# The study of thermal Skyrmions in Yang-Mills theory

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## Introduction

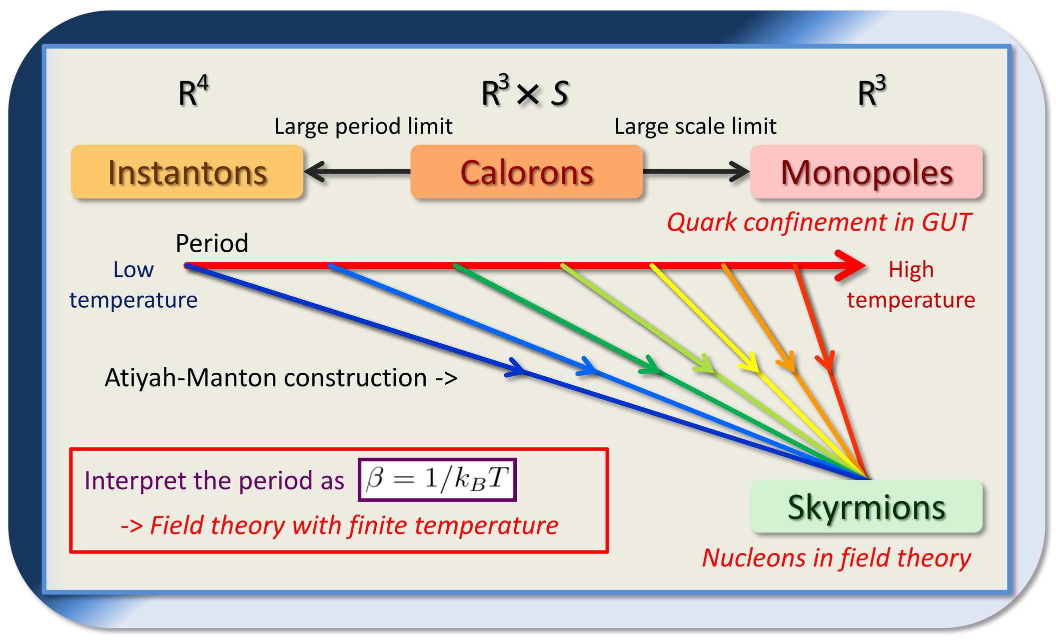
Recently some significant features of calorons are revealed.

The calorons are periodic instantons in Yang-Mills theory, which interpolate between instantons and monopoles in some limits (figure 1). Since the period can be interpreted as temperature in physical context, the study of calorons gives us some knowledge of the field theory with finite temperature. In particular, it is known that the Skyrmions, which are model of nucleons, can be constructed from the instantons with Atiyah-Manton ansatz. If we apply calorons to the construction, we can obtain new Skyrmions with finite temperature.

In this poster we explain the formulations of the above topics.

## Introduction

figure 1. Relations between the solitons in Yang-Mills theory.




## The Skyrme model

The Skyrme model can be written as follows.

$$L_{\mathrm{Skrm}} = \int \left\{ \frac{f_\pi^2}{16} \mathrm{Tr}(\partial_\mu U \partial^\mu U^\dagger) + \frac{1}{32e^2} \mathrm{Tr}([\partial_\mu U U^\dagger, \partial_\nu U U^\dagger][\partial^\mu U U^\dagger, \partial^\nu U U^\dagger]) \right\} \mathrm{d}^3 x.$$

where $f_\pi$ is the pion decay constant, $e$ is a parameter of the model and $U \in \mathrm{SU}(2)$ is the Skyrme field. The field can also be written as

$$U = \frac{1}{f_\pi}(\sigma + \mathrm{i}\boldsymbol{\pi} \cdot \boldsymbol{\tau}), \qquad \sigma^2 + \boldsymbol{\pi}^2 = f_\pi^2.$$

where $\sigma$ is the scalar field, $\boldsymbol{\pi} = (\pi_1, \pi_2, \pi_3)$ are the pion fields and $\boldsymbol{\tau} = (\tau_1, \tau_2, \tau_3)$ are the Pauli matrices. Although this model includes only pions and there is no quark, it is known that the model can describe baryons with soliton solutions, which are called Skyrmions. Note that the number of flavor is $N_f = 2$.

## The Skyrme model

The Skyrme field equation can be written as

$$\partial_\mu \left( R^\mu + \frac{1}{4}[R^\nu, [R_\nu, R^\mu]] \right) = 0.$$

where $R_\mu = \partial_\mu U U^\dagger \in \mathrm{su}(2)$ are the right-currents. We also impose the following boundary condition.

$$U(\boldsymbol{x}, t) \to \mathbf{1}_2 \quad \text{as} \quad |\boldsymbol{x}| \to \infty.$$

Here the map $U$ from $S^3$ to $S^3$ can be classify with an integer *B*, which turns out to be the baryon number. In static case, the baryon number and the energy can be written as follows,

$$B = -\frac{1}{24\pi^2} \int \varepsilon_{ijk} \mathrm{Tr}(R_i R_j R_k) \, \mathrm{d}^3 x,$$
$$E = \frac{1}{12\pi^2} \int \left\{ -\frac{1}{2} \mathrm{Tr}(R_i R_i) - \frac{1}{16} \mathrm{Tr}([R_i, R_j][R_i, R_j]) \right\} \mathrm{d}^3 x$$

## Hedgehogs

In the case of *B*=1, the field can be obtained with following ansatz,

$$U(\boldsymbol{x}) = \exp\{\mathrm{i} f(r) \hat{\boldsymbol{x}} \cdot \boldsymbol{\tau}\} = \cos f + \mathrm{i} \sin f \frac{x_a}{r} \tau_a.$$

Substituting this, one can obtain

$$R_i = \mathrm{i} \left\{ f' \frac{x_i x_a}{r^2} + \frac{\sin 2f}{2r} \left( \delta_{ia} - \frac{x_i x_a}{r^2} \right) - \frac{\sin^2 f}{r} \varepsilon_{iab} \frac{x_b}{r} \right\} \tau_a$$

$$B = -\frac{1}{2\pi^2} \int \frac{\sin^2 f f'}{r^2} \mathrm{d}^3 x \qquad E = \frac{1}{3\pi} \int \left\{ r^2 f'^2 + 2 \sin^2 f (1 + f'^2) + \frac{\sin^4 f}{r^2} \right\} \mathrm{d}r.$$

and the Skyrme field equation is

$$\left( 1 + \frac{2 \sin^2 f}{r^2} \right) f'' + \frac{2f'}{r} + \left( f'^2 - 1 - \frac{\sin^2 f}{r^2} \right) \frac{\sin 2f}{r^2} = 0.$$

which can be solved numerically (figure. 2).

## Hedgehogs

In this case the quantization can be obtained explicitly and the experimental data can be evaluated as table 1.

figure 2. The profile function of *B*=1 hedgehog.

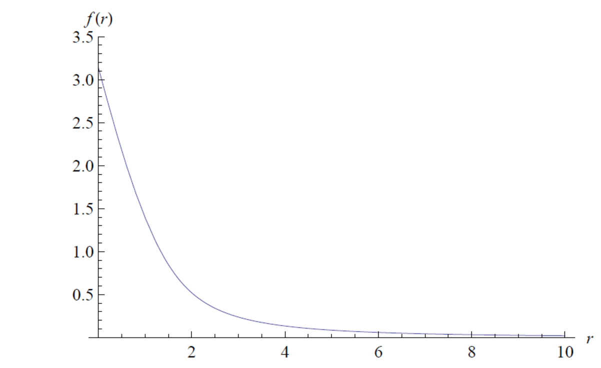


Table 1. Experimental data.

| | A.N.W. | Expt |
|---|---|---|
| $f_\pi$ [MeV] | 129 | 186 |
| $e$ | 5.45 | |
| $M_{\mathrm{N}}$ [MeV] | input | 939 |
| $M_\Delta$ [MeV] | input | 1232 |
| $g_{\mathrm{N}}^{\mathrm{A}}$ | 1.02 | 1.26 |
| $\langle r_E^2 \rangle_0^{1/2}$ [fm] | 0.59 | 0.72 |
| $\langle r_E^2 \rangle_1^{1/2}$ [fm] | $\infty$ | 0.88 |
| $\langle r_M^2 \rangle_0^{1/2}$ [fm] | 0.92 | 0.81 |
| $\langle r_M^2 \rangle_1^{1/2}$ [fm] | $\infty$ | 0.80 |
| $\mu_{\mathrm{p}}$ [n.m.] | 1.87 | 2.79 |
| $\mu_{\mathrm{n}}$ [n.m.] | $-1.31$ | $-1.91$ |

The nucleon mass and the delta mass.

$$M_{\mathrm{N}} = M + \frac{1}{2\lambda}\frac{3}{4}, \qquad M_\Delta = M + \frac{1}{2\lambda}\frac{15}{4},$$
$$\lambda = \frac{16}{3}\pi \left( \frac{1}{f_\pi e^3} \right) \Lambda, \qquad \Lambda = \int r^2 \sin^2 f \left( 1 + f'^2 + \frac{\sin^2 f}{r^2} \right) \mathrm{d}r$$

G.S. Adkins, C.R. Nappi, E. Witten, Nucl.Phys. B228 (1983) 552.

## The Atiyah-Manton construction

It is known that the holonomy or Wilson line of instantons

$$\Omega(\boldsymbol{x}, t) = \pm \mathcal{P} \exp \left( -\int_{-\infty}^{t} A_t(\boldsymbol{x}, t') \, \mathrm{d}t' \right)$$

can be regarded as a Skyrme field, in the sence that if we put $U(\boldsymbol{x}) = \Omega(\boldsymbol{x}, \infty)$ , then

- $U(\boldsymbol{x}) \in \mathrm{SU}(2)$ because $A_t(\boldsymbol{x}, t) \in \mathrm{su}(2)$ .
- $U(\boldsymbol{x})$ satisfy the boundary condition of Skyrmions because of the boundary condition of instantons.
- $U(\boldsymbol{x})$ has the same topological charge of the instantons.

This formulation is called the Atiyah-Manton construction or Atiyah-Manton ansatz (c.f. M.F. Atiyah, N.S. Manton, Phys.Lett. B222 (1989) 438-442).

The holonomy can be calculated analytically in the case of *k*=1, and higher charge cases can also be calculated numerically.

## Skyrmion like configuration from *k*=1 instanton

Some gauge fields of the instantons can be obtained from the following 't Hooft ansatz or CFtHW ansatz,

$$A_t(x) = \frac{\mathrm{i}}{2} \frac{\boldsymbol{\nabla} \rho}{\rho} \cdot \boldsymbol{\tau}$$

In the case of *k*=1, the form can be written as

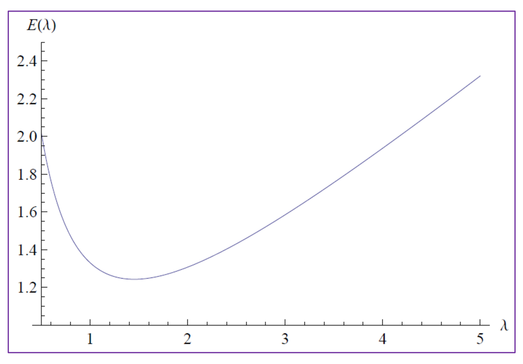
$$\rho(x) = 1 + \frac{\lambda^2}{r^2 + t^2}, \qquad A_t = \mathrm{i} \left( \frac{1}{r^2 + t^2 + \lambda^2} - \frac{1}{r^2 + t^2} \right) \boldsymbol{x} \cdot \boldsymbol{\tau}.$$

In this case the holonomy can be obtained explicitly as follows,

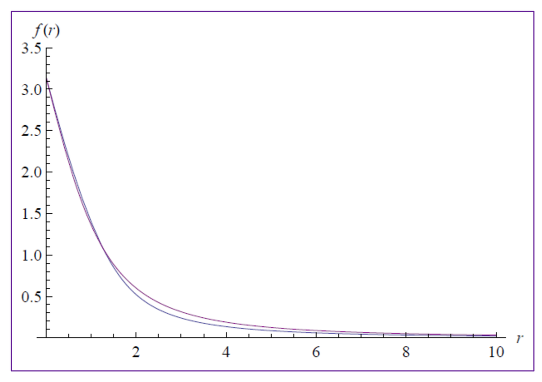
$$U(\boldsymbol{x}, \infty) = \exp\{\mathrm{i} f(r) \hat{\boldsymbol{x}} \cdot \boldsymbol{\tau}\}, \qquad f(r) = \pi \left( 1 - \frac{r}{\sqrt{r^2 + \lambda^2}} \right).$$

The energy depends on the scale parameter $\lambda$ . The true energy is *E*=1.232, while the minimum energy of the solution is *E*=1.243.

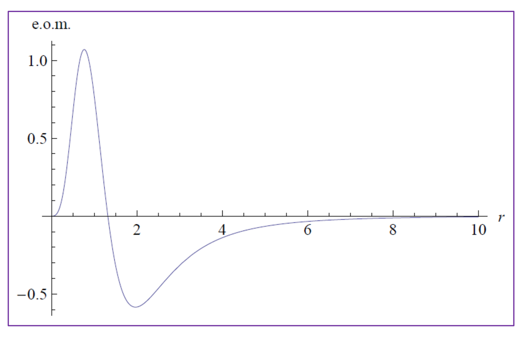
## Skyrmion like configuration from *k*=1 instanton



<- The energy functional.



The profile function with minimum energy. The blue curve is the one of *B*=1 hedgehog solution.

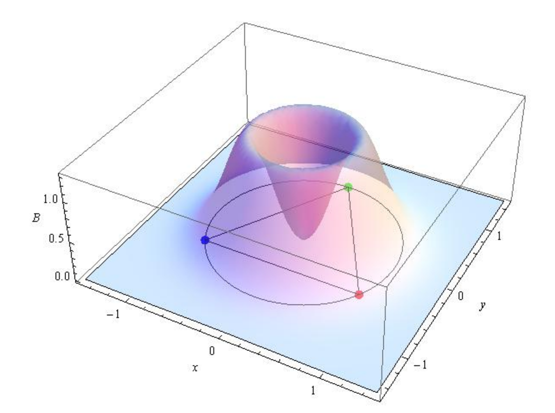


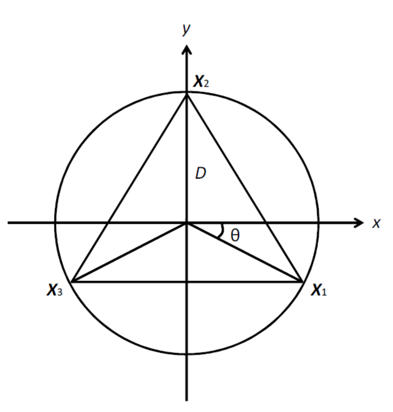
<- The error from the Skyrme field eq. with minimum energy.

## Skyrmion like configuration from *k*=2 instanton

The holonomy of *k*=2 case can also be obtained from CFtHW ansatz. For example, the following superpotential leads the Skyrmion like configuration of torus form or geodesic motion of two Skyrmions.

$$\rho(\boldsymbol{x}, t) = \frac{1}{(\boldsymbol{x} - \boldsymbol{X}_1)^2 + t^2} + \frac{\lambda}{(\boldsymbol{x} - \boldsymbol{X}_2)^2 + t^2} + \frac{1}{(\boldsymbol{x} - \boldsymbol{X}_3)^2 + t^2}$$



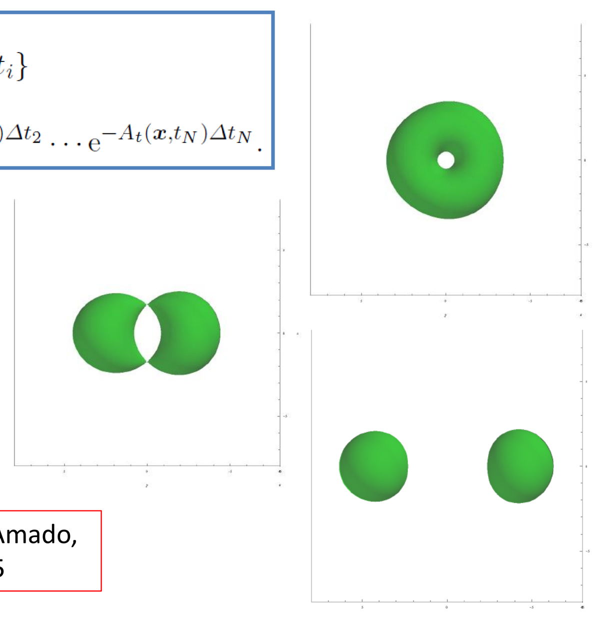



## Skyrmion like configuration from *k*=2 instanton

The holonomy can be computated with following approximation.

$$\Omega(\boldsymbol{x}, \infty) \simeq \pm \prod_{i=1}^{N} \exp\{-A_t(\boldsymbol{x}, t_i) \Delta t_i\}$$
$$= \pm \mathrm{e}^{-A_t(\boldsymbol{x}, t_1)\Delta t_1} \mathrm{e}^{-A_t(\boldsymbol{x}, t_2)\Delta t_2} \cdots \mathrm{e}^{-A_t(\boldsymbol{x}, t_N)\Delta t_N}.$$

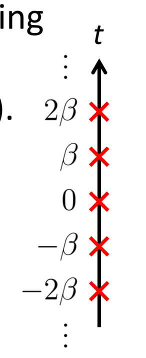
| $1/\lambda$ | $D$ | range | $B$ | $E$ (MeV) |
|---|---|---|---|---|
| 1 | 3.42 | 5.4 | 1.976 | 1585 |
| 2 | 3.76 | 6.0 | 1.977 | 1613 |
| 3 | 4.26 | 7.0 | 1.980 | 1649 |
| 5 | 5.16 | 8.0 | 1.975 | 1672 |
| 10 | 6.92 | 10.0 | 1.970 | 1707 |
| 20 | 9.52 | 14.0 | 1.970 | 1818 |



Hosaka, S.M. Griffies, M. Oka, R.D. Amado, Phys.Lett. B251 (1990) 1-5

## The Atiyah-Manton construction for calorons

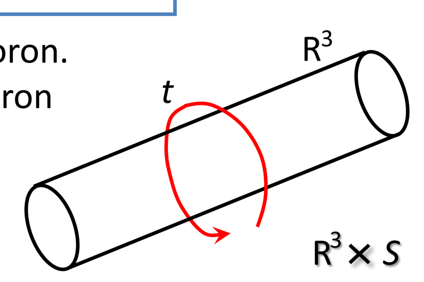
The Atiyah-Manton construction can also be applied for calorons. Calorons are periodic instantons. For example, the following superpotential leads Harrington-Shepard 1-caloron (c.f. B. J. Harrington and H. K. Shepard, Phys. Rev. D 17, 2122 (1978)).




$$\rho(\boldsymbol{x}, t) = 1 + \sum_{n=-\infty}^{\infty} \frac{\lambda^2}{r^2 + (t - n\beta)^2}$$
$$= 1 + \frac{\mu \lambda^2}{2r} \frac{\sinh(\mu r)}{\cosh(\mu r) - \cos(\mu t)}.$$

where $\beta = 2\pi/\mu$ is the period of the caloron. The holonomy or Wilson loop of the caloron can be written as

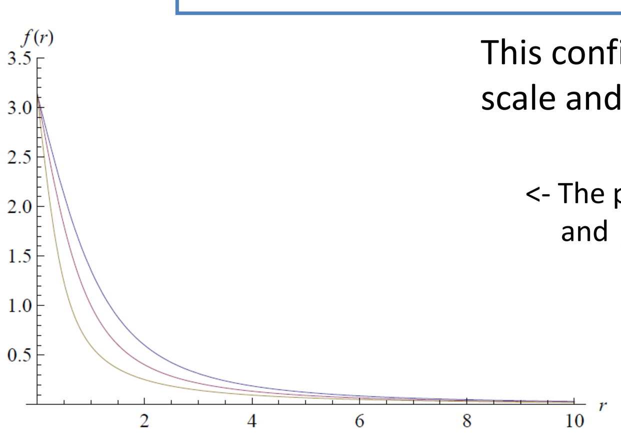
$$\Omega_\beta(x^j) = \mathcal{P} \exp \left[ -\int_0^\beta A_t(x^j, t) \, \mathrm{d}t \right]$$




## Skyrmion like configuration from *k*=1 caloron

In the case of *k*=1, the holonomy can be obtained analytically in the same form as the hedgehog solution but the profile function is

$$f(r) = \pi \left( 1 - \frac{r + \frac{1}{2}\lambda^2(\mu \coth(\mu r) - 1/r)}{\sqrt{r^2 + \frac{1}{4}\mu^2 \lambda^4 + \mu r \lambda^2 \coth(\mu r)}} \right).$$



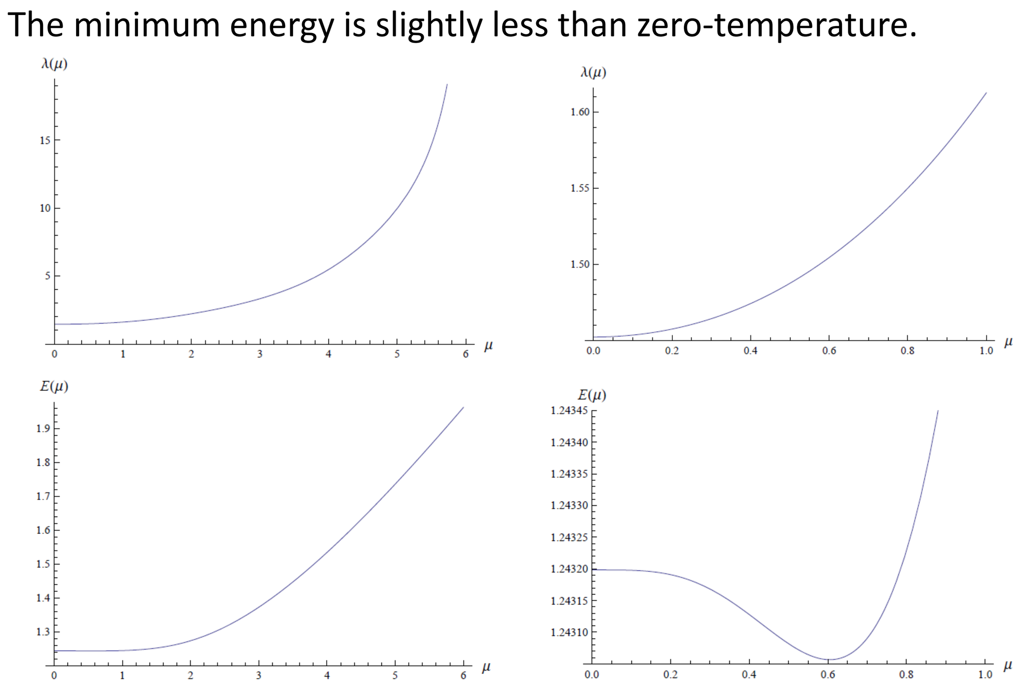
This configuration depends on both scale and period (temperature).

<- The profile function with $\lambda = 1.453$ and $\mu = 0, 2.0, 2\pi/\lambda$.

K. J. Eskola and K. Kajantie, Z. Phys. C 44, 347 (1989).

## Skyrmion like configuration from *k*=1 caloron

The minimum energy is slightly less than zero-temperature.



## Summary

- Skyrmion like configurations can be obtained with Atiyah-Manton ansatz which satisfy the same boundary condition of Skyrmions and have the same topological charge of instantons or calorons.
- Some applications can be considered. For example, the Harrington-Shepard caloron can be generalized to the case of multi-caloron. Moreover, it is known that the calorons with higher charge can be obtained with Nahm construction. We would like to generalize the calculation of holonomy of calorons with higher charge, which may need more precise treatment, e.g. for the periodic boundary condition of calorons.