2 \\ \times \prod_{j,a}\left(2\cosh\frac{\mu_j-\nu_a}{2}\right)^2 \exp\left[-\frac{1}{2g_s}\left(\textstyle\sum_j \mu_j^2 + \sum_a \nu_a^2\right) + p\mu_i\right] \end{aligned} \tag{5}$$

This integral is purely Gaussian one. Then we can integrate and obtain as follows. Especially we list the $N_1 < N_2$ result where we use the Pochhammer symbol $(a)_n = \prod_{j=0}^{n-1}(1 - aq^j),\ q = \mathrm{e}^{-g_s}$.

$$W_{I,p\,\mathsf{ABJ}}^{1/6\,\mathsf{BPS}}(N_1, N_2) = i^{-\frac{\kappa}{2}(N_1^2 + N_2'^2)} \frac{(2\pi)^{\frac{N_1^2 + N_2'^2}{2}}}{N_1!} g_s^{\frac{N_1 + N_2'}{2}} q^{-\frac{1}{2}p^2 - p} (-1)^{N_1(N_2'-1)}$$

$$
\begin{aligned}
&\times \frac{\prod_{j=1}^{N_2'-N_1-1}(q)_j \prod_{j=1}^{N_1-1}(q)_j^{-2}}{G_2(N_1+1)G_2(N_2'+1)} \prod_{i=1}^{N_1} \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \frac{\pi ds_i}{\sin \pi s_i} \ \sum_{k=1}^{N_1} q^{-ps_k+pN_1-pN_2'}(-1)^{\sum_{j=1}^{N_1} s_j} \\
&\times \prod_{j\neq k} \frac{\left(q^{s_j-s_k-p}\right)_1}{\left(q^{s_j-s_k}\right)_1} \prod_{j=1}^{N_1} \frac{\left(q^{s_j+1}\right)_{N_2'-N_1}}{\left(-q^{s_j+1+p\delta_{kj}}\right)_{N_2'-N_1}} \prod_{1\leq j<l\leq N_1} \frac{\left(q^{s_l-s_j}\right)_1^{\,2}\left(-q^{l-j-p\delta_{kj}}\right)_1\left(-q^{l-j+p\delta_{kl}}\right)_1}{\left(-q^{s_l-s_j+p\delta_{kl}}\right)_1\left(-q^{s_l-s_j-p\delta_{kj}}\right)_1} \\
&\times \prod_{j=1}^{N_1} \frac{\left(q^{j-N_2'}\right)_{N_2'-j}(q)_{j-1}\left(-q^{j+p\delta_{kj}}\right)_{N_2'-N_1}}{\left(-q^{j-p\delta_{k(N_1-j+1)}-N_2'}\right)_{N_2'}(q^j)_{N_2'-N_1}}
\end{aligned}
\tag{6}
$$

where $G_2(N)$ is a Barnes G-function. Unfortunately there are some integral which comes from the permutation of Vandermonde derminants. Using this expression we can prove the duality of the BPS Wilson loops under the Seiberg duality. The ABJ Seiberg duality is the following correspondence between the two ABJ theories but with different gauge groups.

$$
U(N_1)_k \times U(N_2)_{-k} = U(2N_1+k-N_2)_k \times U(N_1)_{-k} \tag{7}
$$

By checking the pole and zeros structure and asymptotic behavior of the above integrals we can find the following duality.

$$
W_{I,p}^{1/6\,\mathsf{BPS}}(N_1,N_2) = W_{II,p}^{1/6\,\mathsf{BPS}}(2N_1+k-N_2,N_1) \tag{8}
$$

$$
W_{p=1}^{1/2\,\mathsf{BPS}}(N_1,N_2) = -W_{p=1}^{1/2\,\mathsf{BPS}}(2N_1+k-N_2,N_1) \tag{9}
$$

We further calculated the ABJ matrix model by using the Lens space matrix model. The results are the $\mathsf{min}(N_1,N_2)$ integrals. Then it is preferable to integrate further and obtain the more simple forms. Our results are easy to change the matrix model to the Fermi gas form. Then we must investigate the Fermi gas approach to the ABJ theory. We derive the mapping rule of the ABJ BPS Wilson loop under the Seiberg duality. But this rule is not understood from the brane picture or field theoretical point of view. Then we will pursue them. I would like to thank Tadakatsu Sakai,Sanefumi Moriyama and Masazumi Honda for helpful conversations.

References

[1] A. Kapustin, B. Willett and I. Yaakov, JHEP 1003, 089 (2010) [arXiv:0909.4559], N. Hama, K. Hosomichi and S. Lee, JHEP 1105, 014 (2011).

[2] O. Aharony, O. Bergman and D. L. Jafferis, JHEP 0811, 043 (2008) [arXiv:0807.4924]

[3] N. Drukker, J. Plefka and D. Young, JHEP 0811, 019 (2008) [arXiv:0809.2787]

[4] Awata Hidetoshi, Hirano Shinji and Shigemori Masaki, [arXiv:1212.2966]

[5] M. Marino, P. Putrov, JHEP 1006, 011 (2010). [arXiv:0912.3074]