Superconformal field theories and cyclic homology

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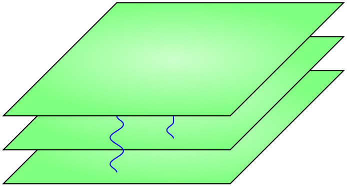
Introduction to AdS/CFT

Consider a stack of N D3 branes filling $\mathbb{R}^{1,3}$ in $\mathbb{R}^{1,3} \times \mathbb{C}^3$.



At low energies, the open string degrees of freedom decouple from the bulk. The resulting theory on the brane world-volume is $\mathcal{N} = 4$ super Yang-Mills.

Goal: Prove AdS/CFT

Less ambitious goal:

Prove part of AdS/CFT for a subset of protected BPS operators and observables.

This talk:

Show that the BPS operators agree under the correspondence.

- Based on joint work with J. Schmude, Y. Tachikawa [arXiv:1207.0573, ATMP to appear]
- and work in progress

AdS/CFT Cartoon

Gauge Theory

$\mathbb{R}^{3,1} \times X_6$

N $D3$ branes

X_6 Calabi-Yau 6-manifold

Gravity Theory

$AdS_5 \times L_5$

N units of RR-flux

L_5 Sasaki-Einstein 5-manifold

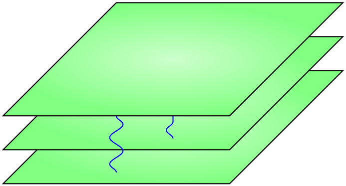


Figure: N D3-branes

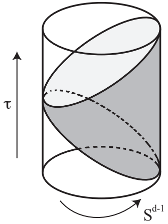


Figure: AdS Space-Time

Protected operators in $\mathcal{N} = 4$ SYM

$\mathcal{N} = 4$ SYM has three adjoint chiral scalar superfields Φ^1, Φ^2, Φ^3. Their interactions are described by the superpotential

$$W = \operatorname{Tr} \Phi^1 \left[\Phi^2, \Phi^3\right].$$

Consider an operator of the form

$$\mathcal{O} = T^{z_1 z_2 \cdots z_k} = \operatorname{Tr} \Phi^{z_1} \Phi^{z_2} \ldots \Phi^{z_k}.$$

If $T^{z_1 z_2 \cdots z_k}$ is symmetric in its indices, then the operator is in a short representation of the superconformal algebra. If $T^{z_1 z_2 \cdots z_k}$ is not symmetric, then the operator is a descendant, because the commutators $[\Phi^{z_i}, \Phi^{z_j}]$ are derivatives of the superpotential W [Witten '98].

Matching protected operators in $\mathcal{N} = 4$ SYM

Under the AdS/CFT dictionary, a scalar excitation Φ in AdS obeying

$$(\Box_{AdS_5} - m^2)\Phi = 0$$

with asymptotics $\rho^{-\Delta}$ near the boundary of AdS ($\rho \to \infty$) is dual to an operator of scaling dimension

$$m^2 = \Delta(\Delta - d) \to \Delta_\pm = \frac{d}{2} \pm \sqrt{\frac{d^2}{4} = m^2}$$

Matching protected operators in $\mathcal{N} = 4$ SYM

The operator

$$\mathcal{O} = \mathrm{Tr}\,\Phi^{z_1}\Phi^{z_2}\ldots\Phi^{z_k}$$

has conformal dimension k and is dual to a supergravity state of spin zero and mass

$$m^2 = k(k-4).$$

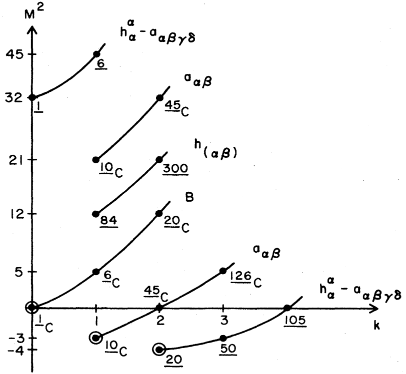


FIG. 2. Mass spectrum of scalars.

Figure: From Kim-Romans-van Nieuwenhuizen [Phys.Rev. D32 (1985) 389]

Goal: Test AdS/CFT by small deformations

$\mathcal{N} = 4$ SYM has superpotential

$$W = \mathrm{Tr}\left(XYZ - XZY\right).$$

What happens when we deform it by giving a mass to one of the scalars

$$W = \mathrm{Tr}\left(XYZ - XZY + mZ^2\right)$$

or deform the coupling constants?

$$W = \mathrm{Tr}\left(qXYZ - q^{-1}XZY\right)$$

Can we still match the spectrum of protected operators?

Goal: Match Closed String States in the large-N limit

Gauge Theory

$\mathbb{R}^{3,1} \times X_6$

Closed strings:

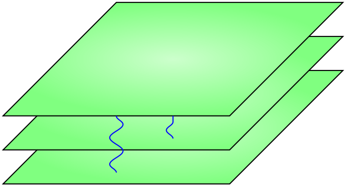
$HC_\bullet(\mathbb{C}Q/\partial W)$

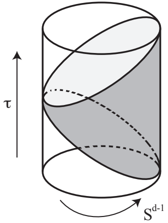
Gravity Theory

$AdS_5 \times L_5$

Closed strings:

$HP_\bullet(X, \pi = 0)$





The Superconformal Algebra

The 4D superconformal algebra combines both the conformal algebra and $\mathcal{N} = 1$ supersymmetry algebra. The conformal algebra consists of Lorentz generators $M_{\mu\nu}$, momenta P_{μ}, special conformal generators K_{μ} and a dilatation D.

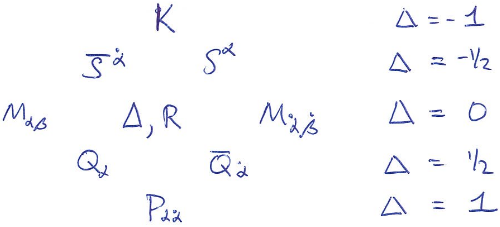


Figure: Generators of the Superconformal Algebra

The Superconformal Index

The SCI is a 4D analog of the Witten index in quantum mechanics

Defined as

$$\mathcal{I}(\mu_i) = \mathrm{Tr}(-1)^F e^{-\beta\delta} e^{-\mu_i \mathcal{M}_i}$$

- The trace is over the Hilbert space of states on S^3
- $\mathcal{Q}$ is one of the Poincare supercharges
- $\mathcal{Q}^\dagger$ is the conjugate conformal supercharge
- $\delta \equiv \frac{1}{2}\left\{\mathcal{Q}, \mathcal{Q}^\dagger\right\}$
- $\mathcal{M}_i$ are $\mathcal{Q}-$closed conserved charges

Operators contributing to the index

Key commutation relations:

$$\{Q_\alpha, Q^{\dagger\beta}\} = E + 2M_\alpha^\beta + \frac{3}{2}r$$
$$\{\overline{Q}_{\dot\alpha}, \overline{Q}^{\dagger\dot\beta}\} = E + 2\overline{M}_{\dot\alpha}^{\dot\beta} - \frac{3}{2}r$$

Operators for which $\overline{\mathcal{Q}}^{\dot\alpha}\mathcal{O} = 0$ are called chiral primaries. Operators contributing to the (right-handed) index have $\delta = \{\mathcal{Q}, \mathcal{Q}^\dagger\} = 0$. Choosing $\mathcal{Q} = \overline{Q}_{\dot -}$, operators contributing to the index satisfy

$$E - 2j_2 - \frac{3}{2}r = 0. \tag{0.1}$$

The 4D Letter Index

Letter	(j_1, j_2)	$\mathcal{I}$
ϕ	$(0,0)$	t^{3r}
$\overline{\psi}_2$	$(0,1/2)$	$-t^{3(2-r)}$
$\partial_{\pm-}$	$(\pm 1/2, 1/2)$	$t^3 y^{\pm 1}$

Letter	(j_1, j_2)	$\mathcal{I}$
λ_1	$(1/2, 0)$	$-t^3 y$
λ_2	$(-1/2, 0)$	$-t^3 y^{-1}$
$\overline{f}_{22}$	$(0,1)$	t^6
$\partial_{\pm-}$	$(\pm 1/2, 1/2)$	$t^3 y^{\pm 1}$

- Fields contributing to the index, from a chiral multiplet (left) and from a vector multiplet (right) [1]

[1] [F. Dolan, H. Osborn],[A. Gadde, L. Rastelli, S. S. Razamat, W. Yan]

Ginzburg's DG Algebra

Letter	(j_1, j_2)	$\mathcal{I}$
ϕ	$(0,0)$	t^{3r}
$\overline{\psi}_2$	$(0,1/2)$	$-t^{3(2-r)}$

Letter	(j_1, j_2)	$\mathcal{I}$
$\overline{f}_{22}$	(0,1)	t^6

Table: Fields contributing to the index, from a chiral multiplet (left) and from a vector multiplet (right), after the cancellation of W_α and the spacetime derivatives ∂_μ are taken into account.

Ginzburg's DG algebra is a free differential-graded algebra

$$\mathfrak{D} = \mathbb{C}\langle x_1, \ldots, x_n, \theta_1, \ldots, \theta_n, t_1, \ldots, t_m\rangle$$

where $\phi, \overline{\psi}_2, \overline{f}_{22}$ correspond to x, θ, t respectively.

SUSY Transformations

The differential $\mathcal{Q}$ on Ginzburg's DG algebra

$$\begin{aligned} \mathcal{Q}\phi_e &= 0, \\ \mathcal{Q}\bar{\psi}_{e,2} &= \partial W(\phi_e)/\partial\phi_e, \\ \mathcal{Q}\bar{f}_{v,22} &= \sum_{h(e)=v} \phi_e\bar{\psi}_{e,2} - \sum_{t(e)=v} \bar{\psi}_{e,2}\phi_e. \end{aligned}$$

Let $[\mathfrak{D},\mathfrak{D}]$ be a $\mathbb{C}$-linear space spanned by commutators. The basis of $\mathfrak{D}_{\mathrm{cyc}} = \mathfrak{D}/(\mathbb{C}+[\mathfrak{D},\mathfrak{D}])$ corresponds to the set of closed path of $\hat{Q}$, or equivalently, the single-trace operators formed from ϕ_e, $\bar{\psi}_{e,2}$ and $\bar{f}_{v,22}$.

Cyclic homology and the superconformal index

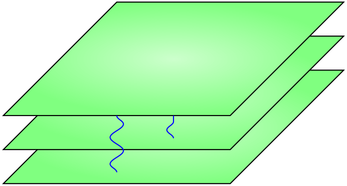
Consider single-trace operators, up to the pairing given by the supersymmetry transformation $\mathcal{Q}$. This corresponds to taking the homology $H_*(\mathfrak{D}_{\text{cyc}}, \mathcal{Q})$. This homology is known as (reduced) cyclic homology of the algebra $\mathfrak{D}$, and is usually denoted by $\overline{HC}_*(\mathfrak{D})$.

The single-trace index is the Euler characteristic of cyclic homology

$$\mathcal{I}_{s.t.}(t) \doteq \operatorname{Tr}(-1)^F t^{3R}|\mathfrak{D}_{\text{cyc}} = \sum_i (-1)^i \operatorname{Tr} t^{3R}|\overline{HC}_i(\mathfrak{D}).$$

A Simple Example

N D3 branes filling $\mathbb{R}^{1,3}$ in $\mathbb{R}^{1,3} \times \mathbb{C}^3$.



$\mathcal{N} = 4$ SYM has superpotential

$$W = \mathrm{Tr}\,(XYZ - XZY)$$

where X, Y, Z are adjoint-valued chiral superfields.

Superpotential algebra

$$\mathcal{A} = \mathbb{C}\langle x, y, z\rangle/(xy - yx, yz - zy, zx - xz) \cong \mathbb{C}[x, y, z]$$

Example: $\mathcal{N} = 4$ SYM

Consider an operator which is not symmetric in x, y, z. For example $\mathcal{O} = xyz - xzy$. Since the operator is not symmetric, it is $\mathcal{Q}$-closed, $\mathcal{O} = \mathcal{Q}\left(x\bar{\psi}_x\right)$ so it vanishes in $\mathcal{Q}$-cohomology. Continuing in this way, we can find all of the protected operators. For example there are six operators $x^2, y^2, z^2, xy + yx, xz + zx, yz + zy$ of conformal dimension 2.

Operators in $\mathcal{N} = 4$ Super Yang-Mills

For $X = \mathbb{C}^3, L^5 = S^5$. The corresponding gauge theory is $\mathcal{N} = 4$ SYM, whose superpotential algebra is

$$\mathcal{A} = \mathbb{C}\langle x, y, z\rangle/(xy - yx, yz - zy, zx - xz) \cong \mathbb{C}[x, y, z]$$

	1	t^2	t^4	t^6	t^8	t^{10}	t^{12}	...
HC_0	1	3	6	10	15	21	28	...
HC_1	0	0	3	8	15	24	35	...
HC_2	0	0	0	1	3	6	10	...
$\mathcal{I}(t)$	1	3	3	3	3	3	3	...

Table: Cyclic homology group dimensions for $\mathcal{N} = 4$ SYM

Elements $\mathcal{O} \in HC_0(\mathcal{A}) = \mathcal{A}/[\mathcal{A}, \mathcal{A}]$ are of the form

$$\mathcal{O} = \operatorname{Tr} x^i y^j z^k, \qquad i, j, k \in \mathbb{N}_{\geq 0}$$

Large-N superconformal index

The large-N superconformal index was first computed as a large-N matrix integral by mathematicians [P. Etingof, V. Ginzburg] and independently by physicists [A. Gadde, L. Rastelli, S. S. Razamat, W. Yan].

The index can also simply computed as the Euler characteristic of a free dg-algebra [P. Etingof, V. Ginzburg].

Superconformal index and Sasaki-Einstein manifolds

The advantage of reformulating the gauge theory index in terms of cyclic homology is that the cyclic homology groups can be directly related to the supergravity index using the HKR isomorphism and its generalisations. For any local Calabi-Yau X, we have

$$\mathbb{C} \oplus \overline{HC}_0(\mathfrak{D}) = H^0(\wedge^0 \Omega'_X) \oplus H^1(\wedge^1 \Omega'_X) \oplus H^2(\wedge^2 \Omega'_X), \quad (0.2)$$

$$\overline{HC}_1(\mathfrak{D}) = H^0(\wedge^1 \Omega'_X) \oplus H^1(\wedge^2 \Omega'_X), \quad (0.3)$$

$$\mathbb{C} \oplus \overline{HC}_2(\mathfrak{D}) = H^0(\wedge^2 \Omega'_X), \quad (0.4)$$

We conclude that the single-trace index is

$$1 + \mathcal{I}_{s.t.}(t) = \sum_{0 \le p-q \le 2} (-1)^{p-q} \operatorname{Tr} t^{3R} | H^q(\wedge^p \Omega'_X). \quad (0.5)$$

This agrees with the field theory computation and is a non-trivial test of AdS/CFT.

The β-deformation

However, we would like to go beyond Sasaki-Einstein geometries. The β-deformation of $\mathcal{N} = 4$ super Yang-Mills theory is a quiver gauge theory with potential $W = qxyz - q^{-1}xzy$ where $q = e^{i\beta}$. The F-term relations are

$$xy = q^{-2}yx$$
$$yz = q^{-2}zy$$
$$zx = q^{-2}xz$$

The cyclic homology groups were computed by Nuss and Van den Bergh.

Chiral Primaries in the β-deformation

Consider an operator $\mathcal{O} = \text{Tr}\, l_1 l_2 \dots l_n$, where l_i is one of the letters x, y, or z. Suppose that l_1 is an x. The F-term conditions imply that

$$\mathcal{O} = \text{Tr}\, l_1 l_2 \dots l_{n-1} l_n = q^{2(|z|-|y|)}\, \text{Tr}\, l_n l_1 l_2 \dots l_{n-1},$$

where $|x|, |y|$, and $|z|$ are the total number of x's, y's, and z's in the operator $\mathcal{O}$. Thus the single-trace chiral primaries have charges $(k, 0, 0), (0, k, 0), (0, k, 0), (k, k, k)$ [D. Berenstein, V. Jejjala, R. G. Leigh]. [2] For q a k-th root of unity, the cyclic homology groups jump.

[2] For $G = SU(N)$ there are additional chiral primaries $\text{Tr}\, xy, \text{Tr}\, xz$ and $\text{Tr}\, yz$. This agrees with the perturbative one-loop spectrum of chiral operators found in [D. Z. Freedman, U. Gursoy].

Operators in the β-deformation

Cyclic homology gives a prediction for the spectrum of protected operators in the β-deformation. The corresponding gravity solution was found by Lunin and Maldacena.

	1	t^2	t^4	t^6	t^8	t^{10}	t^{12}	$\dots$
HC_0	1	3	3	4	3	3	4	$\dots$
HC_1	0	0	0	2	0	0	2	$\dots$
HC_2	0	0	0	1	0	0	1	$\dots$
$\mathcal{I}(t)$	1	3	3	3	3	3	3	$\dots$

Table: Cyclic homology group dimensions for the β-deformation

Further applications of cyclic homology

For CY-3 algebras

$$HC_j(\mathcal{A}) = 0 \text{ for } j > 2$$

This corresponds to the AdS dual theory having no particles of spin higher than 2.

$$HC_2(\mathcal{A}) = Z(\mathcal{A})$$

So the KK-spectrum of gravitons can be computed from the center of the superpotential algebra. For the Pilch-Warner solution, this has been checked explicitly.

Final Remarks

Summary

We have shown how to compare the protected fields on both sides of the AdS/CFT correspondence at large-N.

- Further extension to finite N is possible, although the cyclic homology groups become much harder to compute.

Further directions

Use all fields that contribute to the SCI. This corresponds to proving AdS/CFT under a holomorphic twist of the both the gauge and gravity theories [K. Costello].

- Other twists are also interesting [C. Beem, L. Rastelli, B. C. van Rees].
- Extensions to M-theory compactifications [R.E., J. Schmude].

Thank you for listening!