Surface Defects and Dualities from Superconformal Indices

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Surface defects in gauge theories can be regarded as infra-red limit (and generalization of) of vortices in Abelian Higgs model:

$$\mathcal{L}_{\mathrm{AH}} = -\frac{1}{2e^2} F_{\mu\nu}^2 + |\mathcal{D}_\mu Q_i|^2 - \frac{e^2}{2} \left(Q_i Q_i^\dagger - \nu^2 \right)^2 \qquad (1)$$

We can use Bogomolnyi's trick of "completing the square" to obtain "vortex equations":

$$\frac{1}{e^2} F_{z\bar{z}} = Q_i Q_i^\dagger - \nu^2, \quad \mathcal{D}_z Q_i = 0, \quad z = x^1 + ix^2 \qquad (2)$$

The solutions to these equations represents the minimal energy configurations, and $U(1)$ gauge group is completely broken or "higgsed" asymptotically. The minimal action is given by:

$$S_{\mathrm{vortex}} = -\nu^2 \int dz d\bar{z} F_{z\bar{z}} = 2\pi\nu^2 |k|, \quad k \in \pi_1\left(U(1)\right) \cong \mathbb{Z} \qquad (3)$$

- The integer k is the topological winding number characterizing the map $Q_i : \partial\mathbb{R}^2 \cong S^1 \to U(1) \cong S^1$. Now if take the limit:

$$\nu \to \infty, \quad \rho \to 0, \quad \nu\rho \sim \text{fixed} \quad (\rho = \text{vortex core size}) \quad (4)$$

  Singularities develop in (A_z, Q_i), dynamical vortices become infinitely massive, co-dimension two "Surface Defects".
- We can embed Abelian vortices into higher rank gauge group e. g. $U(1) \subset U(N_c)$,and also couple fermionic degrees of freedom in vortex background.
- Moreover we can embed them in supersymmetric gauge theories in different dimensions, where exact quantities such as "partition function" or "index" can be computed.

- Here we consider vortices/surface defects 4d $\mathcal{N}=2$ $U(N_c)$ SQCD with N_f fundamental flavors, they generally exist in the "Higgs branch" of vacua:

$$A^I_k = 0,\ Q^i_I = v\delta^i_I,\ \tilde{Q}^I_i = 0,\ I,k = 1,\dots,N_c,\ i = 1,\dots,N_f. \tag{5}$$

The gauge+global symmetry breaking pattern is:

$$\nu^2 : U(N_c)\times SU(N_f) \longrightarrow S[U(N_c)\times U(N_f-N_c)]. \tag{6}$$

The breaking of overall $U(1)$ ensures the existence of topological stable, 1+1 dimensional "vortex strings".

- These 2d defects partially break Lorentz hence supersymmetries, however remain invariant under half of supercharges, i. e. 4 out of 8. More precisely, the vortex world volume theory is given by an $\mathcal{N}=(2,2)$ supersymmetric gauge theory, which flows to $\mathcal{N}=(2,2)$ σ-model with target spaces being CP^{N_c-1} or $\mathrm{Gr}(k,N_c)$ (or tangent bundle of).

- If we now add the complex masses $\{\mu_i\}$ for these N_f flavors, Higgs branch is generally lifted, however vortices can still exist at "Root of Baryonic Higgs Branch": :

$$A^I_k = \mathrm{diag}(\mu_1, \ldots, \mu_{N_c}) \;:\; \frac{N_f!}{N_c!(N_f - N_c)!} \text{ isolated vacua} \tag{7}$$

- Breaking 4d $\mathcal{N} = 2$ supersymemtry to $\mathcal{N} = 1$ by introducing superpotential $\mathcal{W}(A)$ for A^I_k, stable vortices can still exist if:

$$\frac{\partial \mathcal{W}(x = \mu_I)}{\partial x} = 0, \quad I = 1, \ldots, N_c \tag{8}$$

- Bulk supersymmetry breaking pattern descends to 2d vortex world volume, $\mathcal{N} = (2,2)$ supersymmetry is also broken down to $\mathcal{N} = (0,2)$ by superpotential $\mathcal{W}(\Sigma)$ for Σ^α_β in $\mathcal{N} = (2,2)$ vector multiplet.
  In fact many interesting 4d phenomena such as IR dualities and Wall-Crossing, also exhibit in 2d vortex world volume.

Super + Conformal Symmetries = Superconformal Symmetry.
In 4d, $SU(2,2|\mathcal{N})$ allow us to define "Superconformal Index":

$$\mathcal{I}^{\mathcal{N}=2}(\mathfrak{a}_i;\mathfrak{p},\mathfrak{q},\mathfrak{t}) = \mathrm{Tr}(-1)^F \mathfrak{p}^{h_{34}-\mathfrak{r}} \mathfrak{q}^{h_{12}-\mathfrak{r}} \mathfrak{t}^{\mathfrak{R}+\mathfrak{r}} e^{i\beta\delta_{\mathcal{N}=2}} \prod \mathfrak{a}_i^{f_i}. \quad (9)$$

For $\mathcal{N}=2$, the bosonic subgroup $SU(2,2|2)$ is

$$SO(2,4)\times SU(2)_R \times U(1)_r \ : \ \mathrm{Cartan}\ (E,h_{12},h_{34},\mathfrak{R},\mathfrak{r}) \quad (10)$$

The trace here is taken over Hilbert space of the states satisfying:

$$\delta_{\mathcal{N}=2} \equiv 2\{\mathcal{Q},\mathcal{Q}^\dagger\} = E-(h_{12}+h_{23})-2\mathfrak{R}+\mathfrak{r}=0 \quad (11)$$

We can also regard $\mathcal{I}^{\mathcal{N}=2}$ as twisted partition function on $S^1\times S^3$, obtained from radial quantization over $R\times S^3$ by compactifying R into S^1 of radius β. The fugacity parameters $\{\mathfrak{a}_i\}$ for other global symmetries such as flavor and baryon can also be inserted.

- We can also define superconformal index for 2d $\mathcal{N}=(2,2)$ SCFT, and it is called "Elliptic Genus", as we now compactify R of $R \times S^1$ into S^1 and have $S^1 \times S^1 \cong T^2$.

$$\mathcal{I}^{\mathcal{N}=(2,2)}(\mathbf{a}_i; q, y) = \mathrm{Tr}(-1)^F q^{H_\mathrm{L}} y^{J_\mathrm{L}} \prod_i \mathbf{a}_i^{f_i}, \quad q = e^{2i\pi\tau} \tag{12}$$

where H_L and J_L are left moving conformal dimension and $U(1)$ R-charge of $SL(2,\mathbb{R})_\mathrm{L} \times U(1)_L \subset SU(1,1|1)_L$.

- The trace is now taken over states satisfying the condition:

$$\delta^{\mathcal{N}=(2,2)} = \{\mathcal{G}, \mathcal{G}^\dagger\} = 2H_\mathrm{R} - J_\mathrm{R} = 0, \tag{13}$$

where H_R and J_R are right-moving conformal dimension and $U(1)$ R-charge. The resultant expression is a modular function of the complex structure τ and other fugacity parameters.

- Breaking of supersymmetry by superpotential can also be implemented in indices/ elliptic genus by identifying the fugacity parameters. For example in 4d:

$$\text{Adding } \mathcal{W}(A) \sim \mathrm{Tr}(A^{K+1}) \rightarrow \mathfrak{t} = (\mathfrak{p}\mathfrak{q})^{\frac{K}{K+1}} \qquad (14)$$

  We now have an $\mathcal{N} = 1$ gauge theory with the same matter contents plus an additional $\mathcal{N} = 1$ adjoint chiral with $U(1)_R$ charge $\frac{2}{K+1}$, e. g. $K = 1$ we obtained $\mathcal{N} = 1$ SQCD.
- We can also define 4d $\mathcal{N} = 1$ superconformal index:

$$\mathcal{I}^{\mathcal{N}=1}(\mathfrak{a}_i; \mathfrak{p}, \mathfrak{q}) = \mathrm{Tr}(-1)^F \mathfrak{p}^{h_{34}+\frac{\tilde{\mathfrak{r}}}{2}} q^{h_{12}+\frac{\tilde{\mathfrak{r}}}{2}} \prod_i \mathfrak{a}_i^{f_i}, \qquad (15)$$

  which provides highly non-trivial checks for various IR dualities such as Seiberg dualities and other generalizations.

We can now consider the superconformal index for 4 dim. $\mathcal{N}=2$ $G=U(N_c)$ plus N_f flavors (hypermultiplets):

$$I^{\mathcal{N}=2}=\frac{\kappa^2}{N_c!}\oint_{\mathbb{T}^{N_c}}\prod_{l=1}^{N_c}\frac{dz_l}{2i\pi z_l}\frac{\prod_{l,k=1}^{N_c}\Gamma(\frac{z_l}{z_k}\frac{pq}{t})}{\prod_{l\neq k}^{N_c}\Gamma(\frac{z_l}{z_k})}\prod_{i=1}^{N_f}\prod_{l=1}^{N_c}\Gamma\left(\frac{z_l}{\mathfrak{a}_i}t^{\frac{1}{2}}\right)\Gamma\left(\frac{\mathfrak{a}_i}{z_l}t^{\frac{1}{2}}\right)$$

where $\{z_l\}$ and $\{a_i\}$ are fugacities for gauge and flavor symmetries. The interesting function here is "Elliptic Gamma Function":

$$\Gamma(x)=\Gamma(x;p,q)=\prod_{r,s\geq 0}\frac{1-x^{-1}p^{r+1}q^{s+1}}{1-xp^rq^s},$$

which has simple poles when $x=p^{-M}q^{-N}$ $M,N\in\mathbb{N}$. The simple poles here are at (from hypermultiplet):

$$z_l=\mathfrak{a}_{i_l}\mathfrak{t}^{1/2}\mathfrak{p}^{n_{i_l}}\mathfrak{q}^{m_{i_l}},\quad l=1,2,\dots,N_c,\quad n_{i_l},m_{i_l}\geq 0 \tag{16}$$

- We can compute the residue at these simple poles using

$$\frac{\Gamma(xp^{-n}q^{-m};p,q)}{\Gamma(x,p,q)} = \frac{p^{\frac{nm(n+1)}{2}}q^{\frac{nm(m+1)}{2}}}{(-x)^{mn}}\prod_{r=1}^{n}\frac{1}{\theta(xp^{-1};q)}\prod_{s=1}^{m}\frac{1}{\theta(xq^{-s};p)}$$

  The pole condition can be regarded as the quantized version of $a_I = \mu_I$, and the integers (n_I, m_I) label the vortex numbers.
- Explicit evaluation of the residues, we precisely obtain two copies of the 2 dim. $\mathcal{N}=(2,2)$ elliptic genus of vortex world volume theory + free 4d hypermultiplets, after the matching of 4d and 2d fugacity parameters.
- In particular $(\mathfrak{p},\mathfrak{q})$, the 4d fugacity parameters for rotations now respectively become the complex structure moduli for the two independent T^2s

- To understand this, we connect with recent progress on calculating exact partition functions on other curved manifolds using localization. For 4d indices, we can absorb the twists due to fugacities by deforming the geometry into:

$$ds^2 = ds^2_{S^3_b} + \tilde{r}^2(b) d\psi^2, \quad ds^2_{S^3_b} = f(\theta)^2 d\theta^2 + l^2 \cos^2\theta d^2\varphi + \tilde{l}^2 \sin^2\theta d\chi$$

where $f(\theta) = \sqrt{l^2 \sin^2\theta + \tilde{l}^2 \cos^2\theta}$, $b^2 = (\tilde{l}/l)$ and $\tilde{r}(b) = \frac{r}{b+1/b}$.

- S^3_b can be factorized into two copies of $S^1 \times_{\mathrm{q}} D^2$, and the partition function on it can be interpreted as equivariant vortex partition function/Higgs Branch Localization.
- Adding another S^1 says the geometry now factorizes into two copies of $T^2 \times D^2$, the complex structure moduli of two T^2 are $(p; q)$. The natural saddle points correspond to vortices/surface operators wrapping on T^2, the decoupling limit reduces 4 dim. index into 2 dim. index.

- This procedure is general, and we can use it to insert surface operators into 4 dim. $\mathcal{N} = 2$ gauge theories, beginning with quiver gauge theories + higgsing. We can also consider extending this to surface operators in 4 dim. $\mathcal{N} = 1$ gauge theories, by adding superpotential $\mathcal{W}(A)$.
- General prescription for inserting surface operators allows us to deduce new dualities between $\mathcal{N} = (0, 2)$ vortex theories. For example, the vortex theories of 4d $\mathcal{N} = 1$ Seiberg duals are related $\mathcal{N} = (0, 2)$ version of "Hopping Transition", i. e. picking up different set of poles from fundamental/anti-fund.
- Finally we can also consider other $\mathcal{N} = (2, 2)/\mathcal{N} = (0, 2)$ surface defects which are not related to IR limit of dynamical vortices, and couple them to 4d gauge theories by "gauging" flavor symmetry group. The 2d duality between them can be recast into invariance under bulk 4d dualities.