Dynamics of $N=1$ gauge theories and M5-branes

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Introduction

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An important point is the relation with M-theory or 6d (2,0) theory: a class of $N=2$ theories, so-called class S theories, is obtained by M5-branes on $\mathbb{R}^{1,3} \times \text{Riemann surface } C$ in $\mathbb{R}^{1,3} \times T^*C \times \mathbb{R}^3$.
Why is it interesting?

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- **explains/proposes** S-duality of class S theories as a symmetry of the Riemann surface $C$. 
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- **enlarges** drastically the $N=2$ theory to a large class by associating the theory with the Riemann surface $C$

- **explains/proposes** S-duality of class $S$ theories as a symmetry of the Riemann surface $C$

- **leads** to a remarkable relation between 4d $N=2$ theories and 2d CFT on the Riemann surface $C$.  
  [Alday-Gaiotto-Tachikawa]
Low energy effective theory

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The low energy effective theory of $N=2$ gauge theory on the Coulomb branch is Abelian $U(1)^n$ theory coupled to some hypermultiplets.

This theory is determined by the Seiberg-Witten curve, which is in this picture a curve in $(x, t) \in T^*C:$

$$x^N + \sum_k \phi_k(t)x^{N-k} = 0$$

where $\phi_k$ is $k$-th differential on $C.$
Aim:
elucidate dynamics of
\( N=1 \) gauge theories
with help of M-theory
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There are much more interesting dynamics in \(N=1\) gauge theory, than \(N=2\) theory: confining phase etc.

➢ how do we describe it in the M-theory?
Aim: elucidate dynamics of \( N=1 \) gauge theories with help of M-theory

There are much more interesting dynamics in \( N=1 \) gauge theory, than \( N=2 \) theory: confining phase etc.

➢ how do we describe it in the M-theory?

The M-theory picture proposes dualities of a wide variety of \( N=2 \) theories.

➢ \( N=1 \) duality? (e.g. Seiberg duality)
Confining phase

Let us consider the $N=1$ mass deformation of $N=2$ theory (in class S) by adjoint chiral mass $W = \mu \text{Tr} \phi^2$. 
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From the Seiberg-Witten theory viewpoint, only the singularities on the Coulomb branch (the loci where the curve degenerates) is $N=1$ vacua when the mass parameter is turned on. [Seiberg-Witten, Douglas-Shenker]
Confining phase

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At the same time, we have another curve, so-called Dijkgraaf-Vafa curve, which determines the gaugino condensate. [Dijkgraaf-Vafa 2002, Cachazo-Douglas-Seiberg-Witten]
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the Seiberg-Witten curve

another curve
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- **the Seiberg-Witten curve**
- **another curve**

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string theoretical observation

\[ v = x_4 + i x_5 \]
\[ w = x_8 + i x_9 \]

\[ D4 \]

NS5 NS5
string theoretical observation

$V = x_4 + i x_5$

$W = x_8 + i x_9$

$w \sim \mu v$

rotation of NS5 (puncture)

mass deformation

NS5

D4

NS5
string theoretical observation

\[ w = x_8 + i x_9 \]
\[ v = x_4 + i x_5 \]

w-direction should enter the second equation describing the vacua of \( N=1 \) theory

[Hori-Ooguri-Oz, Witten, de Boer-Oz]
string theoretical observation

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\end{align*} \]

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NS5 \[ \rightarrow \]
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rotated
The spectral equations give an M-theory geometry

\[ x^N + \sum_k \phi_k(t)x^{N-k} = 0 \]
\( N=1 \) curve

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\]

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\]
The spectral equations give an M-theory geometry

\[ x^N + \sum_{k} \phi_k(t) x^{N-k} = 0 \]

\[ w^N + \sum_{k} V_k(t) w^{N-k} = 0 \]

+ the condition fixing \( \phi_k \) and \( V_k \)
$N=1$ curve

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\[ w^N + \sum_k V_k(t)w^{N-k} = 0 \]

+ the condition fixing $\phi_k$ and $V_k$

A combination of these equations gives the Dijkgraaf-Vafa curve, a curve in $(v, w)$. ($v = xt$)
Generalized Hitchin system

The $N=1$ vacua in confining phase is represented by a generalization of the Hitchin system with commuting fields $\Phi$ and $\varphi$, whose spectral curve is
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\det(x \cdot 1 - \Phi) = 0
\]

cf. Seiberg-Witten curve and Hitchin system [Gaiotto-Moore-Neitzke, Nanopoulos-Xie]
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where $\Phi$ and $\varphi$ are $\text{su}(N)$-valued differential and scalar on $\mathbb{C}$, having prescribed singularity at the puncture.

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Superconformal phase

This picture can also be applied to the case where the theory flows to **the IR \(N=1\) superconformal fixed point.**
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Focusing on the mass deformation of N=2 superconformal theory, e.g., N=2 SU(N) gauge theory with 2N flavors
Superconformal phase

This picture can also be applied to the case where the theory flows to the IR $N=1$ superconformal fixed point.

Focusing on the mass deformation of $N=2$ superconformal theory, e.g., $N=2$ SU($N$) gauge theory with $2N$ flavors.

rotation of punctures
$N=1$ duality from $N=2$ duality

In $N=2$ theory, the S-duality is denoted by
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**$N=1$ duality** from $N=2$ duality

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\[ \begin{array}{cccc}
  a & \bullet & \bullet & \bullet \\
  b & \bullet & \bullet & \bullet \\
  c & \bullet & \bullet & \bullet \\
  d & \bullet & \bullet & \bullet \\
\end{array} \]

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$N=1$ deformation

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  a & \bullet & \bullet & \bullet \\
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$N=1$ duality
PLAN

• Review of $N=2$ gauge theories in class S

• $N=1$ theories in confining phase

• Superconformal phase and $N=1$ dualities
$N=2$ gauge theories in class $S$
$N=2$ theory from M-theory

Consider M-theory geometry

$\mathbb{R}^{1,3} \times T^* C \times \mathbb{R}^3$

cotangent bundle on Riemann surface $C$
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$N$ M5-branes on $\mathbb{R}^{1,3} \times C \times \text{pt}$ with a partially topologically twist (to preserve $N=2$ susy in 4d)
$N=2$ theory from M-theory

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$N$ M5-branes on $\mathbb{R}^{1,3} \times C \times \text{pt}$ with a partially topologically twist (to preserve $N=2$ susy in 4d)

We call the obtained $N=2$ theory as class S which is classified by $N$ and $C$ with punctures:

[Gaiotto, Gaiotto-Moore-Neitzke]
Physical meaning of the Riemann surface
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complex structures \leftrightarrow UV coupling constants
Physical meaning of the Riemann surface

- complex structures ⇔ UV coupling constants
- punctures ⇔ matter fields in 4d (flavor symmetry)
Physical meaning of the Riemann surface

Complex structures $\leftrightarrow$ UV coupling constants

Punctures $\leftrightarrow$ Matter fields in 4d (flavor symmetry)

2 M5-branes on four-punctured sphere is SU(2) theory with 4 flavors with UV coupling constant $q$
Seiberg-Witten theory

The low energy effective theory on the Coulomb branch parametrized by $u_i = \langle \text{tr}\phi^i \rangle$ can be solved by the Seiberg-Witten curve:
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massless BPS particle appears
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The low energy effective theory on the Coulomb branch parametrized by $u_i = \langle \text{tr}\phi^i \rangle$ can be solved by the Seiberg-Witten curve:

$$a_i = \int_{A_i} \lambda_{SW}(u)$$

$$\frac{\partial F}{\partial a_i} = \int_{B_i} \lambda_{SW}(u)$$

massless BPS particle appears
Seiberg-Witten curve: 

**N-sheeted cover of \( \mathbb{C} \)**

The Seiberg-Witten curve is

\[
x^N + \sum_{k=2}^{N} x^{N-k} \phi_k(t) = 0
\]

\( \phi_k \) is the \( k \)-th meromorphic differential with poles at \( t = t_a \) and has moduli which are identified with the Coulomb moduli.

with the differential: \( \lambda_{SW} = x dt \)
Regular singularity and UV SCFT

Focus on the N=2 case the Seiberg-Witten curve is

\[ x^2 = \phi_2(t) \]
Regular singularity and UV SCFT

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\[ x^2 = \phi_2(t) \]

**REGULAR** puncture:

\[ \phi_2(t) \sim \frac{m_a^2}{(t - t_a)^2} \rightarrow \lambda_{SW} \sim \pm \frac{m_a}{t - t_a} \]

: SU(2) flavor symmetry
SU(2) w/ 4 flavors

Let us consider Type IIA construction of this theory

\[ v = x_4 + i x_5 \]

\[ \begin{array}{c}
\text{NS5} \\
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Let us consider Type IIA construction of this theory.

M-theory up-lift: add an $S^1$-direction parametrized by $x_{10}$

$cylinder: x_6 + ix_{10} \quad \Rightarrow \quad t = e^{-(x_6 + ix_{10})}$
SU(2) w/ 4 flavors

2 M5-branes on parametrized by t
SU(2) w/ 4 flavors

2 M5-branes on parametrized by $t$

The M-theory curve is

$$(v - m_1)(v - m_2)t^2 - (1 + q)P_2(v) + q(v - m_3)(v - m_4) = 0$$
SU(2) w/ 4 flavors

2 M5-branes on \[ \text{parametrized by } t \]

The M-theory curve is

\[(v - m_1)(v - m_2)t^2 - (1 + q)P_2(v) + q(v - m_3)(v - m_4) = 0 \]

\[ x^2 + \phi_2(t) = 0, \quad \phi_2 \sim \frac{m^2_\alpha}{t - t_\alpha} \]

\[(x = v/t) \quad t_\alpha = 0, q, 1, \infty \]
SU(2) w/ 4 flavors

$q$: gauge coupling constant of SU(2) $q = e^{2\pi i \tau}$

flavor symmetry: $SU(2)^4 \subset SO(8)$
SU(2) w/ 4 flavors

$q$: gauge coupling constant of SU(2) \( q = e^{2\pi i \tau} \)

flavor symmetry: \( SU(2)^4 \subset SO(8) \)

decoupling SU(2) \( (q \to 0) \)
Building block

The three-punctured sphere is a building block for the $N=2$ theories:
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SU(2): **free** hypermultiplets in tri-fundamental representation of SU(2)$^3$
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SU(2): \textbf{free} hypermultiplets in tri-fundamental representation of SU(2)$^3$

SU(N): \textbf{non-trivial SCFTs} with flavor symmetry associated with the punctures
$T_N$ theory

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- with $SU(N)^3$ flavor symmetry
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- we know (a part of) spectrum of chiral operators

[Gaiotto-Maldacena, Gadde-Lastelli-Razamat-Yan, KM-Tachikawa-Yonekura-Yan]
$T_N$ theory

$T_N$ theory is a 4d $N=2$ SCFT

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- we know (a part of) spectrum of chiral operators
  [Gaiotto-Maldacena, Gadde-Lastelli-Razamat-Yan, KM-Tachikawa-Yonekura-Yan]

- we do not know Lagrangian except for the $N=2$ case
Asymptotically free theory

The $N=2$ theory with asymptotically-free gauge group is obtained by considering **irregular puncture**

[Gaiotto-Moore-Neitzke]
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For the $N=2$ case,

$$\phi_2(t) \sim \frac{c_a}{(t - t_a)^n} \quad (n>2)$$
Asymptotically free theory

The $N=2$ theory with asymptotically-free gauge group is obtained by considering irregular puncture $[\text{Gaiotto-Moore-Neitzke}]$

For the $N=2$ case,

$$\phi_2(t) \sim \frac{c_a}{(t - t_a)^n} \quad (n>2)$$

$n=3$; trivial,
$n=4$; free hypermultiplets in the doublet of SU(2)
$n>4$; nontrivial SCFTs of Argyres-Douglas type $[\text{Cecotti-Neitzke-Vafa, Cecotti-Vafa, Bonelli-KM-Tanzini}]$
As an example, let us consider $N=2$ SU(2) SYM theory. The Seiberg-Witten curve is

\[ x^2 = \phi_2, \quad \phi_2 = \frac{\Lambda^2}{t} + \frac{u}{t^2} + \frac{\Lambda^2}{t^3} \]
As an example, let us consider $N=2$ SU(2) SYM theory. The Seiberg-Witten curve is

$$x^2 = \phi_2, \quad \phi_2 = \frac{\Lambda^2}{t} + \frac{u}{t^2} + \frac{\Lambda^2}{t^3}$$

This has punctures at $t=0$ and $\infty$ with \textit{irregular} behavior:

$$\phi_2 \sim \frac{1}{t^3}$$
$N=1$ theories in confining phase
$N=1$ deformation

We want to consider $N=1$ deformations of class S theories by adding the adjoint chiral mass terms

$$
\sum_i \mu_i \text{Tr} \phi_i^2
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$$
\sum_i \mu_i \text{Tr} \phi_i^2
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In introduction, we saw that $N=1$ deformation corresponds to the rotation of an NS5-brane:

\[ v = x_4 + ix_5 \]
\[ w = x_8 + ix_9 \]
\( N=1 \) deformation

= rotation of puncture

to w-direction
$N=1$ deformation

= rotation of puncture to w-direction

\[ x^N + \sum_k \phi_k(t) x^{N-k} = 0 \]
$N=1$ deformation

= rotation of puncture to w-direction

\[ x^N + \sum_k \phi_k(t) x^{N-k} = 0 \]

\[ w^N + \sum_k V_k(t) w^{N-k} = 0 \]
\( N=1 \) deformation

\[ = \]

rotation of puncture to \( w \)-direction

\[ x^N + \sum_k \phi_k(t)x^{N-k} = 0 \]

\[ w^N + \sum_k V_k(t)w^{N-k} = 0 \]

At punctures, \( V_k \) has singularity determined by \( w \sim \mu_i xt \)
Proposal

A generalization of the Hitchin system with two commuting $\text{su}(N)$-valued fields $\Phi$ and $\varphi$. 
A generalization of the Hitchin system with two commuting $\text{su}(N)$-valued fields $\Phi$ and $\varphi$.

The spectral curve consists of

\[
\begin{align*}
\det(x \cdot 1 - \Phi) &= 0 \\
\det(w \cdot 1 - \varphi) &= 0 \\
\det(xw \cdot 1 - \Phi \varphi) &= 0
\end{align*}
\]

where $\Phi$ and $\varphi$ have prescribed singularities at the punctures of the Riemann surface.
Let us consider the case with two M5-branes wrapped on $C$. The curve is simply

$$x^2 = \phi_2(t),$$  \quad \phi_2: \text{meromorphic differential} $$

$$w^2 = V_2(t),$$  \quad V_2: \text{meromorphic function}$$
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Since \( \Phi \) and \( \varphi \) commute, they are proportional to each other:

\[ \Phi = F(t)\varphi \]
Rank one case

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Since \( \Phi \) and \( \varphi \) commute, they are proportional to each other: \( \Phi = F(t)\varphi \)

\[ \rightarrow \quad x^2 = F(t)^2 w^2 \]
SU(2) SYM theory

Let us consider the $N=1$ deformation of $N=2$ SU(2) SYM theory $(V, \phi)$ by

$$W = \mu \text{Tr} \phi^2$$
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$$W = \mu \text{Tr} \phi^2$$

At energy below the mass scale $|\mathbf{\mu}|$, the theory is $N=1$ pure SYM theory describing gluino condensation in the IR

$$\langle \lambda_\alpha \lambda^\alpha \rangle = \Lambda_{N=1}^3$$
SU(2) SYM theory

The Seiberg-Witten curve was \((v = xt)\)

\[ v^2 = \frac{\Lambda^2}{t} + u + \Lambda^2 t \]
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The boundary condition at \(t=0\) gives

\[ w^2 \sim \mu^2 v^2 \sim \frac{\mu^2 \Lambda^2}{t} \]

No other singularity of \((\text{meromorphic function}) V_2\)

\[ w^2 = \frac{\mu^2 \Lambda^2}{t} + a \quad \text{a: unknown constant} \]
SU(2) SYM theory

The condition from the commuting fields reads

\[ \frac{w^2}{v^2} = F^2(t) = \frac{at + (\mu \Lambda)^2}{\Lambda^2 t^2 + ut + \Lambda^2} \]
SU(2) SYM theory

The condition from the commuting fields reads

\[ \frac{w^2}{v^2} = F^2(t) = \frac{at + (\mu \Lambda)^2}{\Lambda^2 t^2 + ut + \Lambda^2} \]

This gives \( a = 0 \) and \( u = \pm 2\Lambda^2 \)

Namely, the \textbf{N=1 curve} is simply

\[ w^2 = \frac{\mu^2 \Lambda^2}{t} \]
SU(2) SYM theory

These are indeed right values as follows:

1. $u = \pm 2\Lambda^2$ are the loci on the Coulomb branch where the massless monopole or dyon appears.

By the mass deformation, the supersymmetric vacua are only these points.

[Seiberg-Witten]
SU(2) SYM theory

by eliminating $t$, we get from the two equations:

$$w^2 - W'(v)w \pm \mu^2 \Lambda^2 = 0 \quad (W'(v) = \mu v)$$
SU(2) SYM theory

by eliminating $t$, we get from the two equations:

$$w^2 - W'(v)w \pm \mu^2 \Lambda^2 = 0 \quad (W'(v) = \mu v)$$

This is the curve obtained from the *matrix model* [Dijkgraaf-Vafa 2002], or from the *Konishi anomaly equation* [Cachazo-Douglas-Seiberg-Witten]

$$w^2 - W'(v)w + \mu S = 0$$

with $S \equiv \langle \lambda_\alpha \lambda^\alpha \rangle = \pm \mu \Lambda^2 = \pm \Lambda_N^{3N=1}$
In summary,

\[ t=\infty \quad t=0 \]
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Seiberg-Witten curve

\[ x^2 = \phi_2(t), \]

the quadratic differential \( \phi_2 \) is singular at \( t=0 \) and \( \infty \)
In summary,

The quadratic differential $\phi_2$ is singular at $t=0$ and $\infty$.

The meromorphic function $V_2$ is singular only at $t=0$.

Seiberg-Witten curve

$$x^2 = \phi_2(t),$$

the quadratic differential $\phi_2$ is singular at $t=0$ and $\infty$.

$N=1$ curve

$$w^2 = V_2(t),$$

the meromorphic function $V_2$ is singular only at $t=0$. 
Application to other cases

This method can be applied to other SU(2) gauge theories, e.g. SU(2) \times SU(2) quiver gauge theory etc.
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This method can be applied to other $SU(2)$ gauge theories, e.g. $SU(2) \times SU(2)$ quiver gauge theory etc.

For higher rank theory (with $SU(N)$ gauge group), there is no systematic way to solve the model, because there is no easy expression denoting the two commuting fields.
Application to other cases

This method can be applied to other SU(2) gauge theories, e.g. SU(2)xSU(2) quiver gauge theory etc.

For higher rank theory (with SU(N) gauge group), there is no systematic way to solve the model, because there is no easy expression denoting the two commuting fields.

But still we can solve case by case, e.g. SU(N) SYM theory, the \( T_N \) theory coupled to SU(N) gauge group etc.
Superconformal phase
and $N=1$ dualities
Global symmetry: $\text{SU}(N)^3 \times \text{U}(1)_R \times \text{U}(1)_J$

\[ R = R_{N=2}/2 + I_3, \quad J = R_{N=2} - 2 I_3, \quad I_3 \subset \text{SU}(2) \]
**T\(_N\) theory**

Global symmetry: SU(\(N\))^3 \times U(1)_R \times U(1)_J

\[ R = R_{N=2}/2 + l_3, \quad J = R_{N=2} - 2 l_3, \quad l_3 \subset SU(2) \]

Chiral primary operators:

<table>
<thead>
<tr>
<th></th>
<th>SU((N))(_A)</th>
<th>SU((N))(_B)</th>
<th>SU((N))(_C)</th>
<th>U(1)_R</th>
<th>U(1)_J</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\mu_A)</td>
<td>adj</td>
<td></td>
<td></td>
<td>1</td>
<td>-2</td>
</tr>
<tr>
<td>(\mu_B)</td>
<td></td>
<td>adj</td>
<td></td>
<td>1</td>
<td>-2</td>
</tr>
<tr>
<td>(\mu_C)</td>
<td></td>
<td></td>
<td>adj</td>
<td>1</td>
<td>-2</td>
</tr>
</tbody>
</table>

\[ \text{tr} \mu^k_A = \text{tr} \mu^k_B = \text{tr} \mu^k_C \]
$T_N$ theories coupled to $N=1$ vector multiplet

Let us denote the $T_N$ theory as
$T_N$ theories coupled to $N=1$ vector multiplet

Let us denote the $T_N$ theory as

We couple a pair of the $T_N$ theories to $N=1$ vector multiplet:

$\mathcal{F} = J_1 - J_2$
$N=1$ duality

\[ W = \text{tr} \Phi (\mu - \mu') \]

\[ \tau \leftrightarrow -1/\tau \]
$N=1$ duality

\[ W = \text{tr} \Phi (\mu - \mu') + m \text{tr} \Phi^2 \]

\[ W = c(2 \text{tr} \mu \mu' - \text{tr} \mu^2 - \text{tr} \mu'^2) \]

\[ c \sim 1/m \]
**N=1 duality**

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\[ = 2 \hat{c} \text{tr} \hat{\mu} \hat{\mu}' + \frac{1}{4 \hat{c}} \text{tr} M_B^2 + \text{tr} M_B \hat{\mu}_B \]

\[ + \frac{1}{4 \hat{c}} \text{tr} M_C^2 + \text{tr} M_C \hat{\mu}_C \]

\[ (\text{tr} \hat{\mu}^2 = \text{tr} \hat{\mu}_B^2, \quad \text{tr} \hat{\mu}'^2 = \text{tr} \hat{\mu}_C^2) \]
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independent
exactly marginal
couplings

S-dual
\( \tau \leftrightarrow -1/\tau \)

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\[ N=1 \text{ duality} \]

\[ W = \text{tr} \Phi (\mu - \mu') + m \text{tr} \Phi^2 \]

\[ W = c (2 \text{tr} \mu \mu' - \text{tr} \mu^2 - \text{tr} \mu'^2) \]

\[ c \sim 1/m \]

\[ \text{independent} \]
\[ \text{exactly marginal} \]
\[ \text{couplings} \]

\[ \hat{c} \sim 1/c \]

\[ W = \hat{c} \text{tr} (2 \hat{\mu} \hat{\mu}' - \hat{\mu}^2 - \hat{\mu}'^2) \]
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\[ W = c \, \text{tr} \mu \mu' \]

\[ \mu_{A,D} = M_{A,D} \]

\[ M_A \rightarrow \]

\[ W = \frac{1}{c} \text{tr} \hat{\mu} \hat{\mu}' + \text{tr} M_B \hat{\mu}_B + \text{tr} M_C \hat{\mu}_C \]

\[ \mu_{B,C} = M_{B,C} \]
\[ W = c \, \text{tr} \mu \mu' \]

\[ \mu_{A,D} = M_{A,D} \]

\[ M_A - N \]

\[ M_A = \underbrace{N} \]

\[ W = \frac{1}{c} \text{tr} \hat{\mu} \hat{\mu} + \ldots \]

\[ \mu_{X} = M_{X} \]

\[ M_B - N \]

\[ M_B = \underbrace{N} \]

\[ W = \frac{1}{c} \text{tr} \hat{\mu} \hat{\mu}' + \text{tr} M_B \hat{\mu}_B + \text{tr} M_C \hat{\mu}_C \]

\[ \mu_{B,C} = M_{B,C} \]

\[ M_C - N \]

\[ M_C = \underbrace{N} \]

\[ W = c \, \text{tr} \hat{\mu} \hat{\mu}' + \sum_{X=A,B,C,D} \text{tr} M_X \hat{\mu}_X \]

\[ M_D - N \]

\[ M_D = \underbrace{N} \]
‘colored’ puncture
‘colored’ puncture
‘colored’ puncture
M-theory interpretation?

The $N=1$ dualities may be able to understand as a symmetry of colored-punctured Riemann surface.

no puncture case [Bah-Wecht, Bah-Beem-Bobev-Wecht]
M-theory interpretation?

The $N=1$ dualities may be able to understand as a symmetry of *colored*-punctured Riemann surface.  

![The derivation of the adjoint fields from the M-theory point of view is lacking.](image)

cf. generalized Hitchin system viewpoint [Xie]

but the derivation of the adjoint fields from the M-theory point of view is lacking.
M-theory interpretation?

The $N=1$ dualities may be able to understand as a symmetry of colored-punctured Riemann surface. But the derivation of the adjoint fields from the M-theory point of view is lacking. cf. generalized Hitchin system viewpoint [Xie]

But **the derivation of the adjoint fields** from the M-theory point of view is lacking. cf. generalized Hitchin system viewpoint [Xie]

Furthermore, the meaning of the Riemann surface is not so clear in the $N=1$ set-up, compared to the $N=2$ one: just can be used to read off the UV $N=1$ theory.
We give a nilpotent vev $\langle \mu \rangle = \rho(\sigma^+)$, where $\rho$ specifies the embedding $\rho: SU(2) \rightarrow SU(N)$ and characterized by a partition $\Lambda$.

Focus on the embedding corresponding to $\Lambda=(N-1,1)$:

```
T_N
```

free bifundamental hypermultiplet
Seiberg duality
Seiberg duality

SQCD with $N_f = 2N$

Higgsing by $\langle \mu_{A,D} \rangle = \rho(\sigma^+)$

$M_B \rightarrow M_C$
Seiberg duality

SQCD with $N_f = 2N$

Higgsing by $\langle \mu_{A,D} \rangle = \rho(\sigma^+)$
In the Seiberg dual, there are \((2N)^2\) mesons. The above superpotential is mass terms of \(2(N^2 + 1)\) mesons.
New duality of SQCD
New duality of SQCD

\[ \langle \mu_{A,D} \rangle = \rho(\sigma^+) \]

\[ \langle M_{A,D} \rangle = \rho(\sigma^+) \]

SQCD with \( N_f = 2N \)

non-conventional theory
Conclusion

We have considered $N=1$ dynamics of gauge theories.

✧ generalized Hitchin system describes the dynamics of $N=1$ theories in confining phase.
✧ dualities of $N=1$ theories via $N=2$ S-dualities

Future directions

✪ Higher rank theory in confining phase
✪ M-theoretical interpretation of $N=1$ dualities
✪ Duality of asymptotically free theories
✪ Other phases: $N=1$ Coulomb phase from M-theory
✫ Some relation with 2d theory on C???
Thank you very much for your attention!