Twisted $\mathcal{N} = 4$ Super Yang-Mills Theory in Ω-background

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1. Introduction

Ω-background deformation [Moore-Nekrasov-Shatashvili]

$$\Omega\text{-background for } \mathcal{N}=2 \text{ theory} \quad (\leftarrow \text{6D } \mathcal{N}=(1,0) \text{ theory})$$

$$= \underbrace{\text{6D curved background}}_{\text{to introduce } \epsilon_1,\ \epsilon_2} + \underbrace{\text{Sp}(1) \text{ R-symmetry Wilson line}}_{\text{to preserve SUSY}}$$

- 6D Ω-background metric (on $\mathbb{R}^4 \times \mathbf{T}^2$)

$$ds^2 = 2d\bar{z}dz + (dx^\mu + \Omega^{\mu\nu}x_\nu d\bar{z} + \bar{\Omega}^{\mu\nu}x_\nu dz)^2.$$

Here constant antisymmetric matrix $\Omega^{\mu\nu}$ and $\bar{\Omega}^{\mu\nu}$ commute with each other.

Hence they can be taken of the form

$$\Omega^{\mu\nu} = \frac{1}{2\sqrt{2}} \begin{pmatrix} 0 & \epsilon_1 & 0 & 0 \\ -\epsilon_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\epsilon_2 \\ 0 & 0 & \epsilon_2 & 0 \end{pmatrix}, \quad \bar{\Omega}^{\mu\nu} = \frac{1}{2\sqrt{2}} \begin{pmatrix} 0 & \bar{\epsilon}_1 & 0 & 0 \\ -\bar{\epsilon}_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\bar{\epsilon}_2 \\ 0 & 0 & \bar{\epsilon}_2 & 0 \end{pmatrix}.$$

- $\bar{Q}_0 \to \bar{Q}$, $\quad \bar{Q}^2 =$ (gauge transf.) $+$ ($\mathrm{U}(1)^2$ rotation by ϵ_1,ϵ_2)
- $S_0 = \bar{Q}_0 \Xi_0 \to S = \bar{Q}\Xi = \bar{Q}(\Xi_0 + \cdots)$

equivariant localization $\Longrightarrow$ exact computation of path integral

- Nekrasov partition function

Generalization of Ω-background for $\mathcal{N}=4$ super Yang-Mills theory [Ito-H.N.-Saka-Sasaki], [Ito-H.N.-Sasaki 2012]

generalized Ω-background
$=$ 10D curved background $+$ SU(4) R-symmetry Wilson line

In 10D, SU(4) $=$ SO(6) R-symmetry is the subgroup of local Lorentz symmetry (No R-symmetry in 10D SYM).

$\Rightarrow$ contribution to spin (and Affine) connection ($\sim$ torsion)

$\Rightarrow$ deformation of parallel (or Killing) spinor equation

$$\nabla_{\mathcal{M}}\zeta = 0 \quad \longrightarrow \quad \widehat{\nabla}_{\mathcal{M}}\zeta = \left(\nabla_{\mathcal{M}} + \frac{1}{4}K_{\mathcal{M},NP}\Gamma^{NP}\right)\zeta = 0.$$

Classification of parallel spinors and supersymmetry in the theory

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4. Summary

2. Ω-background and SUSY condition

Ω-background for $\mathcal{N}=2$ theory
$=$ 6D curved background $+$ SU(2) R-symmetry Wilson line

generalized Ω-background
$=$ 10D curved background $+$ SU(4) R-symmetry Wilson line
(con)torsion

- 10D metric for generalized Ω-background (on $\mathbb{R}^4 \times \mathbb{T}^6$)

$$ds^2_{10\mathrm{D}} = (dx^\mu + \Omega^\mu_a dx^a)^2 + dx^a dx^a, \quad \Omega^\mu_a = \Omega^\mu{}_{\nu a} x^\nu.$$

Here $\Omega_{\mu\nu a} = -\Omega_{\nu\mu a}$ are constant and commute with each other.
(μ, ν, . . . : 4D indices, a, b, . . . : 6D indices)

$$\text{commuting } \Omega_{\mu\nu a} \;\longrightarrow\; \Omega_{\mu\nu a} = \frac{1}{2\sqrt{2}} \begin{pmatrix} 0 & \epsilon_{1a} & 0 & 0 \\ -\epsilon_{1a} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\epsilon_{2a} \\ 0 & 0 & \epsilon_{2a} & 0 \end{pmatrix}.$$

R-symmetry Wilson line: $\mathcal{A}_{bc} = \mathcal{A}_{a,bc}\, dx^a$, $\mathcal{A}_{a,bc}$: constant

Action

$$S_\Omega = \int d^4x \, \frac{1}{\kappa g^2} \mathrm{Tr}\Big[\frac{1}{4} F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}(D_\mu \varphi_a - F_{\mu\nu}\Omega_a^\nu)^2 + \frac{1}{4}\Big[[\varphi_a, \varphi_b] + i\Omega_a^\mu D_\mu \varphi_b - i\Omega_b^\mu D_\mu \varphi_a - i\Omega_a^\mu \Omega_b^\nu F_{\mu\nu} - i\big({\mathcal{A}_{a,b}}^c - {\mathcal{A}_{b,a}}^c\big)\varphi_c\Big]^2 + (\text{fermions})\Big].$$

The original Ω-background is reproduced by

$$\Omega_{\mu\nu} = \frac{1}{\sqrt{2}}\big(\Omega_{\mu\nu 1} - i\Omega_{\mu\nu 2}\big), \quad \bar{\Omega}_{\mu\nu} = \frac{1}{\sqrt{2}}\big(\Omega_{\mu\nu 1} + i\Omega_{\mu\nu 2}\big),$$

$$\Omega_{\mu\nu 3} = \cdots = \Omega_{\mu\nu 6} = 0,$$

SU(4) R-symmetry Wilson line $\longrightarrow$ SU(2) R-symmetry Wilson line.

- parallel spinor condition and topological twist

$$\partial_\mu \zeta = 0, \quad \partial_a \zeta + \frac{1}{4}\big(\Omega_{\mu\nu a}\Gamma^{\mu\nu} + K_{a,bc}\Gamma^{bc}\big)\zeta = 0.$$

$\Longrightarrow$ 4D and 6D rotation should be canceled.

$$\Omega_{\mu\nu a} \in \mathsf{U}(1)_L \times \mathsf{U}(1)_R \subset \mathsf{SU}(2)_L \times \mathsf{SU}(2)_R.$$

In order to cancel 4D rotation, we restrict the contorsion s.t.

$$\mathcal{A}_{a,bc} \in \mathsf{U}(1)_{L'} \times \mathsf{U}(1)_{R'} \subset \mathsf{SU}(2)_{L'} \times \mathsf{SU}(2)_{R'} \subset \mathsf{SU}(4).$$

	$U(1)_L$	$U(1)_R$	$U(1)_{L'}$	$U(1)_{R'}$
$\zeta_\alpha^{A'}$	$\pm\frac{1}{2}$	0	0	$\pm\frac{1}{2}$
$\zeta_\alpha^{\hat{A}}$	$\pm\frac{1}{2}$	0	$\pm\frac{1}{2}$	0
$\bar{\zeta}_{A'}^{\dot{\alpha}}$	0	$\pm\frac{1}{2}$	0	$\pm\frac{1}{2}$
$\bar{\zeta}_{\hat{A}}^{\dot{\alpha}}$	0	$\pm\frac{1}{2}$	$\pm\frac{1}{2}$	0

cancellation of U(1) charges

$\rightarrow$ identification of SU(2)'s

$\rightarrow$ topological twist

SUSY condition can be solved for each topological twist.
(We assume generic ϵ_1, ϵ_2.)

- half twist $(\mathrm{SU}(2)_R \sim \mathrm{SU}(2)_{R'})$ $\Rightarrow$ $\mathcal{N}=2^*$ theory

  parameters : $\epsilon_1,\ \epsilon_2,\ \bar{\epsilon}_1,\ \bar{\epsilon}_2,\ m,\ \bar{m}$ (m, $\bar{m}$: $\mathcal{N}=2^*$ mass)

  one scalar and one tensor supercharges are preserved.

- Vafa-Witten twist $(\mathrm{SU}(2)_R \sim \mathrm{diag}(\mathrm{SU}(2)_{L'} \times \mathrm{SU}(2)_{R'}))$

  parameters : $\epsilon_1,\ \epsilon_2,\ \bar{\epsilon}_1,\ \bar{\epsilon}_2,\ \hat{\epsilon}_1,\ \hat{\epsilon}_2,\ \check{\epsilon}_1,\ \check{\epsilon}_2$

  two scalar and two tensor supercharges are preserved.

- Marcus twist $(\mathrm{SU}(2)_L \sim \mathrm{SU}(2)_{L'},\ \mathrm{SU}(2)_R \sim \mathrm{SU}(2)_{R'})$

  parameters : $\epsilon_1,\ \epsilon_2,\ \bar{\epsilon}_1,\ \bar{\epsilon}_2$ (special case of half twist)

  two scalar and two tensor supercharges are preserved.

3. Off-shell scalar supersymmetry

(a) half twist

$$A_\mu \to A_\mu, \qquad \varphi_a \to (\varphi, \bar{\varphi}, \varphi^{\dot{\alpha}\hat{A}}),$$
$$\Lambda_\alpha^A \to (\Lambda_\mu, \Lambda_\alpha^{\hat{A}}), \qquad \bar{\Lambda}_A^{\dot{\alpha}} \to (\bar{\Lambda}, \bar{\Lambda}_{\mu\nu}^-, \bar{\Lambda}_{\hat{A}}^{\dot{\alpha}})$$

$\bar{\Lambda}_{\mu\nu}^-$ and $\Lambda_\alpha^{\hat{A}}$ need auxiliary fields for off-shell closure.

auxiliary fields

$$\bar{Q}\bar{\Lambda}_{\mu\nu} = -2F_{\mu\nu}^- - i(\bar{\sigma}_{\mu\nu})^{\dot{\beta}}{}_{\dot{\alpha}}[\varphi^{\dot{\alpha}\hat{A}}, \bar{\varphi}_{\hat{A}\dot{\beta}}] + 2D_{\mu\nu},$$
$$\bar{Q}\Lambda_\alpha^{\hat{A}} = \sqrt{2}(\sigma^\mu)_{\alpha\dot{\alpha}} D_\mu \varphi^{\dot{\alpha}\hat{A}} + 2K_\alpha^{\hat{A}}.$$

Algebra of scalar supercharges

$$\bar{Q}^2 = 2\sqrt{2}\big(\,\delta_{\rm gauge}(\varphi) + \delta_{\rm Lorentz}(\epsilon_1,\,\epsilon_2) + \delta_{\rm flavor}(m)\,\big),$$

$\delta_{\rm gauge}(\varphi)$: gauge transformation by φ.

$\delta_{\rm Lorentz}(\epsilon_1,\,\epsilon_2)$: $\mathsf{U}(1)_L \times \mathsf{U}(1)_{R+R'}$ rotation by parameter $\epsilon_1,\,\epsilon_2$.

$\delta_{\rm flavor}(m)$: $\mathsf{U}(1)_{L'}$ rotation by parameter m.

The action S is written as $\bar{Q}$-exact form up to the topological term:

$$S = \bar{Q}\Xi + \int d^4x \frac{1}{\kappa g^2}\mathrm{Tr}\left[\frac{1}{4}F_{\mu\nu}\tilde{F}^{\mu\nu}\right].$$

Here

$$\Xi = \int d^4x \frac{1}{\kappa g^2} \mathrm{Tr} \Big[-\frac{1}{2} F^-_{\mu\nu} \bar{\Lambda}^{\mu\nu} - \frac{1}{4} H_{\mu\nu} \bar{\Lambda}^{\mu\nu} - \frac{1}{2\sqrt{2}} \Lambda^\mu \left(D_\mu \bar{\varphi} - F_{\mu\nu} \bar{\Omega}^\nu \right)$$

$$+ \frac{i}{4} \bar{\Lambda} \left([\varphi, \bar{\varphi}] + i \Omega^\mu D_\mu \bar{\varphi} - i \bar{\Omega}^\mu D_\mu \varphi + i \bar{\Omega}^\mu \Omega^\nu F_{\mu\nu} \right)$$

$$- \frac{i}{2} \bar{\Lambda}^{\dot{\alpha}\hat{A}} \left([\bar{\varphi}, \bar{\varphi}_{\hat{A}\dot{\alpha}}] + i \bar{\Omega}^\mu D_\mu \bar{\varphi}_{\hat{A}\dot{\alpha}} - \frac{i}{2} \bar{\Omega}_{\mu\nu} (\bar{\sigma}^{\mu\nu})^{\dot{\beta}}{}_{\dot{\alpha}} \bar{\varphi}_{\hat{A}\dot{\beta}} + \bar{M}^{\hat{B}}{}_{\hat{A}} \bar{\varphi}_{\hat{B}\dot{\alpha}} \right)$$

$$+ \frac{1}{2} \Lambda^\alpha_{\hat{A}} G^{\hat{A}}_\alpha - \frac{1}{\sqrt{2}} \Lambda^\alpha_{\hat{A}} (\sigma^\mu)_{\alpha\dot{\alpha}} D_\mu \varphi^{\dot{\alpha}\hat{A}} - \frac{i}{4} \bar{\Lambda}^{\mu\nu} (\bar{\sigma}_{\mu\nu})^{\dot{\beta}}{}_{\dot{\alpha}} [\varphi^{\dot{\alpha}\hat{A}}, \bar{\varphi}_{\hat{A}\dot{\beta}}] \Big],$$

$$H_{\mu\nu} = D_{\mu\nu} - F^-_{\mu\nu} - \frac{i}{2} (\bar{\sigma}_{\mu\nu})^{\dot{\beta}}{}_{\dot{\alpha}} [\varphi^{\dot{\alpha}\hat{A}}, \bar{\varphi}_{\hat{A}\dot{\beta}}], \quad G^{\hat{A}}_\alpha = K^{\hat{A}}_\alpha + \frac{1}{\sqrt{2}} (\sigma^\mu)_{\alpha\dot{\alpha}} D_\mu \varphi^{\dot{\alpha}\hat{A}},$$

$$\bar{M}^{\hat{B}}{}_{\hat{A}} = \begin{pmatrix} \bar{m} & 0 \\ 0 & -\bar{m} \end{pmatrix}.$$

(b) Vafa-Witten twist [Vafa-Witten]

$$A_\mu \to A_\mu, \qquad \varphi_a \to (\varphi, \bar{\varphi}, \hat{\varphi}, \hat{\varphi}^-_{\mu\nu}),$$

$$\Lambda^A_\alpha \to (\Lambda_\mu, \hat{\Lambda}_\mu), \qquad \bar{\Lambda}^{\dot{\alpha}}_A \to (\bar{\Lambda}, \bar{\Lambda}^-_{\mu\nu}, \hat{\bar{\Lambda}}, \hat{\bar{\Lambda}}^-_{\mu\nu})$$

Λ_μ, $\hat{\Lambda}_\mu$, $\bar{\Lambda}^-_{\mu\nu}$ and $\hat{\bar{\Lambda}}^-_{\mu\nu}$ need auxiliary fields for off-shell closure.

$\Longrightarrow$ auxiliary fields: K_μ, $D^-_{\mu\nu}$

Algebra of scalar supercharges

$$\bar{Q}^2 = 2\sqrt{2}\big(\, \delta_{\mathrm{gauge}}(\varphi) + \delta_{\mathrm{Lorentz}}(\epsilon_1,\ \epsilon_2)\,\big),$$

$$\hat{\bar{Q}}^2 = -2\sqrt{2}\big(\, \delta_{\mathrm{gauge}}(\bar{\varphi}) + \delta_{\mathrm{Lorentz}}(\bar{\epsilon}_1,\ \bar{\epsilon}_2)\,\big),$$

$$\{\bar{Q}, \hat{\bar{Q}}\} = 2\sqrt{2}\big(\, \delta_{\mathrm{gauge}}(\hat{\varphi}) + \delta_{\mathrm{Lorentz}}(\hat{\epsilon}_1,\ \hat{\epsilon}_2)\,\big).$$

The action is written in the exact form with respect to the two scalar supercharges simultaneously:

$$S = \bar{Q}\hat{\bar{Q}}\mathcal{F} + \int d^4x \, \frac{1}{\kappa g^2} \mathrm{Tr}\left[\frac{1}{4} F_{\mu\nu} \tilde{F}^{\mu\nu}\right].$$

Here

$$\begin{aligned}\mathcal{F} = \int d^4x \, \frac{1}{\kappa g^2} \mathrm{Tr}\Big[& -\frac{1}{2\sqrt{2}} \hat{\varphi}^{\mu\nu} F^-_{\mu\nu} + \frac{1}{8} \bar{\Lambda}^{\mu\nu} \hat{\bar{\Lambda}}_{\mu\nu} + \frac{1}{8} \Lambda^\mu \Lambda_\mu - \frac{1}{8} \bar{\Lambda} \hat{\bar{\Lambda}} + \frac{i}{24\sqrt{2}} \hat{\varphi}^{\mu\nu} [\hat{\varphi}_\mu{}^\lambda, \hat{\varphi}_{\lambda\nu}] \\ & + \frac{1}{16\sqrt{2}} \hat{\varphi}^{\mu\nu} (\hat{\Omega}^{\rho,}{}_{\mu\sigma} D_\rho \hat{\varphi}_\nu{}^\sigma - \hat{\Omega}^{\rho,}{}_{\nu\sigma} D_\rho \hat{\varphi}_\mu{}^\sigma - i \hat{\Omega}_{\mu\nu,\rho\sigma} \hat{\varphi}^{\rho\sigma} + i \hat{\Omega}_{\rho\sigma,}{}^{\rho\sigma} \hat{\varphi}_{\mu\nu}) \\ & + \frac{3}{2\sqrt{2}} \hat{\Omega}^{[\rho,\mu\nu]} \left(A_{[\mu} F_{\nu\rho]} - \frac{i}{3} A_{[\mu} A_\nu A_{\rho]} \right) \Big],\end{aligned}$$

$$\hat{\Omega}^{\rho,}{}_{\mu\nu} = \hat{\Omega}^{\rho\sigma,}{}_{\mu\nu} \, x_\sigma, \quad \hat{\Omega}_{12,12} = -\hat{\Omega}_{12,34} = -\frac{1}{\sqrt{2}} \check{\epsilon}_1, \quad \hat{\Omega}_{34,12} = -\hat{\Omega}_{34,34} = \frac{1}{\sqrt{2}} \check{\epsilon}_2.$$

(c) Marcus twist (GL twist) [Marcus], [Kapustin-Witten]

$$A_\mu \to A_\mu, \qquad \varphi_a \to (\varphi, \bar{\varphi}, \varphi_\mu),$$
$$\Lambda_\alpha^A \to (\Lambda_\mu, \Lambda, \Lambda_{\mu\nu}^+), \qquad \bar{\Lambda}_A^{\dot{\alpha}} \to (\bar{\Lambda}, \bar{\Lambda}_{\mu\nu}^-, \bar{\Lambda}_\mu)$$

Λ, $\bar{\Lambda}$, $\Lambda_{\mu\nu}^+$ and $\bar{\Lambda}_{\mu\nu}^-$ need auxiliary fields for off-shell closure.

$\Longrightarrow$ auxiliary fields: K, $K_{\mu\nu}^+$, $D_{\mu\nu}^-$

On-shell algebra of scalar supercharges

$$Q^2 = \bar{Q}^2 = 2\sqrt{2}\big(\,\delta_{\text{gauge}}(\varphi) + \delta_{\text{Lorentz}}(\epsilon_1,\,\epsilon_2)\,\big),$$
$$\{Q, \bar{Q}\} = 0.$$

The first equation holds off-shell but the second does not hold off-shell on the fields $\Lambda_{\mu\nu}^+$ and $\bar{\Lambda}_{\mu\nu}^-$.

linear combination of scalar supercharges [Kapustin-Witten]

$$\mathcal{Q} = uQ + v\bar{Q}, \quad u, v \in \mathbb{C},$$

$$\mathcal{Q}^2 = 2\sqrt{2}(u^2 + v^2)\big(\,\delta_{\text{gauge}}(\varphi) + \delta_{\text{Lorentz}}(\epsilon_1,\, \epsilon_2)\,\big).$$

The above algebra holds off-shell.

$\mathcal{Q}$-exactness of the action

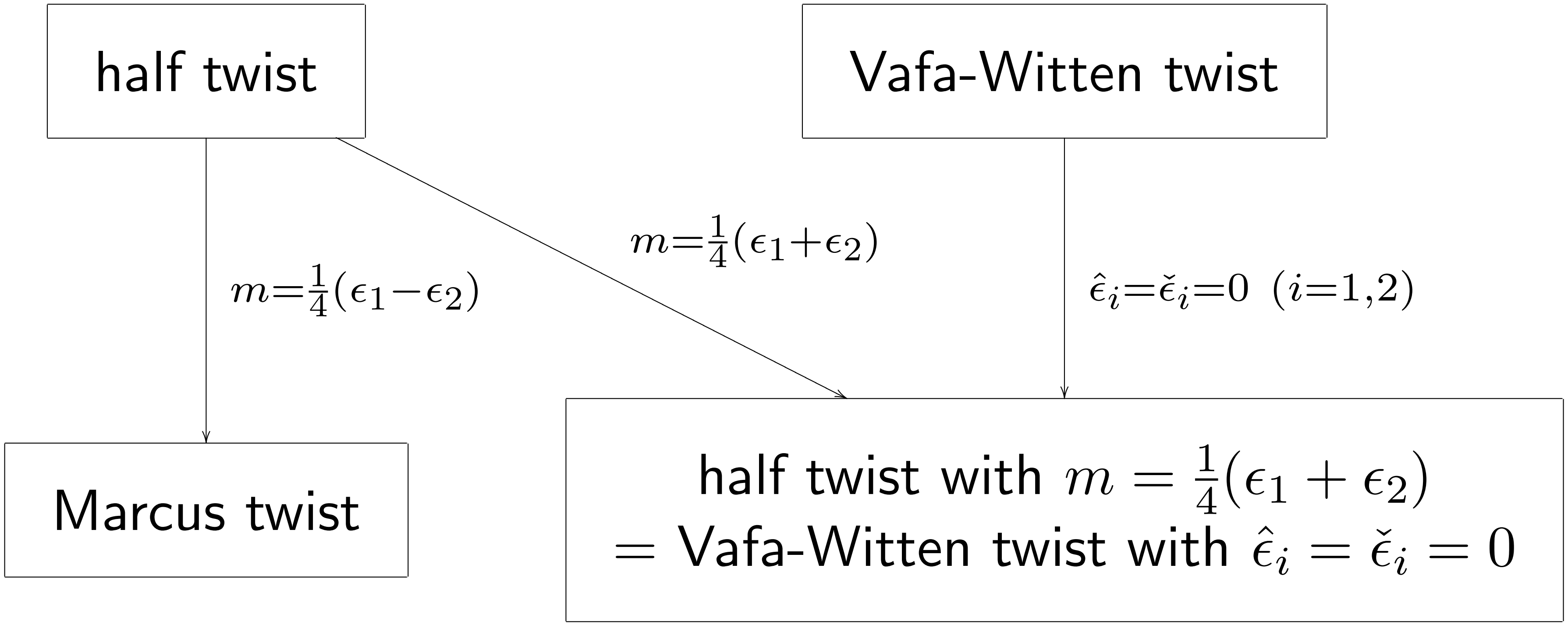
- $u^2 + v^2 \neq 0$ case

$$S = (\mathcal{Q}\text{-exact term}) + \int d^4x \, \frac{1}{\kappa g^2} \mathrm{Tr}\left[\frac{u^2 - v^2}{4(u^2 + v^2)} F_{\mu\nu}\tilde{F}^{\mu\nu}\right].$$

- $u^2 + v^2 = 0$ case

$$S = (\mathcal{Q}\text{-exact term}) + (\mathcal{Q}\text{-closed term, not } \mathcal{Q}\text{-exact}).$$

Relation among three topological twists



cf. [Pestun], [Pestun-Okuda], etc.

4. Summary

summary

1. We have constructed off-shell scalar supercharge(s) in Ω-deformed $\mathcal{N}=4$ super Yang-Mills.

2. The action is written as the exact form with respect to the supercharge(s) except the case of Marcus twist with $u^2+v^2=0$.

future work

- Nekrasov-Shatashvili limit

- extension to more complicated backgrounds
  [Festuccia-Seiberg], [Dumitrescu-Festuccia-Seiberg],
  [Hama-Hosomichi], [Klare-Zaffaroni] etc.

- auxiliary fields in 10D formalism [Berkovits], etc.

- embedding to superstring/SUGRA
  R-R 3-form (in instanton (D(-1)) effective action)
  [Ito-H.N.-Sasaki], [Ito-H.N.-Saka-Sasaki]

  (cf. [Hellerman-Orlando-Reffert], [Reffert], [Nakayama-Ooguri])