Doubled D-branes in generalized geometry

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22 July YITP

Outline

Motivation

Doubled D-branes in doubled geometry

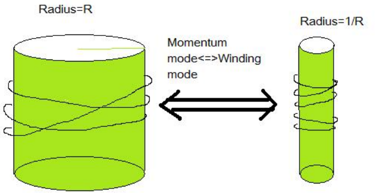
Dirac strautures and Courant algebroids in Generalized geometry

Doubled D-branes as Courant algebroids

Conclusion

T-duality

Geometrically, T-duality arises from compactifying a theory on a circle with radius R, and such a theory describes the same physics as a theory compactified on a circle with radius $1/R$ with the winding mode and momentum mode exchanged.



Motivation

Doubled geometry:

A doubled torus with coordinates $(x, \tilde{x})$

$\swarrow$ Physical space $\qquad\qquad$ T dual $\searrow$

Generalized geometry:

$$
\begin{array}{ccc}
(TE \oplus T^*E)_{S^1} & \xrightarrow{\cong} & (T\hat{E} \oplus T^*\hat{E})_{\hat{S}^1} \\
 & {\scriptstyle \pi} \searrow \quad \swarrow {\scriptstyle \tilde{\pi}} & \\
 & TM &
\end{array}
$$

In doubled formalism, half of the components obey the Dirichlet boundary condition while the other half obey the Neumann boundary condition. I.e. A D-brane and it's T-dual can be described simultaneously in doubled geometry–A doubled D-brane.

In generalized geometry, D-branes are described by objects called Dirac structures.

We conjecture that a doubled D-brane in doubled geometry is equivalent to a Courant algebroid in generalized geometry.

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In doubled geometry (Hull et al.), the key component is a $2n \times 2n$-matrix $\mathbb{H}$ called a generalized metric which transform as an $O(n,n)$-tensor:

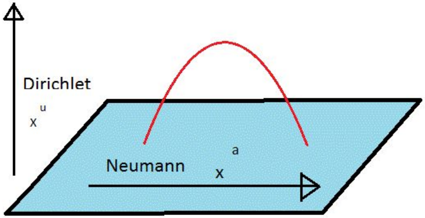
$$\mathbb{H} = \begin{pmatrix} G - BG^{-1}B & BG^{-}1 \\ -G^{-1}B & G^{-1} \end{pmatrix},$$

and also an $O(n,n)$-invariant constant matrix $\mathbb{L}$ conveniently chosen as

$$\mathbb{L} = \begin{pmatrix} 0 & \mathbb{I}_{n\times n} \\ \mathbb{I}_{n\times n} & 0 \end{pmatrix}.$$

Doubled coordinate is defined by $\mathbb{X}^I = (X^i, \tilde{X}_i)$ where $\tilde{X}^i = (\tilde{X}_a, X^\nu)$.

D-branes



Neumann boundary condition:

$$\partial_1 X^a|_{\partial\Sigma} = 0.$$

Dirichlet boundary condition:

$$\delta X^\mu|_{\partial\Sigma} = 0.$$

T-duality exchanges Dirichlet and Neumann boundary conditions.

Doubled D-brane

On the doubled space, we can define the corresponding projectors (Albertsson, Kimura and Reid-Edwards, 2009):

- Dirichlet projector: Π,
  Neumann projectors: $\bar{\Pi}$,
- Projectors by definition: $\Pi + \bar{\Pi} = \mathbb{I}$,
- The projectors need to satisfy the following conditions
  1. Normal condition: $\Pi^2 = \Pi$, and $\bar{\Pi}^2 = \bar{\Pi}$.
  2. Orthoganality condition: $\bar{\Pi}\mathbb{H}\Pi = 0$.
  3. Integrability condition: $\bar{\Pi}^K{}_I \bar{\Pi}^L{}_J \partial_{[K} \bar{\Pi}^M_{L]} = 0$.

Boundary conditions

The Dirichlet projector is used to express the Dirichlet boundary conditions in a covariant way:

$$\Pi^t \partial_0 \mathbb{X}|_{\partial_\Sigma} = 0,$$

While the Neumann projectors give rise to Neumann boundary conditions:

$$\bar{\Pi}^t \mathbb{H} \partial_1 \mathbb{X}|_{\partial_\Sigma} = 0.$$

Here the self-duality condition is used to eliminate half of the degrees of freedom:

$$\partial_\alpha \mathbb{X}^I = \epsilon_\alpha{}^\beta \mathbb{L}^{IJ} \mathbb{H}_{JK} \partial_\beta \mathbb{X}^K.$$

T-duality transformation

Let $h \in O(n, n; Z)$.
The doubled coordinate, generalized metric and Dirichlet/Neumann projectors transform under T-duality via

$$
\begin{aligned}
\mathbb{H}_{IJ} \mapsto \quad & \tilde{\mathbb{H}}_{IJ} = (h^{-1}\mathbb{H}h)_{IJ}, \\
\mathbb{X}^I \mapsto \quad & \tilde{\mathbb{X}}^I = (h^{-1})^I{}_J \mathbb{X}^J, \\
\Pi \mapsto \quad & \tilde{\Pi} = h^{-1}\Pi\, h, \\
\bar{\Pi} \mapsto \quad & \tilde{\bar{\Pi}} = h^{-1}\bar{\Pi}\, h.
\end{aligned}
$$

2-Dimensional Example

Consider a 2-dimensional model with

$$\mathbb{H} = \begin{pmatrix} R^2 & 0 \\ 0 & R^{-2} \end{pmatrix}, \quad \mathbb{L} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

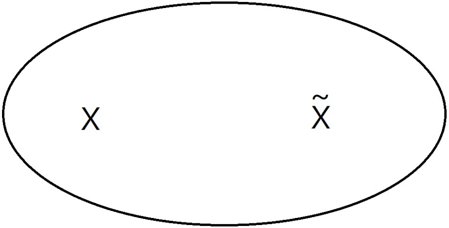
and double coordinates $X = (x, \tilde{x})^t$.
The possible allowed Dirichlet projectors are

$$\Pi^{(1)} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \Pi^{(2)} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \bar{\Pi}^{(1)}.$$

The self-duality condition $\partial_\alpha \mathbb{X}^I = \epsilon_\alpha{}^\beta \mathbb{L}^{IJ} \mathbb{H}_{JK} \partial^\beta \mathbb{X}^K$ gives

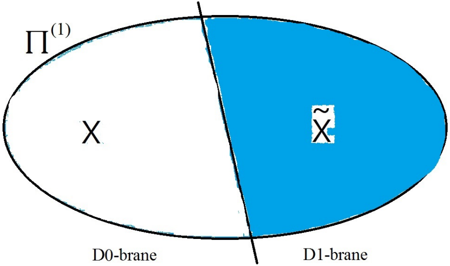
$$\partial_0 x = R^{-2} \partial_1 \tilde{x}, \quad \partial_0 \tilde{x} = R^2 \partial_1 x.$$

Case I: Solving the boundary conditions along with the self-duality condition for $\Pi^{(1)}$, we find



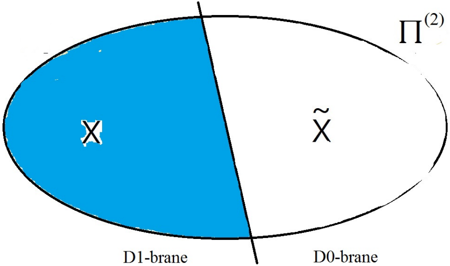
$$\partial_0 x = 0, \quad \partial_1 \tilde{x} = 0,$$

Case I: Solving the boundary conditions along with the self-duality condition for $\Pi^{(1)}$, we find



$$\partial_0 x = 0, \quad \partial_1 \tilde{x} = 0,$$

Case II: Solving the boundary conditions along with the self-duality condition for $\Pi^{(2)}$, we find



$$\partial_1 x = 0, \quad \partial_0 \tilde{x} = 0,$$

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Generalized geometry

Generalized geometry was first developed by Hitchin to unify both symplectic geometry and complex geometry.
Generalized geometry has been of great interest due to emerging connections with areas of mathematical physics, for instance:

- Relation with string theory, ex. B-field symmetries,
- Connection with Mirror symmetry
- Adaptation of T-duality to generalized geometry

Reference:

Hitchin math.DG/0209099
Gualtieri math.DG/0401221
Cavalcanti math.DG/0501406

Generalized tangent space

Let M be a manifold. $TM \oplus T^*M$ is called the Generalized tangent space of M.
There are two natural operations on $TM \oplus T^*M$:
(1) $TM \oplus T^*M$ has a natural symmetric non-degenerate bilinear form defined by

$$\langle X + \xi, Y + \eta \rangle = \frac{1}{2}(\imath_Y \xi + \imath_X \eta)$$

where $X, Y \in \Gamma(TM)$, and $\xi, \eta \in \Gamma(T^*M)$.
(2) Courant bracket:
The canonical bracket originally introduced by Courant is:

$$[\![x + \xi, y + \eta]\!] = [x, y] + \mathcal{L}_x \eta - \mathcal{L}_y \xi + \frac{1}{2} d(\imath_y \xi - \imath_x \eta).$$

Properties of the Courant bracket

- A Courant bracket is skew-symmetric
- It does not satisfy the Jacobi-identity:
  Let $A, B, C \in \Gamma(TM) \oplus \Gamma(T^*M)$, and $f \in C^\infty M$, define

  $$\mathrm{Jac}(A, B, C) = [\![[\![A, B]\!], C]\!] + cycl = d\mathrm{Nij}(A, B, C),$$

  here

  $$\mathrm{Nij}(A, B, C) = \frac{1}{3}(\langle [\![A, B]\!], C\rangle + cycl.)$$

- It does not in general satisfy the Leibnitz rule: Let $\rho : TM \oplus T^*M \to TM$ be an anchor, then

  $$[\![A, fB]\!] = f[\![A, B]\!] + (\rho(A)f)B - \langle A, B\rangle df.$$

- $\exists$ an automorphism defined by $B \in \wedge^2 T^*M$, $dB = 0$.

B-field transform and β-transform

Let B be a smooth 2-form which maps $TM \to T^*M$ via the interior product $x \mapsto \imath_x B$. There is an infinitesimal transformation given by

$$e^B = \begin{pmatrix} 1 & 0 \\ B & 1 \end{pmatrix} : x + \xi \mapsto x + \xi + \imath_X B$$

β-transform is another important symmetry given by $\beta \in \wedge^2(TM)$:

$$e^\beta = \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix} : x + \xi \mapsto x + \xi + \imath_\xi \beta.$$

e^B and e^β are both elements of the special orthogonal group $SO(TM \oplus T^*M) \cong SO(n,n)$ which preserves the natural pairing $\langle \, , \, \rangle$.

Courant algebroid

A Courant algebroid over a manifold M is a vector bundle $E \to M$ equipped with
- a nondegenerate symmetric bilinear form $\langle \cdot, \cdot \rangle$,
- a Courant bracket,
- an anchor $\rho : E \to TM$.

Example:
$TE \oplus T^*E$ with the natural pairing, trivial anchor map and the original Courant bracket is a Courant algebroid.

Dirac structure

$L \in TM \oplus T^*M$ is a Dirac structure if (1) L is maximally isotropic, (2) L is involutive, i.e. $[\![\Gamma(L), \Gamma(L)]\!] \in \Gamma(L)$.
Examples of Dirac structures are:

- TM and T^*M.
- $\oplus_p T_p \oplus_q T^*_q$, where $p + q = dim(M)$.
- $e^B(TM)$.
- $e^\beta(T^*M)$.

In generalized geometry, D-branes are described by Dirac structures.

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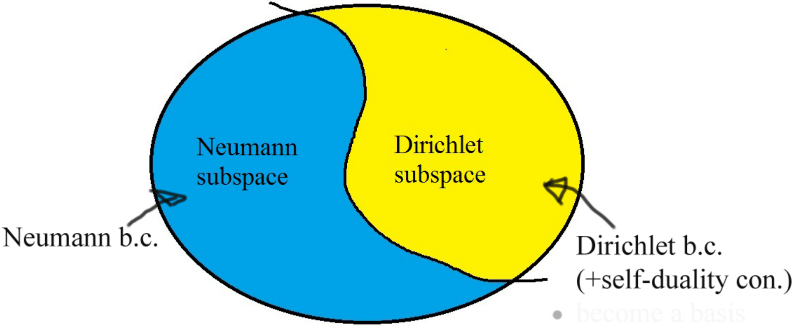
Doubled D-branes in $TM \oplus T^*M$



Procedures: (1) The Neumann boundary condition

$$\bar{\Pi}^t \mathbb{H} \partial_1 \mathbb{X}|_{\partial_\Sigma} = 0$$

gives us a basis on the tangent part of the doubled space, i.e.

$$X := x_I (\bar{\Pi}^t \mathbb{H} \partial_{\mathbb{X}})^I.$$

while the Dirichlet boundary condition

$$\Pi^t \partial_0 \mathbb{X}|_{\partial_\Sigma} = 0$$

gives us a basis on the cotangent part of the doubled space, i.e.

$$\Xi := \xi^I (\Pi^t d\mathbb{X})_I.$$

(2) $\{X + \Xi\}$ requiring

$$\partial_{\tilde{x}_i} \mapsto dx_i, \quad d\tilde{x}_i \mapsto \partial_{x_i}$$

becomes a basis of $TM \oplus T^*M$.

A 4-dimensional Example

Let us consider a 4-dimensional example.
We start with flat metric and constant B-field, i.e.

$$G = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & b \\ -b & 0 \end{pmatrix},$$

it follows that the generalized metric is given by

$$\mathbb{H} = \begin{pmatrix} 1+b^2 & 0 & 0 & b \\ 0 & 1+b^2 & -b & 0 \\ 0 & -b & 1 & 0 \\ b & 0 & 0 & 1 \end{pmatrix}.$$

A 4-dimensional Example

Solutions	D-brane	T-dual	Generalized space
Π_1	D_1 $(\{X, Y\})$	$\bar{\Pi}_1$	$e^B(TM) \oplus e^\beta(T^*M)$
Π_2	D_2	Π_3	$TM \oplus e^\beta(T^*M)$
Π_3	D_0	Π_2	$e^B(TM) \oplus T^*M$
Π_4	D_2	Π_5	$TM \oplus e^{\beta'}(T^*M)$
Π_5	D_0	Π_4	$e^{B'}(TM) \oplus T^*M$
Π_6	D_1 (X)	Π_7	$e^B(T_Y) \oplus T_X \oplus e^\beta(T^*_X) \oplus T^*_Y$
Π_7	D_1 (Y)	Π_6	$T_X \oplus e^\beta(T^*_Y) \oplus T^*_X \oplus e^B(T_X) \oplus T_Y$
Π_8	D_1 $(\{X, Y\})$	$\bar{\Pi}_1$	$e^\theta(TM) \oplus e^\Theta(T^*M)$

A 6-dimesional Example

Similarly we consider a 6-dimensional example.
We start with flat metric and constant B-field, i.e.

$$G = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & b\,z & -b\,y \\ -b\,z & 0 & b\,x \\ b\,y & -b\,x & 0 \end{pmatrix},$$

it follows that the generalized metric is given by $\mathbb{H} =$

$$\begin{pmatrix} 1 + b^2y^2 & -b^2xy & -b^2xz & 0 & bz & -by \\ -b^2xy & 1 + b^2z^2 + b^2x^2 & -b^2yz & -bz & 0 & bx \\ -b^2xz & -b^2yz & 1 + b^2x^2 + b^2y^2 & by & -bx & 0 \\ 0 & -bz & by & 1 & 0 & 0 \\ bz & 0 & -bx & 0 & 1 & 0 \\ -by & bx & 0 & 0 & 0 & 1 \end{pmatrix}$$

A 6-dimensional example

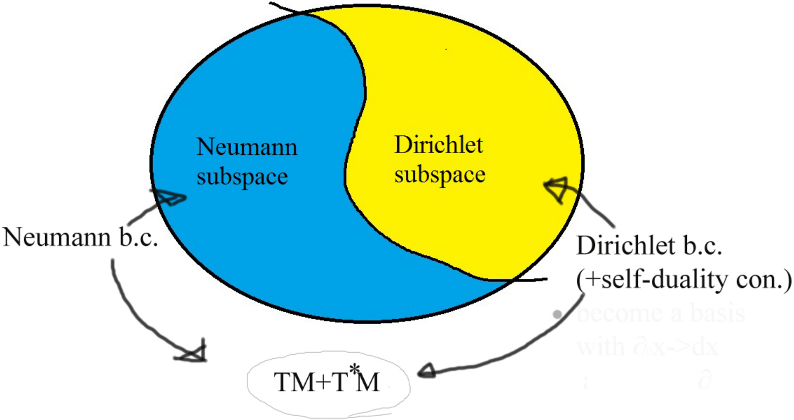
Solutions	D-brane	T-dual	Generalized space
Π_1	D_0	Π_2	$e^B(TM) \oplus T^*M$
Π_2	D_3	Π_1	$TM \oplus e^\beta(T^*M)$
Π_3	$D_1\ (X)$	Π_4	$e^\beta(T^*_X) \oplus T^*_{Y,Z} \oplus e^B(T_{Y,Z}) \oplus T_X$
Π_4	$D_2\ (\{Y,Z\})$	Π_3	$e^B(T_X) \oplus T_{Y,Z} \oplus e^\beta(T^*_{Y,Z}) \oplus T^*_X$
Π_5	$D_1\ (Z)$	Π_6	$e^B(T_{X,Y}) \oplus T_Z \oplus e^\beta(T^*_Z) \oplus T^*_{X,Y}$
Π_6	$D_2\ (\{X,Y\})$	Π_5	$e^\beta(T^*_{X,Y}) \oplus T^*_Z \oplus e^B(T_Z) \oplus T_{X,Y}$
Π_7	$D_1\ (Y)$	Π_6	$e^\beta(T^*_Y) \oplus T^*_{X,Z} \oplus e^B(T_{X,Z}) \oplus T_Y$
Π_8	$D_2(\{X,Z\})$	Π_7	$e^B(T_Y) \oplus T_{X,Z} \oplus e^\beta(T^*_{X,Z}) \oplus T^*_Y$

Observations

We observe that:

- Doubled D-branes in doubled geometry are equivalent to a Courant algeboid composed of a pair of Dirac structures in generalized geometry, which can be further classified into the following categories:
    - $TM \oplus T^*M$
    - $e^B(TM) \oplus T^*M$
    - $TM \oplus e^\beta(T^*M)$
    - $L \oplus \tilde{L}$ where $L = \oplus_p T_p \oplus_q T_q^*$ and $\tilde{L} = e^B(\oplus_p T_p^*) \oplus e^\beta(\oplus_q T_q)$ requiring $p + q = dim(M)$ and $dB = d\beta = 0$.
- Π and $\tilde{\Pi} \equiv \bar{\Pi}$ in doubled geometry corresponds to $TM \leftrightarrow T^*M$ in generalized geometry.
- B-trasnform reduces the dimension of a D-brane while β-transform increases the dimension of a D-brane.

Conclusion


Neumann subspace
Dirichlet subspace
Neumann b.c.
Dirichlet b.c.
(+self-duality con.)
TM+T*M

Thank you!