Functional Renormalization Group approach

and

gauge symmetry in QED

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Introduction and Summary

◇ Functional Renormalization Group (FRG): A convincing nonperturbative approach to field theory and condensed matter physics Introduces momentum cutoff Λ

$\Rightarrow$ Dynamics described by RG flow of couplings $g(\Lambda)$ in theory space.

Gauge symmetry ?

Even in the presence of Λ, gauge symmetry realized as a quantum symmetry by imposing Ward-Takahashi (WT) identity for Wilson action S_Λ

$$\Sigma_\Lambda \sim \left(\partial S_\Lambda / \partial \phi^A\right) \delta\phi^A - \mathrm{Str}\left(\partial \delta\phi / \partial \phi\right) = 0$$

(1) symmetry tr. $\delta\phi^A$ depend on S_Λ
(2) non-trivial Jacobian factor in functional measure $\mathcal{D}\phi$

$\Sigma_\Lambda = 0$ defines gauge invariant subspace in theory space.

$\Rightarrow$ Expression of symmetry in FRG.

♢ Two fundamental equations we have to solve:

i) WT identity $\Sigma_\Lambda = 0$

ii) RG flow eq. $\partial_t \Gamma_\Lambda = \mathrm{Str}\,(\partial_t R)\left[\partial^2\Gamma_\Lambda/\partial\Phi\partial\Phi + R\right]^{-1}$ $(\Lambda\partial_\Lambda = \partial_t)$

We discuss an exact solution to $\Sigma_\Lambda = 0$ for suitably truncated Wilson action in QED.

- Need to introduce higher dimensional interactions with form factors (momentum dependent 4-fermi couplings).
- Take account of full momentum dependence in WT and flow eq. without using derivative (momentum) expansion.

♢ Results

- Exact evaluation of photon 2-point functions
- Relation which corresponds to $Z_1 = Z_2$
- Relations between form factors in 4-fermi couplings and photon propagator

♢ Plan of the talk

[1] Derivation of the WT identity

[2] WT identity in QED

[3] Exact solution to WT identity

[4] Momentum dependent flow eq.

[5] Outlook

Derivation of the WT identity

◇ Consider a gauge-fixed theory described by

$$\mathcal{S}[\varphi] = \frac{1}{2}\varphi\cdot D\cdot\varphi + \mathcal{S}_I[\varphi], \qquad \varphi\cdot D\cdot\varphi = \int_p \varphi^A(-p)D_{AB}(p)\varphi^B(p).$$

We rewrite its partition function as

$$\begin{aligned}
\mathcal{Z}_\varphi[J] = \int \mathcal{D}\varphi \exp\left(-\mathcal{S}[\varphi] + J\cdot\varphi\right) \\
= \int \mathcal{D}\varphi\mathcal{D}\phi \exp\left(-\mathcal{S}[\varphi] + J\cdot\varphi\right) \exp\Bigg[-\frac{1}{2}\Big(\phi - K\varphi - J(1-K)D^{-1}\Big)\cdot\frac{D}{K(1-K)}\cdot \\
\times\Big(\phi - K\varphi - (-)^{\epsilon(J)}D^{-1}(1-K)J\Big)\Bigg] \\
= N_J\int \mathcal{D}\phi\mathcal{D}\chi \exp\Bigg[-\frac{1}{2}\Big(\phi\cdot K^{-1}D\cdot\phi + \chi\cdot(1-K)^{-1}D\cdot\chi\Big) - \mathcal{S}_I[\phi+\chi] + J\cdot K^{-1}\phi\Bigg]
\end{aligned}$$

where $\chi = \varphi - \phi$. Source terms introduced to cancel $J\cdot\varphi$. We have introduced an IR cutoff Λ through a positive function that behaves as

$$K(p) \equiv K(p^2/\Lambda^2) \to \quad \begin{cases} 1 & (p^2 < \Lambda^2) \\ 0 & (p^2 > \Lambda^2) \end{cases}$$

For cut-off function, we take e.g. $K(p) = e^{-p^2/\Lambda^2}$.

The Wilson action is defined by

$$S_\Lambda[\phi] = \frac{1}{2}\phi^A (K^A)^{-1}\ D_{AB}\phi^B + S_{I,\Lambda}[\phi]\ ,$$

where the interaction part is given by a functional integral

$$\begin{aligned} \exp -S_{I,\Lambda}[\phi] &= \int \mathcal{D}\chi \exp -\Big[\frac{1}{2}\chi \cdot (\Delta_H)^{-1} \cdot \chi + \mathcal{S}_I[\phi + \chi]\Big] \\ \Delta_H &= (1-K)D^{-1} \end{aligned}$$

The partition function for the Wilson action,

$$Z_\phi[J] = \int \mathcal{D}\phi \exp\Big[-S_\Lambda[\phi] + J \cdot K^{-1}\phi\Big] \ ,$$

is related to that for the original one by

$$\mathcal{Z}_\varphi[J] \quad = \quad N_J Z_\phi[J] \ ,$$

where the normalization factor is given by

$$N_J \quad = \quad \exp\frac{1}{2}\left[-(-)^{\epsilon(J_A)} J_A \left(\frac{1-K}{K}\right)^A \left(D^{-1}\right)^{AB} J_B\right] \ .$$

♢ We define the WT operator

$$\Sigma_\Lambda[\phi] = K^A \left[\frac{\partial^r S_\Lambda}{\partial\phi^A}\delta\phi^A - (-)^{\epsilon_A}\frac{\partial^l \delta\phi^A}{\partial\phi^A}\right],$$

This corresponds to change of the Wilson action $\oplus$ change of the functional measure under BRS tr. $\delta\phi$. Therefore, cancellation of these two contributions

$$\Sigma_\Lambda[\phi] = 0$$

signal for the presence of BRS (quantum) symmetry.

To find $\delta\phi$, take functional average of the WT op. for the original theory with standard BRS symmetry

$$\begin{aligned} \Sigma[\varphi] &= \frac{\partial^r \mathcal{S}}{\partial\varphi^A}\delta\varphi^A - (-)^{\epsilon_A}\underbrace{\frac{\partial^l \delta\varphi^A}{\partial\varphi^A}}_{=0} \\ \delta\varphi^A &= R^A{}_B\varphi^B \ . \end{aligned}$$

where $R^A{}_B$ are field independent coeffients, and $\delta\varphi^A$ stand for $classical$ (conventional) BRS transformations for linear symmetry. Use them as a "seed" for quantum symmetry.

Perform change of variables $\varphi^A \to \varphi^{A\prime} = \varphi^A + \delta\varphi^A\lambda$ with Grassmann odd constant λ.

Invariance of partition function leads to

$$
\mathcal{Z}_{\varphi'}[J] = \mathcal{Z}_{\varphi}[J] + \int \mathcal{D}\varphi \left[-\frac{\partial^r \mathcal{S}}{\partial \varphi^A}\delta\varphi^A + \frac{\partial}{\partial \varphi^A}\delta\varphi^A + J\cdot\delta\varphi \right] \lambda \exp\left(-\mathcal{S}[\varphi] + J\cdot\varphi\right)
$$
$$
= \mathcal{Z}_{\varphi}[J] + \int \mathcal{D}\varphi \ \left[\left(-\Sigma[\varphi] + J_A \ R^A_B \ \varphi^B\right)\right] \lambda \exp\left(-\mathcal{S}[\varphi] + J\cdot\varphi\right) = \mathcal{Z}_{\varphi}[J]
$$

It gives

$$
\int \mathcal{D}\varphi \ \Sigma[\varphi] \exp\left(-\mathcal{S}[\varphi] + J\cdot\varphi\right) = \int \mathcal{D}\varphi [J_A R^A{}_B \varphi^B] \exp\left(-\mathcal{S}[\varphi] + J\cdot\varphi\right)
$$
$$
= J_A R^A{}_B \partial_{J_B} \mathcal{Z}_{\varphi}[J] = J_A \ R^A{}_B \partial_{J_B} \left(N_J Z_\phi[J]\right)
$$
$$
= J_A R^A{}_B \Big(Z_\phi[J] \ \partial_{J_B} N_J + N_J \ \partial_{J_B} Z_\phi[J] \Big)
$$
$$
= N_J \int \ \mathcal{D}\phi \ \Sigma[\phi] \ \exp\left(-S[\Phi] + J\cdot K^{-1}\Phi\right)
$$

where $\partial_{J_B} N_J$ generates a non-trivial modification. We find

$$\delta\phi^A = R^A{}_B \left[\phi^B\right]_\Lambda, \qquad \left[\phi^B\right]_\Lambda = \phi^B - (\Delta_H)^{BC} \frac{\partial^l S_{I,\Lambda}}{\partial\phi^C}$$

where $\left[\phi^A\right]_\Lambda$ are "composite operators" for fields ϕ^A. They obey RG flow equations:

$$\partial_t \mathcal{O}_\Lambda[\phi] = -\frac{\partial^r S_{I,\Lambda}}{\partial\phi^A}(\partial_t\Delta_H)^{AB}\frac{\partial^l \mathcal{O}_\Lambda}{\partial\phi^B} + \frac{1}{2}(-)^{\epsilon_A(1+\epsilon_{\mathcal{O}})}(\partial_t\Delta_H)^{AB}\frac{\partial^l\partial^r\mathcal{O}_\Lambda}{\partial\phi^B\partial\phi^A}\,.$$

We obtain general expression of the WT op. for linear gauge symmetry

$$\Sigma_\Lambda[\phi] = K^A\left\{\frac{\partial^r S_\Lambda}{\partial\phi^A}R^A{}_B\left[\phi^B\right]_\Lambda + (-)^{\epsilon_A}R^A{}_B\,(\Delta_H)^{BC}\,\frac{\partial^l\partial^r S_{I,\Lambda}}{\partial\phi^C\partial\phi^A}\right\}\,.$$

Note that $K^A\partial S_\Lambda/\partial\phi^A,\ [\phi^A]_\Lambda,\ \Sigma_\Lambda$ are composite operators.

WT identity for QED

$\diamondsuit$ Consider the Wilson action $S_\Lambda[\phi] = S_{0,\Lambda} + S_{I,\Lambda}$ for the fields $\phi^A = (a_\mu,\ \bar{\psi}_{\hat{\alpha}},\ \psi_\alpha,\ c,\ \bar{c})$.

The kinetic part of the Wilson action is given by

$$\begin{aligned} S_{0,\Lambda} &= \frac{1}{2}(K^A)^{-1} Z_A \phi^A D_{AB} \phi^B \\ &= \int_p K^{-1}(p) \left[\frac{Z_3}{2} a_\mu(-p) p^2 \left\{ \delta_{\mu\nu} - \left(1 - (Z_3 \xi_0)^{-1}\right) \frac{p_\mu p_\nu}{p^2} \right\} a_\nu(p) + \bar{c}(-p) i p^2 c(p) \right] \\ &+ \int_p K^{-1}(p) Z_2 \bar{\psi}(-p) \not{p} \psi(p) \ , \end{aligned}$$

where we have introduced the renormalization constants, $Z_2,\ Z_3$. The classical BRS tr.

$$\begin{aligned} \delta_{cl}\ a_\mu(p) &= -i p_\mu c(p), & \delta_{cl}\ \bar{c}(p) = \xi_0^{-1} p_\mu a_\mu(p) \\ \delta_{cl}\ \psi(p) &= -i\ e_0 \int_q \psi(q) c(p-q), & \delta_{cl}\ \bar{\psi}(-p) = i\ e_0 \int_q \bar{\psi}(-q) c(q-p) \ , \end{aligned}$$

fix the coefficients $R^A{}_B$ in our general formula for quantum symmetry. Here, $e_0,\ \xi_0$ are gauge coupling and gauge fixing parameters which are constants.

The WT operator for QED is constructed as

$$\begin{aligned}
\Sigma_\Lambda[\phi] &= \int_p \Big\{ \frac{\partial S_\Lambda}{\partial a_\mu(p)}(-ip_\mu)c(p) + \frac{\partial^r S_\Lambda}{\partial \bar{c}(p)}\xi_0^{-1} p_\mu a_\mu(p) \Big\} \\
&\quad -i\ e_0 \int_{p,\ q} \Big\{ \frac{\partial^r S_\Lambda}{\partial \psi_\alpha(q)} \frac{K(q)}{K(p)} \psi_\alpha(p) - \frac{K(p)}{K(q)} \bar{\psi}_{\hat{\alpha}}(-q) \frac{\partial^l S_\Lambda}{\partial \bar{\psi}_{\hat{\alpha}}(-p)} \Big\} c(q-p) \\
&\quad -i\ e_0 \int_{p,\ q} U_{\beta\hat{\alpha}}(-q,p) \Big\{ \frac{\partial^l S_\Lambda}{\partial \bar{\psi}_{\hat{\alpha}}(-p)} \frac{\partial^r S_\Lambda}{\partial \psi_\beta(q)} - \frac{\partial^l \partial^r S_\Lambda}{\partial \bar{\psi}_{\hat{\alpha}}(-p) \partial \psi_\beta(q)} \Big\} c(q-p)\ ,
\end{aligned}$$

where

$$U(-q,p) = Z_2^{-1}\left[K(q)\frac{1-K(p)}{\not{p}} - K(p)\frac{1-K(q)}{\not{q}} \right]$$

Exact solution to WT identity

To construct interaction part $S_{I,\Lambda}[\phi]$, we first specify its 1PI part, namely Legendre effective action $\Gamma_{I,\Lambda}[\Phi]$, imposing for simplicity chiral symmetry on the fermionic sector. We introduce some form factors in 4-fermi interactions:

$$
\begin{aligned}
\Gamma_{I,\Lambda}[\Phi] &= \int_p \left[\frac{1}{2} Z_3 A_\mu(-p)\mathcal{M}_{\mu\nu}(p)A_\nu(p) + Z_2\sigma(p)\bar{\Psi}(-p)\not{p}\Psi(p)\right] \\
&-eZ_2Z_3^{1/2}\int_{p,q}\bar{\Psi}(-p)\not{A}(p-q)\Psi(q) + \frac{Z_2^2}{2\Lambda^2}\int_{p_1,\cdots,p_4}(2\pi)^4\delta^4(p_1+p_2+p_3+p_4) \\
&\times\left[h_S(s,u)\left\{\left(\bar{\Psi}\Psi\right)^2 - \left(\bar{\Psi}\gamma_5\Psi\right)^2\right\} + h_V(s,u)\left\{\left(\bar{\Psi}\gamma_\mu\Psi\right)^2 + \left(\bar{\Psi}\gamma_5\gamma_\mu\Psi\right)^2\right\}\right] \\
&+\frac{Z_2^2}{2\Lambda^4}\int_{p_1,\cdots,p_4}(2\pi)^4\delta^4(p_1+p_2+p_3+p_4) \\
&\times(p_1+p_4)_\mu(p_2+p_3)_\nu h_{V'}(s,u)\left[\left(\bar{\Psi}\gamma_\mu\Psi\right)\left(\bar{\Psi}\gamma_\nu\Psi\right) + \left(\bar{\Psi}\gamma_5\gamma_\mu\Psi\right)\left(\bar{\Psi}\gamma_5\gamma_\nu\Psi\right)\right]
\end{aligned}
$$

where s, u are Mandelstam variables. $S_{I,\Lambda}[\phi]$ constructed by using the Legendre tr. :

$$
\begin{aligned}
S_{I,\Lambda}[\phi] &= \Gamma_{I,\Lambda}[\Phi] + \frac{1}{2}(\Phi - \phi) \cdot (1-K)^{-1} D \cdot (\Phi - \phi) \\
&= \Gamma_{I,\Lambda}[\phi] + \frac{Z_3}{2} \int_p a_\mu(-p) \left[\sum_{n=1} (-)^n \left[\left(\mathcal{M} \Delta_H \right)^n \right]_{\mu\lambda} (p)\ \mathcal{M}_{\lambda\nu}(p) \right] a_\nu(p) \\
&- e Z_2 Z_3^{1/2} \int_{p,q} \sum_{n=1} (-)^n \left[\left(\mathcal{M} \Delta_H \right)^n \right]_{\mu\nu} (p-q) a_\nu(p-q) \\
&- \frac{Z_2^2 e^2}{2} \int_{p_1, \cdots, p_4} (2\pi)^4 \delta^4(p_1 + p_2 + p_3 + p_4) \left(\bar{\psi} \gamma_\mu \psi \right) \left(\bar{\psi} \gamma_\nu \psi \right) \left(\mathbf{\Delta}_G \right)_{\mu\nu} (p_1 + p_2)
\end{aligned}
$$

where additional terms to $\Gamma_{I,\Lambda}$ are 1P reducible contributions. $\left(\mathbf{\Delta}_G \right)_{\mu\nu}(p) = Z_3^{-1}[P^T_{\mu\nu} T(p) + P^L_{\mu\nu} L(p)]$ is full photon propagator constructed with photon 2-point functions

$$
\begin{aligned}
\mathcal{M}_{\mu\nu}(p) &= P^T_{\mu\nu} \mathcal{T}(p) + P^L_{\mu\nu} \mathcal{L}(p), \qquad P^T = \delta_{\mu\nu} - p_\mu p_\nu / p^2, \quad P^L = p_\mu p_\nu / p^2 \\
T(p) &= \frac{1-K}{p^2 + (1-K)\mathcal{T}(p)}, \qquad L(p) = \frac{\xi(1-K)}{p^2 + \xi(1-K)\mathcal{L}(p)}, \qquad \xi = Z_3 \xi_0
\end{aligned}
$$

Substitute $S_\Lambda = S_{0,\Lambda} + S_{I,\Lambda}$ into the WT identity $\Sigma_\Lambda[\phi] = 0$, which can be expanded in polynomial of ϕ^A. Consider two sectors $a \times c$ and $\bar\psi \times \psi \times c$ in this expansion. For simplicity, we assume $\sigma(p) = 0$ for fermionic 2-point function. For $a(p) \times c(-p)$ sector, we have

$$Z_3 p_\nu \mathcal{L}(p) = -e_0\ eZ_2 Z_3^{1/2} \int_q \mathrm{Tr}[U(-p-q,q)\gamma_\nu]$$

For $\bar\psi(p) \times \psi(-q) \times c(q-p)$ sector, we have second WT relation

$$\begin{aligned}
&\left(e_0 Z_2 - eZ_2 Z_3^{1/2}\right)(\not p - \not q) - 2e_0 Z_2^2 \int_k \Bigg[\frac{1}{\Lambda^2}\Big\{\big(h_S - 2h_V\big)[k^2,(p+q)^2] \\
&-2h_V[(p+q)^2,k^2]\Big\} + e^2 T(k^2) + \frac{1}{\Lambda^4}\Big\{2(p-q)^2 h_{V'}[k^2,(p+q)^2] \\
&+k^2 h_{V'}[(p+q)^2,k^2]\Big\}\Bigg] U(-q-k,p+k) \\
&-e_0 Z_2^2 \int_k \left[\frac{2}{\Lambda^4} h_{V'}[(p+q)^2,k^2] + e^2 \frac{1}{k^2}\{T(k^2) - L(k^2)\}\right] \not k U(-q-k,p+k)\not k = 0\ .
\end{aligned}$$

This constraint splits into two conditions: $\mathrm{constant} \times (\not{p} - \not{q})$ and one-loop part. They should separately vanish. The first one gives

$$e_0 = Z_3^{1/2} e \ .$$

This corresponds to the well-known WT relation in the standard realization of gauge symmetry in QED: $Z_1 = Z_2$ for $Z_1 = Z_2 Z_3^{1/2} Z_e$ with $e = Z_e\ e_0$.

On the other hand, one-loop part gives

$$\frac{1}{\Lambda^2}\left\{ \big(h_S - 2h_V\big)[k^2, (p-q)^2] - 2h_V[(p-q)^2, k^2] \right\}$$
$$= e^2\left\{ T[(p-q)^2] - L[(p-q)^2] \right\} - \frac{e^2}{2}\left\{ T(k^2) + L(k^2) \right\}$$
$$\frac{1}{\Lambda^4} h_{V'}[(p+q)^2, k^2] = -\frac{e^2}{2k^2}\left\{ T(k^2) - L(k^2) \right\}$$

These are relations between 4-fermi interactions and photon propagator.

Note that derivative expansion will give , $1 - Z_3^{1/2} e/e_0 \simeq [h_S(0,0) - 4h_V(0,0)]$.

Remarkably, longitudinal component of photon 2-point function $\mathcal{L}$ can be evaluated exactly for a specific cutoff function $K(p) = e^{-p^2/\Lambda^2}$ using some formula for the modified Bessel functions:

$$\int_0^\pi d\theta \; e^{2pk\cos\theta} \sin^2\theta = \frac{\pi}{2pk} I_1(2pk), \qquad \int_0^\infty dk e^{-k^2} I_1(2pk) = \frac{p}{2} \; {}_1F_1(1, 2; p^2)$$

we obtain

$$\mathcal{L}(p^2) = -e^2 \frac{\Lambda^2}{2\pi^2 \bar{p}^4} \left[1 - \exp(-\bar{p}^2/2) - \bar{p}^2 \left(1 - \frac{1}{2} \exp(-\bar{p}^2/2) \right) \right]$$

where we have used $e_0 = Z_3^{1/2} e$ to eliminate e_0, and $\bar{p}^2 = p^2/\Lambda^2$. To fix transverse part $\mathcal{T}$, we use RG flow equations.

Momentum dependent flow equations

♢ For photon 2-point functions $\propto e^2$ in RG flow equation we have

$$\frac{Z_3}{2}\int_p A_\mu(-p)\left[P^T_{\mu\nu}\left\{2\eta_A p^2 - 2\mathcal{T}(p)\right\} + (-2)P^L_{\mu\nu}\mathcal{L}(p)\right]A_\nu(p)$$

$$= -e^2 Z_3 \int_{p,q} 2K'(q)\,(1-K(p+q))^2\,\frac{1}{(p+q)^2}\mathrm{Tr}\left[\not{A}(-p)(\not{p}+\not{q})\not{A}(p)\not{q}\right]$$

Rhs can be exactly evaluated to give

$$\mathrm{rhs} = -Z_3\frac{e^2}{2\pi^2}\int_p A_\mu(-p)\left[P^T_{\mu\nu}\frac{1}{4p^4}\left\{4-\left(4+2p^2-p^4\right)\exp(-p^2/2)\right\}\right.$$

$$\left. -P^L_{\mu\nu}\frac{1}{p^4}\left\{1-p^2-\left(1-\frac{p^2}{2}\right)\exp(-p^2/2)\right\}\right]A_\nu(p)$$

Since p^2 term in transverse part here generates well-know anomalous dimension for photon field $\eta_A = e^2/12\pi^2$, we subtract it to find $\mathcal{T}$

$$
\begin{aligned}
\mathcal{T}(p^2) - \eta_A p^2 &= \frac{e^2}{8\pi^2 p^4}\Big\{4 - \big(4 + 2p^2 - p^4\big)\exp(-p^2/2)\Big\} \\
\mathcal{T}(p^2) &= \frac{\Lambda^2 e^2}{8\pi^2 \bar{p}^4}\Big\{4 + \frac{2\bar{p}^6}{3} - \big(4 + 2\bar{p}^2 - \bar{p}^4\big)\exp(-\bar{p}^2/2)\Big\}\ .
\end{aligned}
$$

$\mathcal{L}$ appeared here is exactly the same as the one obtained by WT identity.

In this way, we fix photon 2-point functions.

Note that a constant mass term is contain in $\mathcal{T}$ and $\mathcal{L}$ as

$$
\mathcal{T} = \mathcal{L} = \frac{3e^2}{16\pi^2}\Lambda^2 + \cdots
$$

Outlook

♢ $\Sigma_\Lambda = 0$ (almost) determines S_Λ.

All 4-fermi couplings expressed in terms of e^2 and photon 2-point functions ?

$\Leftarrow$ careful analysis of flow eq.

♢ Exact evaluation of photon 2-point functions is interesting but only possible in QED with simplified fermionic sector.

$\Rightarrow$ For more complicated cases such as YM theory, need to develop suitable approximation method which replaces derivative expansion.

Taking account of momentum dependence in WT identity and RG flow eq. will give new insights into FRG !