Brane solutions of Hopf soliton in seven dimensions

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I. Introduction

We consider a seven-dimensional brane world model. The brane is described by an localized solution to the extended Skyrme-Faddeev model embedding in the extra dimensions. The solutions are axially symmetric knotted solitons and have non-zero Hopf charge Q_H which is the product of two winding numbers m and n. We consider the case of $Q_H = 2$ ($m = 1$, $n = 2$) and numerically solve the coupled system of the Einstein and Euler-Lagrange equations by Newton-Raphson method. Furthermore, in terms of the inherent chiral character of the solutions, we could analyze the property of the localized chiral fermions in these branes. We may assume the Lagrangian which the chiral components ψ_R or ψ_L interact with the spin connections A_R or A_L.

II. Static Hopfions in the extended Skyrme-Faddeev model

Lagrangian density

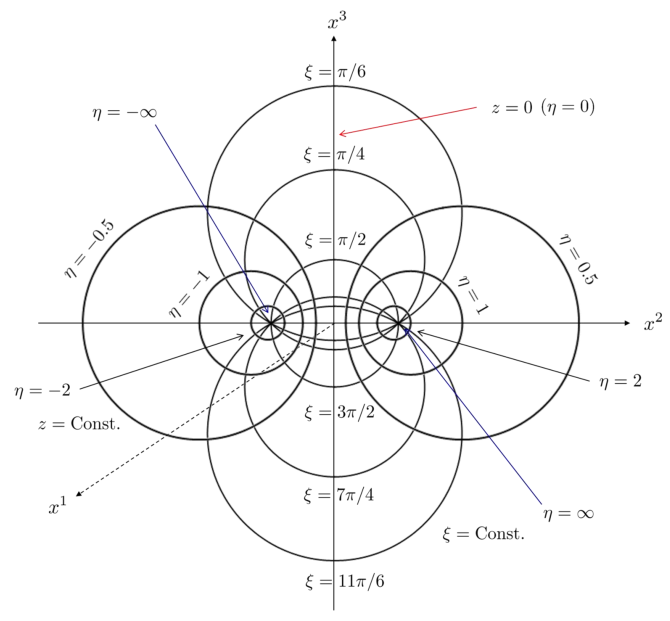
$$\mathcal{L} = A^2\partial_\mu\vec{n}\cdot\partial^\mu\vec{n} - \frac{1}{e^2}(\partial_\mu\vec{n}\times\partial^\nu\vec{n})^2 + \frac{b}{2}(\partial_\mu\vec{n}\cdot\partial^\mu\vec{n})^2$$

$$A>0,\quad e^2<0,\quad b<0,\quad b\,e^2>1$$

Toroidal coordinates

$$x^1 = \frac{r_0}{p}\sqrt{z}\cos\varphi,\quad x^2 = \frac{r_0}{p}\sqrt{z}\sin\varphi,\quad x^3 = \frac{r_0}{p}\sqrt{1-z}\sin\xi$$

$$p \equiv 1-\cos\xi\sqrt{1-z} \qquad z = \tanh^2\eta$$



$$-\infty<\eta<\infty \;\Leftrightarrow\; 0\le z\le 1,\; -\pi\le\theta\le\pi,\;\; 0\le\varphi\le 2\pi$$

Hopf fibration

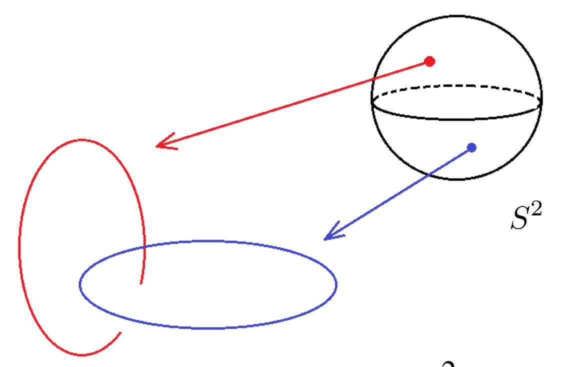
maps from the three dimensional space $\mathbb{R}^3 \sim S^3$ to the target space S^2

$$\mathbb{R}^3\to S^3 : Z_1 = \sqrt{1-g(z,\xi)}\,e^{i\Theta(z,\xi)},\quad Z_2 = \sqrt{g(z,\xi)}\,e^{-in\phi}$$

$$S^3\to S^2 : \frac{Z_1}{Z_2} = \sqrt{\frac{1-g(z,\xi)}{g(z,\xi)}}\,e^{i\Theta(z,\xi)+in\phi} \equiv u(z,\xi)$$

The triplet of scalar fields $\vec{n}$ living on the S^2

$$\vec{n} = \frac{1}{1+|u|^2}\left(u+u^*,\; -i(u-u^*),\; |u|^2-1\right)$$



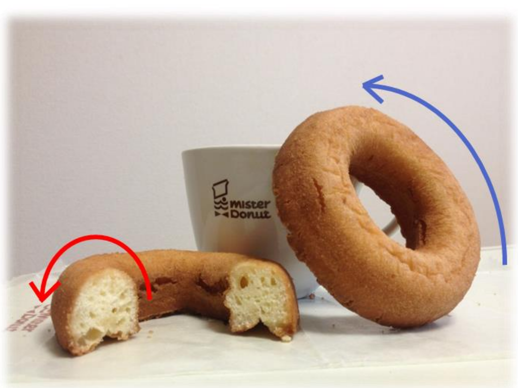
The preimage of a generic point on the target S^2 is a closed loop. If a field has Hopf topological charge Q_H then the two loops consisting of preimages of two generic distinct points on S^2 will be linked exactly Q_H times.

The definition of Hopf topological charge

$$Q_H = \frac{1}{4\pi^2}\int d^3x\,\vec{A}\cdot\left(\vec{\nabla}\wedge\vec{A}\right) \qquad \vec{A} = \frac{i}{2}\sum_{k=1}^{2}\left[Z_k^*\vec{\nabla}Z_k - Z_k\vec{\nabla}Z_k^*\right]$$

We get the Hopf topological charge under the boundary conditions:

$$Q_H = mn$$



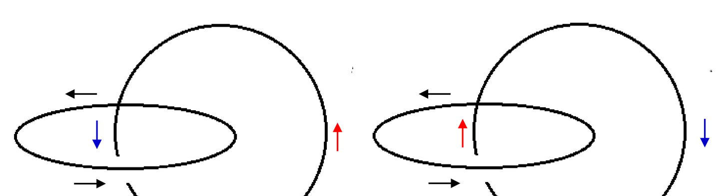
for $0\le z\le 1$

$g(z=0,\xi)=0,\quad g(z=1,\xi)=1$

for $-\pi\le\xi\le\pi$

$\Theta(z,\xi=-\pi) = -m\pi,\quad \Theta(z,\xi=\pi) = m\pi$

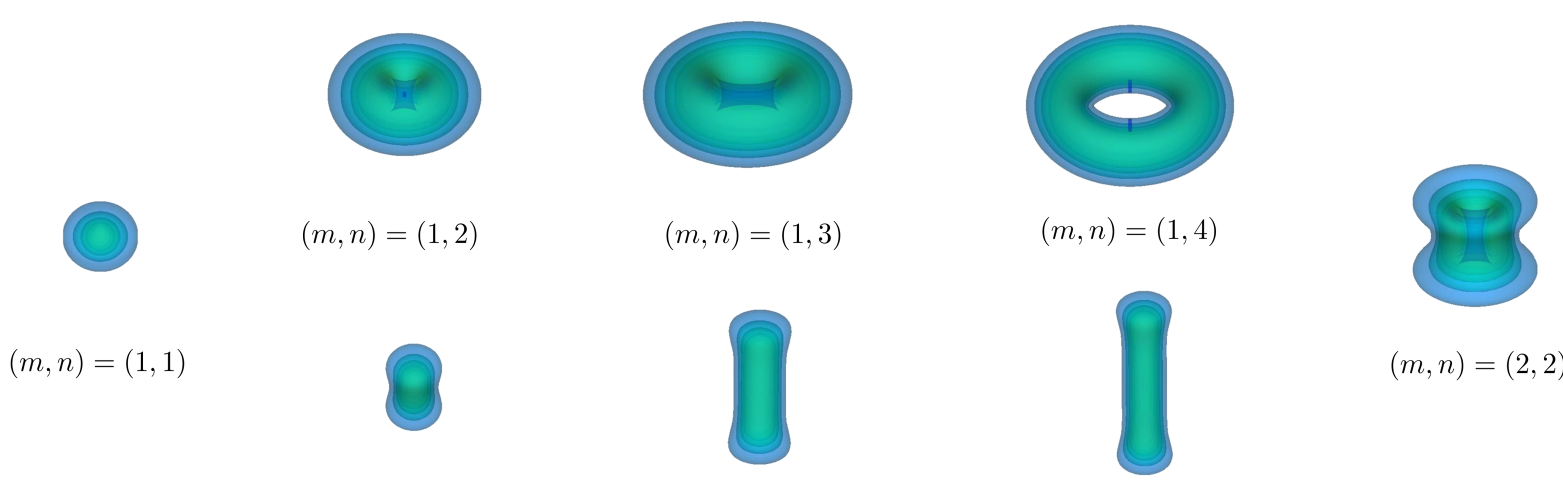
The Hopf topological charge corresponds to the linking number of Hopfions, and can be either positive or negative. This reflects the chiral structure of knotted solitons.



Linking number +2   Linking number -2

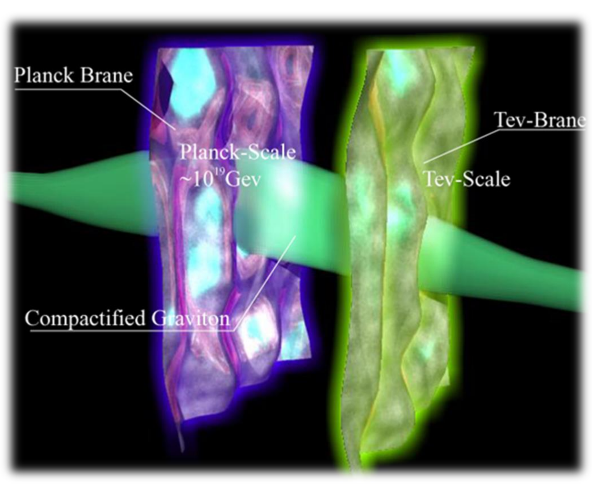
The axial symmetric solutions

Very probably, the solutions with Hopf charge 3 and 4 correspond to excited states. On the other hand, The solutions Hopf charge 1 or 2 may correspond to the minimum of energy. That is in fact what happens in the Skyrme-Faddeev model.



III. Brane World Scenario

Brane World Scenario is the high dimensional theory which solve the hierarchy problem and cosmological constant problem. The standard model particles and forces are confined to a 3-brane. We assume seven dimensional space-time world using toroidal coordinates in the extra dimensions. SM particles exist in the center of the torus which is located in $z = 1$ ($\eta = \infty$).



IV. The model

The actions

$$S = S_{\text{matter}} + S_{\text{gravity}} \qquad S_{\text{matter}} = \int d^7x\,\sqrt{g}\,\mathcal{L} \qquad S_{\text{gravity}} = \int d^7x\sqrt{g}\left(\frac{1}{2\chi_{(7)}}R - \Lambda_{(7)}\right)$$

The metric

$$g_{MN} = M^2(z,\xi)g^{(4)}_{\mu\nu}dx^\mu dx^\nu - \frac{L^2(z,\xi)}{M^2(z,\xi)}\left(\frac{r_0}{p}\right)^2\left[\frac{dz^2}{4z(1-z)} + (1-z)d\xi^2 + zd\varphi^2\right]$$

$$ds^2_{(4)} = g^{(4)}_{\mu\nu}\,dx^\mu dx^\nu = dt^2 - \delta_{ij}e^{2H(t)}dx^i dx^j$$

The ansatz for scalar fields $F = F(z,\xi)$, $\Theta = \Theta(z,\xi)$

$$\vec{n} = (\sin F\cos[\Theta+n\varphi],\; \sin F\sin[\Theta+n\varphi],\; \cos F)\,,\quad \vec{n}\cdot\vec{n} = 1$$

Boundary conditions

$$F(z=0,\xi)=0,\;\; F(z=1,\xi)=\pi,\quad [\partial_z\Theta(z,\xi)]_{z=0} = [\partial_z\Theta(z,\xi)]_{z=1} = 0 \qquad (0\le z\le 1)$$

$$\Theta(z,\xi=0)=0,\;\; \Theta(z,\xi=\pi)=m\pi,\;\; [\partial_\xi F(z,\xi)]_{\xi=0} = [\partial_\xi F(z,\xi)]_{\xi=\pi} = 0 \qquad (0\le\xi\le\pi)$$

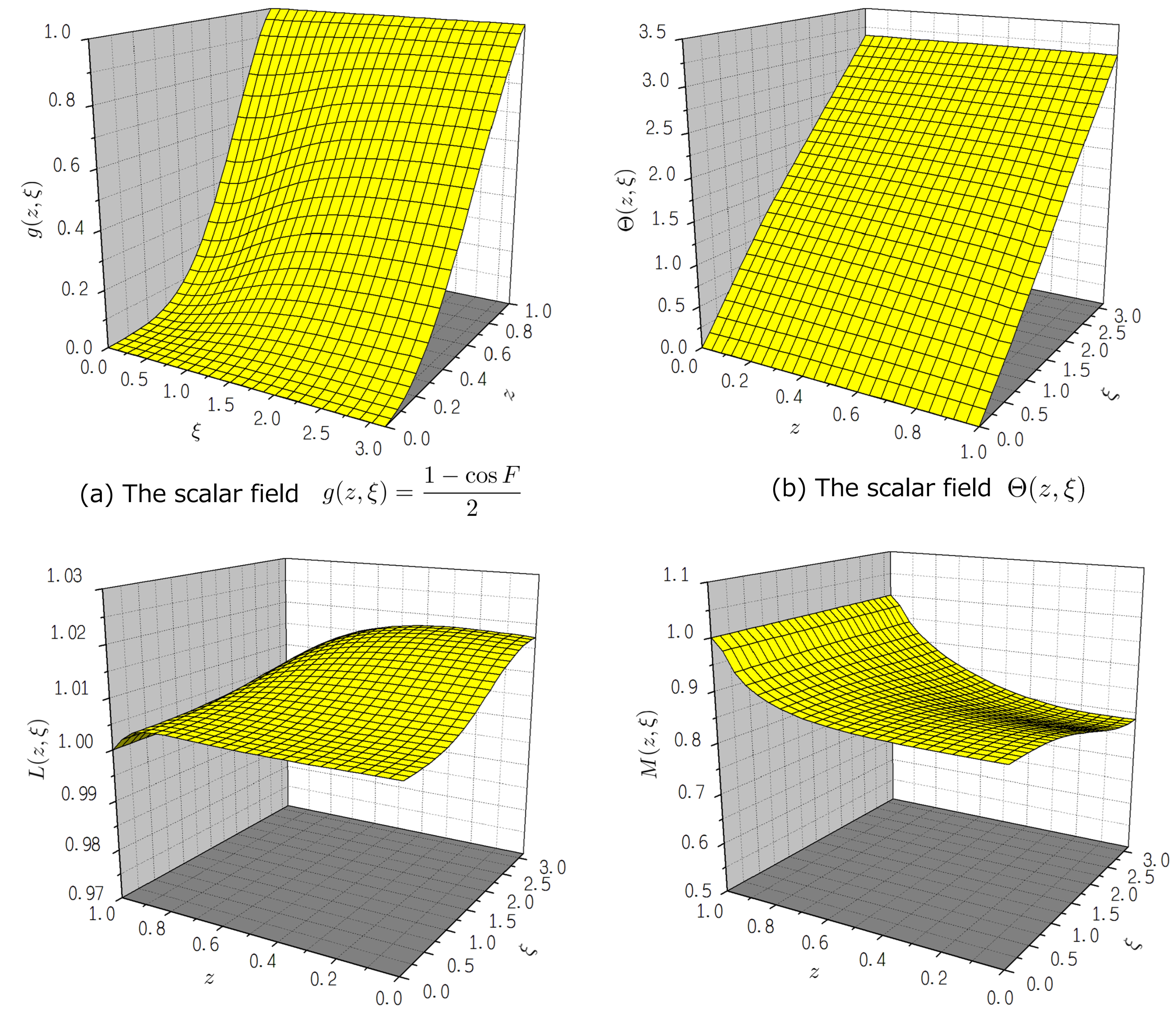
$$L(z=1,\xi)=1,\;\; L(z=1,\xi)=1,\quad [\partial_z L(z,\xi)]_{z=1} = [\partial_z M(z,\xi)]_{z=1} = \text{Const.} \qquad (0\le\xi\le\pi)$$

$$[\partial_\xi L(z,\xi)]_{\xi=0} = [\partial_\xi L(z,\xi)]_{\xi=\pi} = 0,\quad [\partial_\xi M(z,\xi)]_{\xi=0} = [\partial_\xi M(z,\xi)]_{\xi=\pi} = 0 \qquad (0\le z\le 1)$$

The scalar curvature

$$R = \frac{2p^2}{a^2z(1-z)}\frac{M^2}{L^2}\Bigg[2z\frac{\partial_\xi^2M}{M} + 2u\frac{\partial_z^2M}{M} + 2z\frac{\partial_\xi^2L}{L} + 2u\frac{\partial_z^2L}{L} - 2z\frac{\partial_\xi^2p}{p} - 2u\frac{\partial_z^2p}{p} + 5z\left(\frac{\partial_\xi M}{M}\right)^2 + 5u\left(\frac{\partial_z M}{M}\right)^2 - z\left(\frac{\partial_\xi L}{L}\right)^2 - u\left(\frac{\partial_z L}{L}\right)^2 + 3z\left(\frac{\partial_\xi p}{p}\right)^2 + 3u\left(\frac{\partial_z p}{p}\right)^2$$
$$+2z\frac{\partial_\xi M}{M}\frac{\partial_\xi L}{L} + 2u\frac{\partial_z M}{M}\frac{\partial_z L}{L} - 2z\frac{\partial_\xi L}{L}\frac{\partial_\xi p}{p} - 2u\frac{\partial_z L}{L}\frac{\partial_z p}{p} - 2z\frac{\partial_\xi M}{M}\frac{\partial_\xi p}{p} - 2u\frac{\partial_z M}{M}\frac{\partial_z p}{p} + 8z(1-z)(1-2z)\left(\frac{\partial_z L}{L} + \frac{\partial_z M}{M} - \frac{\partial_z p}{p}\right) - 3z(1-z)\Bigg] - \frac{12\gamma}{M^2}$$

V. Numerical solutions



(a) The scalar field $g(z,\xi) = \dfrac{1-\cos F}{2}$ (b) The scalar field $\Theta(z,\xi)$

(c) The metric field $L(z,\xi)$ (d) The metric field $M(z,\xi)$

scalar or metric fields for $\alpha = 0.01,\ \beta = 0.00,\ \text{and}\ \gamma = 0.00.$

Due to re-scaling $z = \tanh^2\eta$, we are able to calculate points near the space infinity. According to the results written above, both metric fields do not diverge at infinity, and we could realize the gravity localization.

VI. Future works - A localizing mode of fermions on the brane

We consider the follow Dirac Hamiltonian

$$\mathcal{H}_{\text{fermion}} = \bar{\Psi}\begin{pmatrix} i\sigma^i(\partial_i + g_LA_i) & 0 \\ 0 & -i\sigma^i(\partial_i - g_RA_i)\end{pmatrix}\Psi$$

where Ψ consists of the two components Ψ_L and Ψ_R. At the quantum level, under the $U(1)$ frame rotation

$$\Psi' = \begin{pmatrix} e^{i\phi} & 0 \\ 0 & e^{-i\phi}\end{pmatrix}\Psi$$

the separate right and left chiral components of the fermion number current suffer from an equal size but opposite sign anomaly

$$\partial_\mu j^\mu_L = -\partial_\mu j^\mu_R = -\frac{1}{32\pi^2}\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F^{\rho\sigma} = -\frac{1}{16\pi^2}\partial_\mu(\epsilon^{\mu\nu\rho\sigma}A_\nu F_{\rho\sigma}).$$

Integrating this equations over space-time, we get

$$N_L = -N_R = -\frac{1}{16\pi^2}\int d^3x\epsilon^{ijk}A_iF_{jk} = -Q_H$$

This means Hopf topological charge corresponds to the chiral fermion number. Using this, we attempt to localize left or right chiral fermions on the brane.

References

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L. Freyhult and A. J. Niemi, *Chirality and fermion number in a knotted soliton background*, Phys. Lett. B 557 (2003) 121