# ABJM行列模型から位相的弦理論へ

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[Hatsuda+M+Okuyama 1207, 1211, 1301]

[HMO+Marino 1306] [HMO+Honda 1306]

# Lessons from '90s

- String Theory As Unified Theory
- String Theory, NOT JUST "A Theory Of Strings"

  $\Rightarrow$ Only Sensible After NonPerturbative Branes
- Various Perturbative String Theories

  $\Rightarrow$ Various Vacua Of "A Unique Theory"
- Further Unification By M-theory

# ABJM for NonPerturbative Strings

ABJM Theory [N=6 Super Chern-Simons Theory]

$\Updownarrow$

$N$ x M2-branes on $C^4/Z_k$ ($\rightarrow CP^3 x S^1$ as $k \rightarrow \infty$)

- To Understand Better

  M-Theory (NonPerturbative Strings)

- Roles of ABJM Theory?

# Yes

- Partition Function

  (Normalization = Free of Physical Poles)
- A Non-Trivial Cancellation Mechanism

  Strings & Membranes: Poles

  Strings + Membranes: Cancel

# Contents

# ABJM Matrix Model

- Partition Function of ABJM Theory
- Due to SUSY, Localized to Matrix Model

$N_1=N_2=N$

$$Z(N) = \frac{1}{N_1!N_2!}\int\prod_{i=1}^{N_1}\frac{d\mu_i}{2\pi}\prod_{k=1}^{N_2}\frac{d\nu_k}{2\pi}\,e^{-(\sum\mu_i^2-\sum\nu_k^2)/2g_s}$$

$$\times\frac{\prod_{i<j}\left(2\sinh\frac{\mu_i-\mu_j}{2}\right)^2\prod_{k<l}\left(2\sinh\frac{\nu_k-\nu_l}{2}\right)^2}{\prod_{i,k}\left(2\cosh\frac{\mu_i-\nu_k}{2}\right)^2}$$

$g_s=2\pi i/k$

# 't Hooft Expansion

- $N \to \infty$ Expansion with $\lambda = N/k$ Fixed
- Genus Expansion

$$F = N^2 F_0(\lambda) + N^0 F_1(\lambda) + N^{-2} F_2(\lambda) + N^{-4} F_3(\lambda) + \ldots$$

$$= N^2 [F_0^{\text{pert}}(\lambda) + \# \, e^{-2\pi\sqrt{2\lambda}} + \# \, e^{-4\pi\sqrt{2\lambda}} + \ldots]$$

$$+ N^0 [F_1^{\text{pert}}(\lambda) + \# \, e^{-2\pi\sqrt{2\lambda}} + \# \, e^{-4\pi\sqrt{2\lambda}} + \ldots]$$

$$+ \ldots \ldots$$

Sum Up To Airy Function!

[Fuji-Hirano-M]

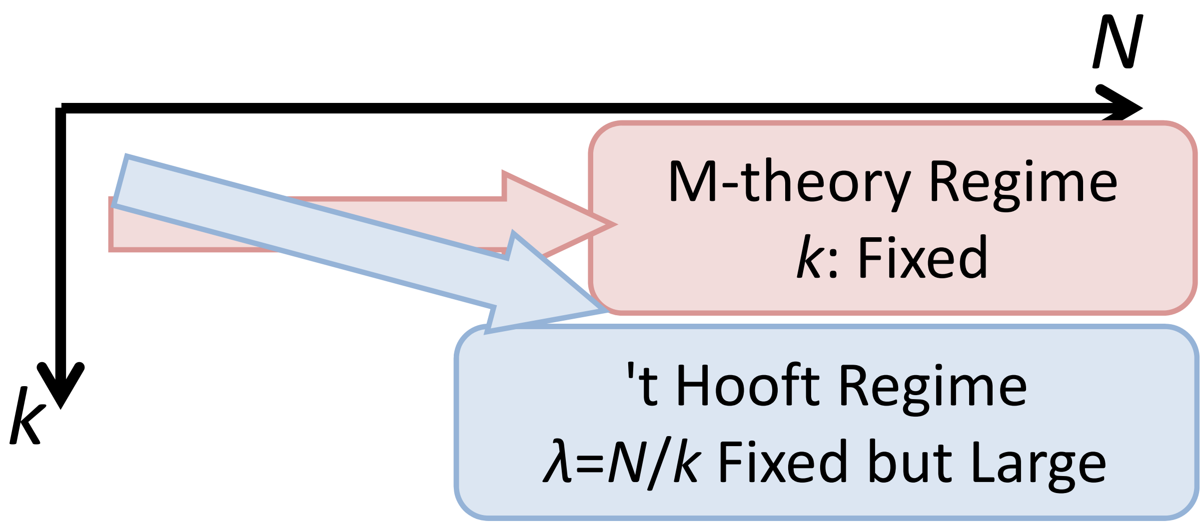
Worldsheet Instanton

$$e^{-2\pi\sqrt{2\lambda}} = \exp[- T_{F1} \, \text{Area}(\mathbf{CP^1})]$$

F-String Wrapping $\mathbf{CP}^1 \subset CP^3$

# M-theory Expansion

- M-theory Background: $C^4/Z_k$
- $N \to \infty$ with $k$ Fixed
- Close To 't Hooft Limit But Not Exactly



# Fermi Gas Formalism

[Marino-Putrov]

- Rewriting Partition Function
- Regarding as Fermi Gas with Density Matrix $\boldsymbol{\rho}$

$$Z(N) = (N!)^{-1} \sum_{\sigma \in S(N)} (-1)^{\sigma} \int \prod_i dq_i \langle q_i | \boldsymbol{\rho} | q_{\sigma(i)} \rangle$$

$$\boldsymbol{\rho} = (2 \cosh q/2)^{-1/2} (2 \cosh p/2)^{-1} (2 \cosh q/2)^{-1/2}$$

$$[p,q] = i\, 2\pi k$$

- Introducing Grand Potential

$$e^{J(\mu)} = \sum_{N=0}^{\infty} Z(N)\, e^{\mu N} = \det (1 + e^{\mu} \boldsymbol{\rho})$$

# WKB (small $k$) expansion

$$J(\mu) = k^{-1} J_0(\mu) + k^1 J_1(\mu) + k^3 J_2(\mu) + k^5 J_3(\mu) + \ldots$$

$$= k^{-1} [\, J_0^{\text{pert}}(\mu) + (\#\mu^2+\#\mu+\#)e^{-2\mu} + (\ldots)\, e^{-4\mu} + \ldots \,]$$

$$+\, k\, [\, J_1^{\text{pert}}(\mu) + (\#\mu^2+\#\mu+\#)e^{-2\mu} + (\ldots)\, e^{-4\mu} + \ldots \,]$$

$$+ \quad \ldots \ldots$$

$$= (\#\mu^3+\#\mu+\#) + (\#\mu^2+\#\mu+\#)e^{-2\mu} + (\ldots)e^{-4\mu} + \ldots$$

**Airy Function Reproduced**

[Marino-Putrov, Honda et al]

**Membrane Instanton**

$$e^{-2\mu} \approx e^{-\pi\sqrt{2Nk}} = \exp[-\, T_{D2}\, \text{Area}(\mathbf{RP}^3)]$$

D2-brane Wrapping $\mathbf{RP}^3 \subset CP^3$

[Drukker-Marino-Putrov]

# Summary

$$J(\mu) = J^{\mathrm{pert}}(\mu) + J^{\mathrm{WS}}(\mu) + J^{\mathrm{MB}}(\mu) + \ldots$$

$$J^{\mathrm{pert}}(\mu) = C\,\mu^3/3 + B\,\mu + A$$

$$J^{\mathrm{WS}}(\mu) = \sum_{m=1}^{\infty} d_k^{(m)}\, e^{-m \times 4\mu/k}$$

$$J^{\mathrm{MB}}(\mu) = \sum_{l=1}^{\infty} (a_k^{(l)}\,\mu^2 + b_k^{(l)}\,\mu + c_k^{(l)})\, e^{-l \times 2\mu}$$

$e^{-2\pi\sqrt{2\lambda}} \approx e^{-4\mu/k}$

- WS Instantons from Topological Strings

$$d_k^{(m)} = \sum_g \sum_{d|m} (-1)^{m/d}\, N^g_d/d\, (2\,\mathrm{Sin}[2\pi d/k])^{2g-2}$$

$N^g_d$: Gopakumar-Vafa Invariant on $F_0=P^1 \times P^1$

# Contents

# Partition Function $Z(N)$ with Fixed $k$

Example: (For $k$=1)

$Z(1) = 1/4$

$Z(2) = 1/16\pi$

$Z(3) = (\pi-3)/2^6\pi$

$Z(4) = (-\pi^2+10)/2^{10}\pi^2$

$Z(5) = (-9\pi^2+20\pi+26)/2^{12}\pi^2$

$Z(6) = (36\pi^3-121\pi^2+78)/2^{14}3^2\pi^3$

$Z(7) = (-75\pi^3+193\pi^2+174\pi-126)/2^{16}3\pi^3$

$Z(8) = (1053\pi^4-2016\pi^3-4148\pi^2+876)/2^{21}3^2\pi^4$

$Z(9) = (5517\pi^4-13480\pi^3-15348\pi^2+8880\pi+4140)/2^{23}3^2\pi^4$

# Exact Values From Numerical Studies

- Define Approximate Grand Potential

$$J(\mu) = \log[\ \textstyle\sum_{N=0}^{N\mathrm{max}} Z(N)\, e^{\mu N}\ ]$$

- Compare with Expectation

$$J(\mu)=(\#\mu^3+\#\mu+\#)+\textstyle\sum_{m=1}^{\infty}\# e^{-m\mathrm{x}4\mu/k}+\textstyle\sum_{l=1}^{\infty}(\#\mu^2+\#\mu+\#)e^{-l\mathrm{x}2\mu}$$

- Read off Exact Values of Coefficients

# Results (Schematically)

$J_{k=1}(\mu) = [\#\mu^2+\#\mu+\#]e^{-4\mu} + [\#\mu^2+\#\mu+\#]e^{-8\mu} + [\#\mu^2+\#\mu+\#]e^{-12\mu} + \ldots$

$J_{k=2}(\mu) = [\#\mu^2+\#\mu+\#]e^{-2\mu} + [\#\mu^2+\#\mu+\#]e^{-4\mu} + [\#\mu^2+\#\mu+\#]e^{-6\mu} + \ldots$

$J_{k=3}(\mu) = [\#]e^{-4\mu/3} + [\#]e^{-8\mu/3} + [\#\mu^2+\#\mu+\#]e^{-4\mu} + \ldots$

$J_{k=4}(\mu) = [\#]e^{-\mu} + [\#\mu^2+\#\mu+\#]e^{-2\mu} + [\#]e^{-3\mu} + \ldots$

...

$J_{k=6}(\mu) = [\#]e^{-2\mu/3} + [\#]e^{-4\mu/3} + [\#\mu^2+\#\mu+\#]e^{-2\mu} + \ldots$

WS(1) WS(2) WS(3)

# Comparison with WS Instanton (Top String)

$J_{k=1}(\mu) = [\#\mu^2+\#\mu+\#]e^{-4\mu} + [\#\mu^2+\#\mu+\#]e^{-8\mu} + [\#\mu^2+\#\mu+\#]e^{-12\mu} + \ldots$

$J_{k=2}(\mu) = [\#\mu^2+\#\mu+\#]e^{-2\mu} + [\#\mu^2+\#\mu+\#]e^{-4\mu} + [\#\mu^2+\#\mu+\#]e^{-6\mu} + \ldots$

$J_{k=3}(\mu) = [\#]e^{-4\mu/3} + [\#]e^{-8\mu/3} + [\#\mu^2+\#\mu+\#]e^{-4\mu} + \ldots$

$J_{k=4}(\mu) = [\#]e^{-\mu} + [\#\mu^2+\#\mu+\#]e^{-2\mu} + [\#]e^{-3\mu} + \ldots$

...

$J_{k=6}(\mu) = [\#]e^{-2\mu/3} + [\#]e^{-4\mu/3} + [\#\mu^2+\#\mu+\#]e^{-2\mu} + \ldots$

WS(1) WS(2) WS(3)

○ : Match ✕ : Divergent △ : Not-Match

# Divergences Cancelled by MB Instanton

$$J_{k=1}(\mu) = [\#\mu^2+\#\mu+\#]e^{-4\mu} + [\#\mu^2+\#\mu+\#]e^{-8\mu} + [\#\mu^2+\#\mu+\#]e^{-12\mu} + \ldots$$

$$J_{k=2}(\mu) = [\#\mu^2+\#\mu+\#]e^{-2\mu} + [\#\mu^2+\#\mu+\#]e^{-4\mu} + [\#\mu^2+\#\mu+\#]e^{-6\mu} + \ldots$$

$$J_{k=3}(\mu) = [\#]e^{-4\mu/3} + [\#]e^{-8\mu/3} + [\#\mu^2+\#\mu+\#]e^{-4\mu} + \ldots$$

$$J_{k=4}(\mu) = [\#]e^{-\mu} + [\#\mu^2+\#\mu+\#]e^{-2\mu} + [\#]e^{-3\mu} + \ldots$$

$$\ldots$$

$$J_{k=6}(\mu) = [\#]e^{-2\mu/3} + [\#]e^{-4\mu/3} + [\#\mu^2+\#\mu+\#]e^{-2\mu} + \ldots$$

WS(1) WS(2) WS(3)

MB(2)

MB(1)

# 1-Membrane Instanton

- Vanishing in $k$=odd
- Canceling Divergence @ $k=2m$
- Matching the WKB data

⇩

$$a_k^{(1)} = -4(\pi^2 k)^{-1} \mathrm{Cos}[\pi k/2]$$
$$b_k^{(1)} = 2\pi^{-1} \mathrm{Cot}[\pi k/2]\, \mathrm{Cos}[\pi k/2]$$
$$c_k^{(1)} = \ldots$$

# How About $\triangle$ ?

- ($l$, $m$) Bound State ?

$$e^{-(m \times 4\mu/k + l \times 2\mu)}$$

Ex: $e^{-3\mu}$ Effects in $k$=4 Sector From

**Both** (0,3) **&** (1,1)

- Contribution from Bound States

$$J(\mu) = J^{\text{pert}}(\mu) + J^{\text{MB}}(\mu) + J^{\text{WS}}(\mu) + J^{\text{bnd}}(\mu)$$

$$J^{\text{bnd}}(\mu) = \sum_{m=1}^{\infty} \sum_{l=1}^{\infty} \#\, e^{-(m \times 4\mu/k + l \times 2\mu)}$$

# Bound States

$$J(\mu) = J^{\text{pert}}(\mu) + J^{\text{MB}}(\mu) + J^{\text{WS}}(\mu) + J^{\text{bnd}}(\mu)$$

Bound States Incorporated by $\mu \to \mu_{\text{eff}}$

$$J^{\text{WS}}(\mu) + J^{\text{bnd}}(\mu) = J^{\text{WS}}(\mu_{\text{eff}})$$

$$\mu_{\text{eff}} = \mu + \sum_{l=1}^{\infty} a_k^{(l)}\, e^{-l \times 2\mu}$$

All in Effective Chemical Potential

$$J(\mu) = J^{\text{pert}}(\mu_{\text{eff}}) + J'^{\text{MB}}(\mu_{\text{eff}}) + J^{\text{WS}}(\mu_{\text{eff}})$$

$$J'^{\text{MB}}(\mu_{\text{eff}}) = \sum_{l=1}^{\infty} (\tilde{b}_k^{(l)}\, \mu_{\text{eff}} + \tilde{c}_k^{(l)})\, e^{-l \times 2\mu\text{eff}}$$

Membrane Instanton in Linear Functions

$\tilde{c}_k^{(l)} = k^2 \partial_k[\tilde{b}_k^{(l)}/\text{k}] \;\Rightarrow\;$ Only $a_k^{(l)}$ & $b_k^{(l)}$ NonTrivial

# Summary

- Exact Values From Numerical Study
- Poles Cancellation Requirement

$$\Downarrow$$

- Explicit First Few Membrane Instantons
- Bound States (Neither from Genus OR WKB)
- No Other Contributions

# Contents

# Refined Topological Strings

$a_k^{(l)}$ & $b_k^{(l)}$ ⇔ A- & B-Periods of Q Spectral Curve!

Classical Spectral Curve

$$-1 + e^{x} + e^{p} + z_1 e^{-x} + z_2 e^{-p} = 0$$

Quantum Spectral Curve

$$(-1+e^{x}+z_1 e^{-x})\Psi(x)+\Psi(x+\hbar)+z_2\Psi(x-\hbar)=0$$

Quantization of Spectral Curve ⇔ Refined Topological String in Nekrasov-Shatashivili Limit

[Mironov-Morozov, Aganagic-Cheng-Dijkgraaf-Krefl-Vafa]

# All Explicitly In Topological Strings

$$J(\mu)=J^{\mathrm{pert}}(\mu^{\mathrm{eff}})+J^{\mathrm{WS}}(\mu^{\mathrm{eff}})+J'^{\mathrm{MB}}(\mu^{\mathrm{eff}})$$

$$J^{\mathrm{pert}}(\mu)=C\mu^3/3+B\mu+A$$

$$J^{\mathrm{WS}}(\mu^{\mathrm{eff}})=F_{\mathrm{top}}(T_1^{\mathrm{eff}},T_2^{\mathrm{eff}},\lambda)$$

$$J'^{\mathrm{MB}}(\mu^{\mathrm{eff}})=(2\pi \mathrm{i})^{-1}\partial_\lambda[\lambda F_{\mathrm{NS}}(T_1^{\mathrm{eff}}/\lambda,T_2^{\mathrm{eff}}/\lambda,1/\lambda)]$$

$F(T_1,T_2,\tau_1,\tau_2)$: Free Energy of Refined Top Strings

$T_1,T_2$: Kahler Moduli

$\tau_1,\tau_2$: Coupling Constants

$T_1^{\mathrm{eff}}=4\mu^{\mathrm{eff}}/k-\mathrm{i}\pi$

$T_2^{\mathrm{eff}}=4\mu^{\mathrm{eff}}/k+\mathrm{i}\pi$

$\lambda=2/k$

Topological Limit $F_{\mathrm{top}}(T_1,T_2,\tau)=\lim_{\tau1\to\tau,\tau2\to-\tau} F(T_1,T_2,\tau_1,\tau_2)$

NS Limit $F_{\mathrm{NS}}(T_1,T_2,\tau)=\lim_{\tau1\to\tau,\tau2\to 0} 2\pi \mathrm{i}\tau_2 F(T_1,T_2,\tau_1,\tau_2)$

# Triple Sine Function

- Triple Sine Function

- Defined from Multiple Zeta Function

- All Poles Cancelled Formally

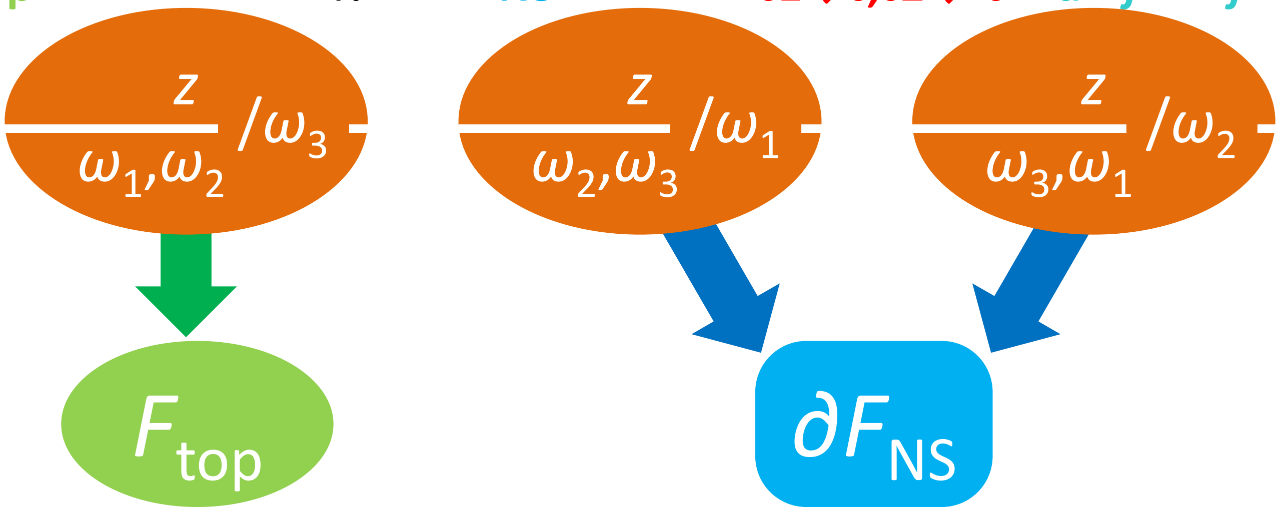
$$\log S_3(z;\boldsymbol{\omega}) = \frac{z}{\omega_1,\omega_2} / \omega_3 + \frac{z}{\omega_2,\omega_3} / \omega_1 + \frac{z}{\omega_3,\omega_1} / \omega_2$$

- Sign Modifications

- Pole Cancellation Still Holds!

# Relation to Triple Sine Function

$$F_{top} + (2\pi i)^{-1}\, \partial_\lambda [\, \lambda\, F_{NS}\, ] = \lim_{\tau 1 \to \tau, \tau 2 \to -\tau} \sum_d \sum_j N^d{}_j \log S_3$$




- Before The Limit? Ellipsoid $S^3$? Squashed $S^3$?
- Similar to M5 Partition Function? Duality???

# Contents

# Summary

- Instanton Bound States of WS & MB

Neither from Genus Exp OR WKB Exp

- An Explicit Formula for ABJM Matrix Model

Perturbative: Cubic Function

WS Instanton: Top String

MB Instanton: Refined Top String in NS Limit

- Partition Function (& Wilson Loop)

# Punch Lines

- M-theory / Nonperturbative Strings

  - String Theory, NOT JUST A Theory Of Strings

    Only Sensible After NonPerturbative Branes

  - ABJM Matrix Model

    Poles from WS & MB Cancel Only After Sum

- Cancellation Mechanism

  - Raison D'Etre for M-theory
  - First Few Membrane Instantons
  - Infinite Cancellation in General Formula

Thank you for your attention.