# Thermodynamic Limit of the Nekrasov-type Formula for E-string Theory

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# Introduction

**Nekrasov Formula**

(Instanton Partition Function) (N.A.Nekrasov '04)

Seiberg-Witten description

(N.Seiberg-E.Witten '94)

Low Energy Theories of 4D N=2 SYM

$$\hbar \to 0$$

**Nekrasov formula** → **SW description**

(N.A.Nekrasov '04, N.A.Nekrasov-A.Okounkov '03)

# Introduction

**Nekrasov-type Expression**

(BPS partition function) (K.Sakai '12)

Seiberg-Witten description

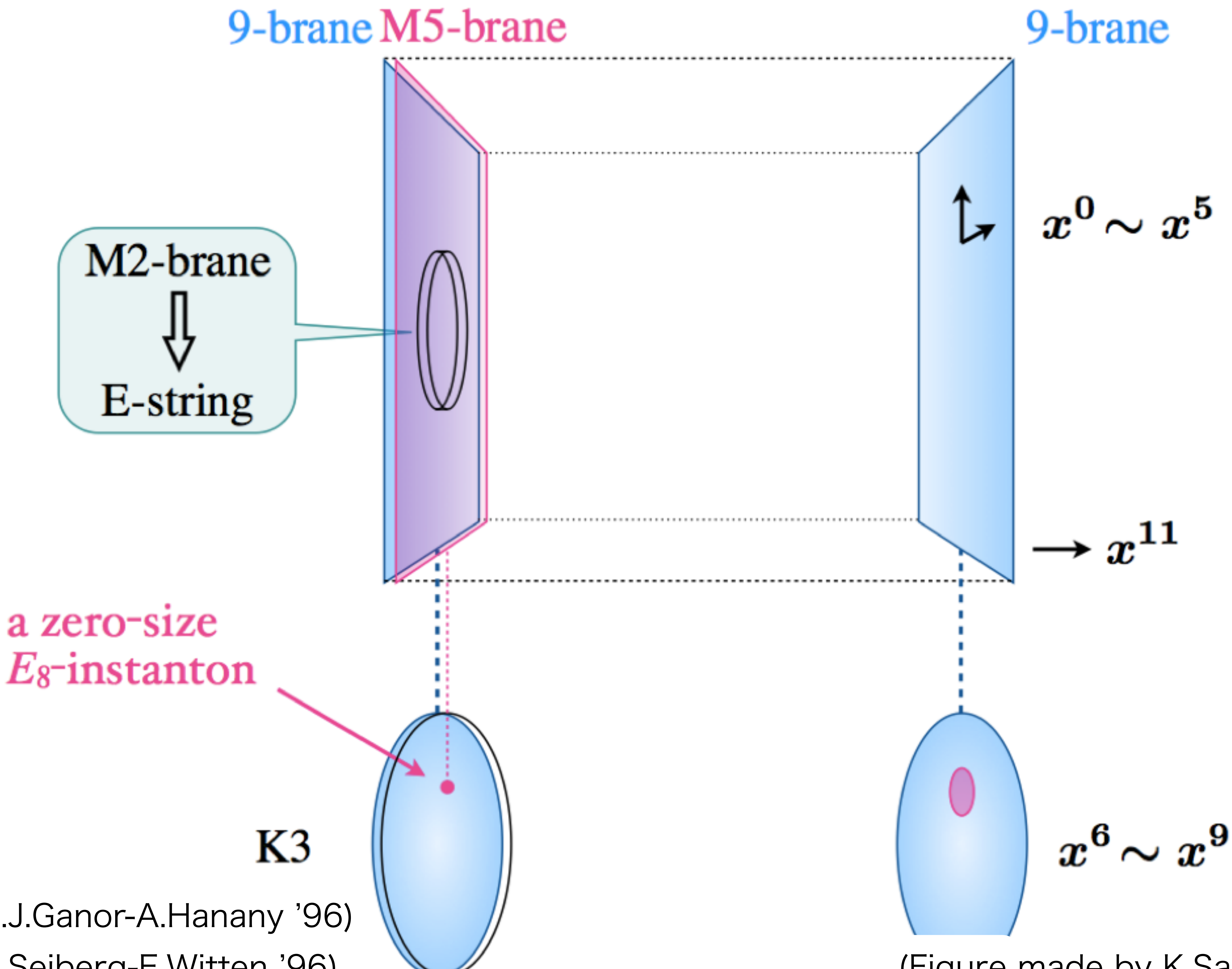
A Low Energy Theory of 4D N=2 SYM
for E-string theory on $\mathbb{R}^4 \times T^2$

(O.J.Ganor '97)

**Nekrasov-type expression** $\xrightarrow{\hbar \to 0}$ **SW description?**

# OUTLINE

1. E-string Theory and SW description
2. The Nekrasov-type Expression for E-string Theory
3. Reproducing SW description
4. Summary




(O.J.Ganor-A.Hanany '96)
(N.Seiberg-E.Witten '96)
(Figure made by K.Sakai)

# E-string Theory

(O.J.Ganor-A.Hanany '96)
(N.Seiberg-E.Witten '96)

6D (1,0) theory with a tensor multiplet

Torus Compactification

4D N=2 U(1) Gauge Theory (O.J.Ganor '97)

## SW-curve and Prepotential for E-string theory

(with E8 symmetry)

$$y^2 = 4x^3 - \frac{E_4(\tau)}{12}u^4 x - \frac{E_6(\tau)}{216}u^6 + 4u^5$$

$$\frac{\partial F_0}{\partial \varphi} = 8\pi^3 i(\varphi_D - \tau\varphi)$$

$\tau$ : complex structure of the torus

$u$ : Coulomb modulus

$\varphi$ : Higgs vev

$E_4, E_6$ : Eisenstein Series

$\varphi_D$ : dual Higgs vev

(O.J.Ganor-D.Morrison-N.Seiberg '96)

# The Nekrasov-type Expression for E-string Theory

(K.Sakai '12)

$$Z = \sum_{\mathbf{R}} (-e^{2\pi i \varphi})^{|\mathbf{R}|} \prod_{k=1}^{3} \prod_{(i,j)\in R_k} \frac{\vartheta_1(\frac{1}{2\pi}(a_k + (j-i)\hbar), \tau)^6}{\prod_{l=1}^{3} \vartheta_1(\frac{1}{2\pi}(a_k - a_l + h_{k,l}(i,j)\hbar), \tau)^2}$$

UV gauge coupling
in usual gauge theories

$\mathbf{R} = (R_1, R_2, R_3)$ : 3-sets of partition

$h_{k,l}(i,j)$ : relative hook-length

diagonal components of Higgs vev
in usual gauge theories

for E-string theory

$\varphi$ : Higgs vev

$a_k$ : loci on the torus fixed as $a_1 = \pi,\ a_2 = -\pi - \pi\tau,\ a_3 = \pi\tau$

$\tau$ : complex structure of the torus (=IR gauge coupling)

E-string theory has a distinct physical interpretation for the parameters from usual gauge theories

# Reproducing SW description

(N.A.Nekrasov '04, N.A.Nekrasov-A.Okounkov '03)

$$F_0 = (2\hbar^2 \ln Z)|_{\hbar=0} \qquad Z = \sum_{\mathbf{R}} e^{2\pi i \varphi |\mathbf{R}|} Z_{\mathbf{R}}$$

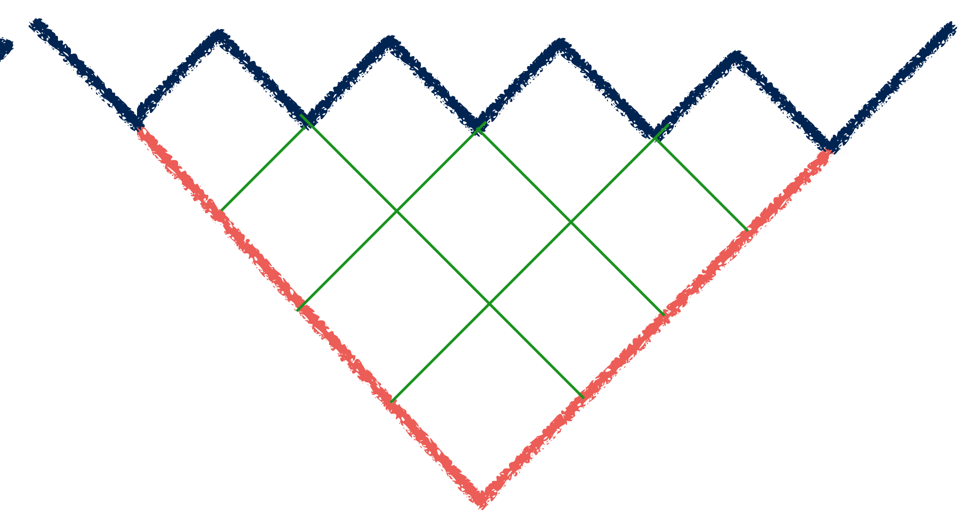
$$Z_{\mathbf{R}} = \exp\left[ -\frac{1}{4}\int dz dw f''(z) f''(w) \gamma(z-w;\hbar) + 3\int dz f''(z)\gamma(z;\hbar) + \sum_{k,l=1}^{3} \gamma(a_k - a_l;\hbar) - 6\sum_{k=1}^{3}\gamma(a_k;\hbar) \right]$$

This can be viewed as the partition function of a matrix model

$\gamma(z;\hbar)$ : a certain function

$f(z)$ : a profile function

$f''(z)$ : a density function



determine the most dominant profile in the limit $\hbar \to 0$

(a saddle point approximation)

# Solving the saddle point equation (e.o.m. in a matrix model)

**Resolvent** $\omega(z) := \Omega'(z)$

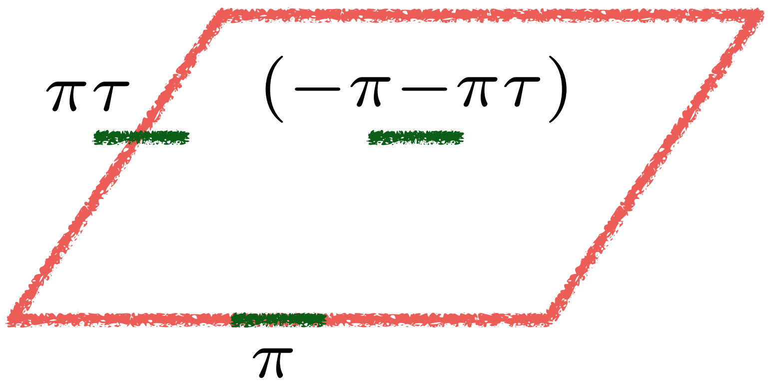
**antiderivative** $\Omega := \int_{\mathcal{C}} f''(w) \ln \vartheta_1\left(\frac{z-w}{2\pi}\right) dw - 6 \ln \vartheta_1\left(\frac{z}{2\pi}\right)$

$$2\pi i f''(z) = \omega(z - i\epsilon) - \omega(z + i\epsilon), \quad z \in \mathcal{C}$$

$\mathcal{C}$ : support of $f''$

$$\int_{\mathcal{C}} dw f''(w) \gamma_0(z-w) - 6\gamma_0(z) - \pi i \varphi z^2 = 0, \quad z \in \mathcal{C}$$

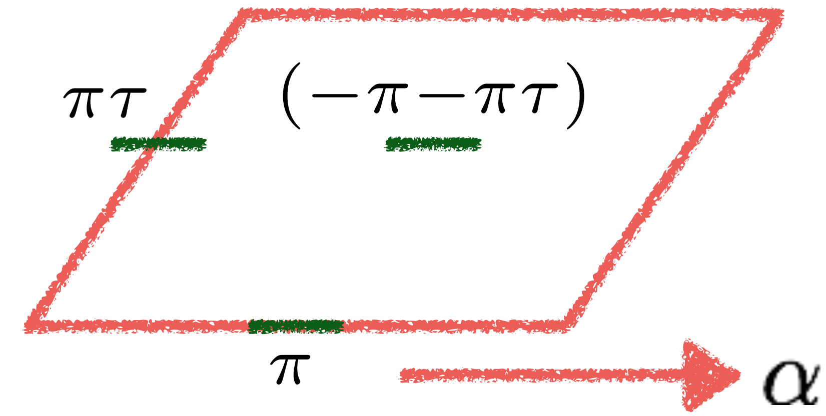
$$\Leftrightarrow \quad \frac{1}{2}\big(\Omega(z - i\epsilon) + \Omega(z + i\epsilon)\big) - 2\pi i \varphi = 0$$




$$\boxed{\Omega(z) = 2\ln\left(\sqrt{-\tfrac{1}{4}u\wp'(z)^2} + \sqrt{-\tfrac{1}{4}u\wp'(z)^2 - 1}\right) dz + 2\pi i \varphi}$$

# An integral over $\alpha$-cycle

$$\frac{1}{4\pi^2 i}\oint_\alpha \Omega(z)dz = 0 \mod \mathbb{Z}$$




$$\frac{\partial\varphi}{\partial u} = \frac{i}{4\pi^2 u}\oint_\alpha \frac{\wp'(z)dz}{\sqrt{\wp'(z)^2+4u^{-1}}}$$

SW-curve with genus four

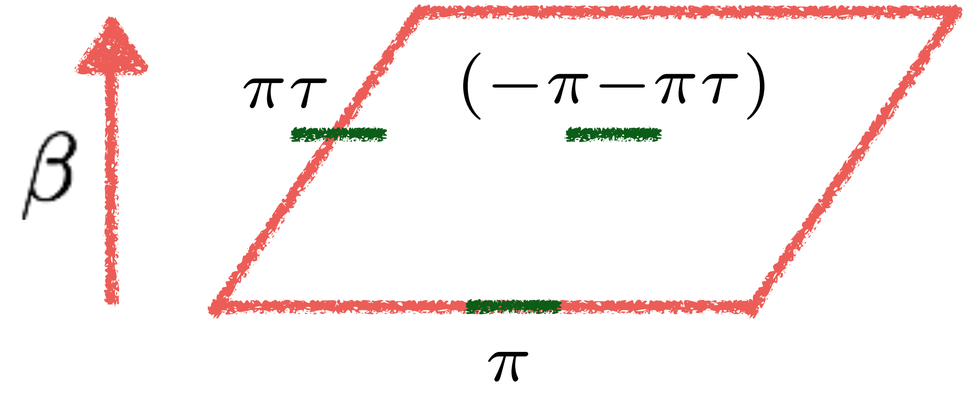
a variable change $\wp(z) = u^{-2}x$

$$\frac{\partial\varphi}{\partial u} = \frac{i}{4\pi^2}\oint_{\tilde\alpha} \frac{dx}{y}$$

$$y^2 = 4x^3 - \frac{E_4}{12}u^4 x - \frac{E_6}{216}u^6 + 4u^5$$

SW-curve with genus one

# An integral over $\beta$-cycle




$$\frac{1}{4\pi^2 i\tau}\oint_\beta \Omega(z)dz = \begin{cases} \dfrac{i}{8\pi^3\tau}\left(\dfrac{\partial F_0}{\partial \varphi}\right) \quad \mod \mathbb{Z} \\ \underline{\dfrac{1}{2\pi^2 i\tau}\oint_\beta \ln\left(\sqrt{-\frac{1}{4}u\wp'(z)^2}+\sqrt{-\frac{1}{4}u\wp'(z)^2-1}\right)dz+\varphi} \end{cases}$$

$$\equiv \varphi_D$$

$$\frac{\partial F_0}{\partial \varphi} = 8\pi^3 i(\varphi_D - \tau\varphi)$$

the SW-description has reproduced from the Nekrasov-type Expression

# Summary

We proved that

$$Z=\sum_{\mathbf{R}}(-e^{2\pi i\varphi})^{|\mathbf{R}|}\prod_{k=1}^{3}\prod_{(i,j)\in R_k}\frac{\vartheta_1(\frac{1}{2\pi}(a_k+(j-i)\hbar),\tau)^6}{\prod_{l=1}^{3}\vartheta_1(\frac{1}{2\pi}(a_k-a_l+h_{k,l}(i,j)\hbar),\tau)^2}$$

IS

the Nekrasov-type formula for E-string theory

**Similarly, SW description for E-string theory with E6 or E7 symmetry can be correctly reproduced.**