A twistorial model for massive spinning particles

Satoshi Okano (D2, Nihon U.) in collaboration with Shinichi Deguchi (IQS, Nihon U.) @ YITP, July 25, 2014

0. Introduction

- Various classical-mechanical models for spinning particle:

✓ Rigid body model, Barut-Zanghi model (commutative variables)
✓ *"Spinning particle"* model, Superparticle model
(anti-commutative variables)

- Twistor theory

Twistor theory is a useful tool to describe *massless* particles.

The action for *massless spinning* particles is given by

$$S_0=\int d\tau\left[i\bar{Z}_A\dot{Z}^A+a\left(\bar{Z}_AZ^A-2s\right)\right],\ \dot{}=\frac{d}{d\tau},\ Z^A=Z^A(\tau),\ a=a(\tau),$$

(T.Shirafuji, 1983, I. Bars and M. Picon, 2006; S. Deguchi, T. Egami and J. Note, 2010)

where, s is a real constant specifying the helicity.
The model is investigated in terms of spacetime and spinor variables.

(S. Deguchi, S. Negishi, S.O. and T. Suzuki, Int. J. Mod. Phys. A 29 (2014) 1450044)

- By contrast, to describe a **massive particle**, it is necessary to introduce more than two twistor variables

$$Z_i{}^A\equiv(\omega_i{}^\alpha,\pi_{i\dot{\alpha}})\,,\qquad i=1,2,\ldots N\ .$$

with a SU(N) symmetry.

...What's the physical meaning of SU(N) symmetry ?

Penrose, Perjes and Hughston attempted to classify elementary particles.

$$\left\{\begin{array}{ll}\text{SU(2)}\Rightarrow\text{weak isospin}: & (e^-\,,\ \nu_e)\\ \text{SU(3)}\Rightarrow\text{flavor symmetry}: & (u\,,\ d\,,\ s)\\ \vdots & \vdots\end{array}\right.$$

(Penrose 1975, Parjes 1975, Hughston 1979)

We present a model for massive spinning particles and then investigate the validity of the attempt within the model.

1. A model for *massive spinning* particles

- We first consider the following SU(2) invariant action:

$$S=\int d\tau\left[i\bar{Z}^i_A\dot{Z}^A_i+a\left(\bar{Z}^i_AZ^A_i-2s\right)+\bar{Z}^i_Ab_i{}^jZ^A_j+\frac{f}{2}\left(\bar{\pi}^{i\alpha}\pi_i^{\dot{\alpha}}\bar{\pi}^k_\alpha\pi_{k\dot{\alpha}}-m^2\right)\right].$$

$b_i{}^j(\tau)=b^r(\tau)\sigma_{ri}{}^j$: SU(2) gauge fields, $\quad\sigma_r(r=1,2,3)$: Pauli matrices

$$\left(Z^A_i\to U_i{}^j(\tau)Z^A_j,\quad \bar{Z}^i_A\to\bar{Z}^j_AU^\dagger{}_j{}^i(\tau)\,,\ b\to U(\tau)bU^\dagger(\tau)-i\dot{U}(\tau)U^\dagger(\tau)\,,\quad U\in SU(2).\right)$$

But, this action describes only massive *spinless* particles due to too strong constraints.

- Coset construction and modifying the action

We consider the coset space SU(2)/U(1) and representative elements $V(\xi,\bar{\xi})$.
The left action of U on V generates a transformation in accordance with

$$V\to V(\xi',\bar{\xi}')=U(\tau)V(\xi,\bar{\xi})\Theta^{-1}(\tau),$$

where $\Theta(\tau):=\exp\left(i\vartheta(\tau)\sigma_3\right)$, and ϑ is a real gauge function.
Using V, we define the new variables

$$\tilde{Z}^A_i:=V^\dagger{}_i{}^jZ^A_j\,,\quad \tilde{\bar{Z}}{}^i_A:=\bar{Z}^j_AV_j{}^i\,,\quad \tilde{b}:=V^\dagger bV-i\dot{V}^\dagger V.$$

With the new variables, the SU(2) transformation reads

$$\tilde{Z}_i{}^A\to\Theta_i{}^j\tilde{Z}^A_j\,,\quad \tilde{\bar{Z}}{}^i_A\to\tilde{\bar{Z}}{}^j_A\Theta^\dagger{}_j{}^i,\quad \tilde{b}^1\to\tilde{b}^1\cos2\vartheta+\tilde{b}^2\sin2\vartheta\,,\quad \tilde{b}^2\to-\tilde{b}^1\sin2\vartheta+\tilde{b}^2\cos2\vartheta,$$
$$\tilde{b}^3\to\tilde{b}^3+\dot{\vartheta}.$$

We modify the action with the new variables as

$$S=\int d\tau\left[i\bar{Z}^i_A\dot{Z}^A_i+a\left(\bar{Z}^i_AZ^A_i-2s\right)+b^3\left(\bar{Z}^i_A\sigma_{3i}{}^jZ^A_j-2t\right)\right.$$
$$\left.+b^{\imath}\left(\bar{Z}^i_A\sigma_{\imath i}{}^jZ^A_j+\frac{k}{2}b^{\imath}\right)+\frac{f}{2}\left(\bar{\pi}^{i\alpha}\pi_i^{\dot{\alpha}}\bar{\pi}^k_\alpha\pi_{k\dot{\alpha}}-m^2\right)\right].\qquad(\imath=1,2)$$

where t is a real constant.

- Rewrite S in terms of spacetime and spinor variables

We express ω^α_i as $\omega^\alpha_i=ix^{\alpha\dot{\alpha}}\pi_{i\dot{\alpha}}+\psi^\alpha_i$, and then the action $S(x^{\alpha\dot{\alpha}},\bar{\pi}^i_\alpha,\pi_{i\dot{\alpha}},\psi^\alpha_i,\bar{\psi}^{i\dot{\alpha}},f,a,b^3,b^{\imath})$ can be written as $S=\int d\tau L$ with

$$L=-\dot{x}^{\alpha\dot{\alpha}}\bar{\pi}^i_\alpha\pi_{i\dot{\alpha}}-i(\psi^\alpha_i\dot{\bar{\pi}}^i_\alpha-\bar{\psi}^{i\dot{\alpha}}\dot{\pi}_{i\dot{\alpha}})+a(\bar{\pi}^i_\alpha\psi^\alpha_i+\bar{\psi}^{i\dot{\alpha}}\pi_{i\dot{\alpha}}-2s)$$
$$+b^3\sigma_{3i}{}^j\left(\bar{\pi}^i_\alpha\psi^\alpha_j+\bar{\psi}^{i\dot{\alpha}}\pi_{j\dot{\alpha}}-2t\right)+b^{\imath}\left(\bar{\pi}^i_\alpha\sigma_{\imath i}{}^j\psi^\alpha_j+\bar{\psi}^{i\dot{\alpha}}\sigma_{\imath i}{}^j\pi_{j\dot{\alpha}}+kb^{\imath}/2\right)$$
$$+\frac{f}{2}\left(\bar{\pi}^{i\alpha}\pi_i^{\dot{\alpha}}\bar{\pi}^j_\alpha\pi_{j\dot{\alpha}}-m^2\right).$$

We investigate our model based on this Lagrangian.

2. Canonical quantization

- Classification of the constraints in Hamiltnian formalism.

5 first-class constraints
4 second-class constraints ⟹ Define the Dirac bracket.

- Canonical quantization

Dirac bracket ⟹ Canonical commutation relation.
The first-class constraints ⟹ Define the physical states : $\hat{\phi}|\Phi\rangle=0.$

$$\left(-i\frac{\partial}{\partial x^{\alpha\dot{\alpha}}}+\bar{\pi}^i_\alpha\pi_{i\dot{\alpha}}\right)\Phi=0\,,\qquad\left(\bar{\pi}^{i\alpha}\pi_i^{\dot{\alpha}}\bar{\pi}^j_\alpha\pi_{j\dot{\alpha}}-m^2\right)\Phi=0\,,$$
$$\left(\bar{\pi}^i_\alpha\frac{\partial}{\partial\bar{\pi}^i_\alpha}-\pi_{i\dot{\alpha}}\frac{\partial}{\partial\pi_{i\dot{\alpha}}}-2s\right)\Phi=0,\qquad\sigma_{3i}{}^j\left(\bar{\pi}^i_\alpha\frac{\partial}{\partial\bar{\pi}^j_\alpha}-\pi_{i\dot{\alpha}}\frac{\partial}{\partial\pi_{j\dot{\alpha}}}-2t\right)\Phi=0\,,$$
$$\frac{\partial}{\partial a}\Phi=\frac{\partial}{\partial f}\Phi=\frac{\partial}{\partial b^3}\Phi=0\,.$$

- The solutions of these simultaneously eqs.:

$$\Phi^{i_1\cdots i_m}_{\alpha_1\cdots\alpha_m;\,j_1\cdots j_n\dot{\alpha}_1\cdots\dot{\alpha}_n}=\bar{\pi}^{i_1}_{\alpha_1}\cdots\bar{\pi}^{i_m}_{\alpha_m}\pi_{j_1\dot{\alpha}_1}\cdots\pi_{j_n\dot{\alpha}_n}e^{-ix^{\beta\dot{\beta}}\bar{\pi}^k_\beta\pi_{k\dot{\beta}}},$$

with $s=\dfrac{m_1+m_2-n_1-n_2}{2}$ and $t=\dfrac{m_1-m_2-n_1+n_2}{2}$, $(m:=m_1+m_2,\ n:=n_1+n_2),$

where, (m_1,m_2,n_1,n_2) are respectively the # of $(\bar{\pi}^1_\alpha,\bar{\pi}^2_\alpha,\pi_{1\dot{\alpha}}\pi_{2\dot{\alpha}})$ in the solution.

3. The physical meaning of the SU(2) transformation

- Derivation of the Dirac equation

The solutions in the cases $s=\pm1/2,\quad t=\pm1/2$ are

$$(s,t)=(1/2,1/2):\quad \Phi^1_\alpha=\bar{\pi}^1_\alpha\phi,\qquad(s,t)=(-1/2,1/2):\quad \Phi_{1\dot{\alpha}}=\pi_{1\dot{\alpha}}\phi,$$
$$(s,t)=(1/2,-1/2):\quad \Phi^2_\alpha=\bar{\pi}^2_\alpha\phi,\qquad(s,t)=(-1/2,-1/2):\quad \Phi_{2\dot{\alpha}}=\pi_{2\dot{\alpha}}\phi.$$
$$\phi:=\exp\left(-ix^{\beta\dot{\beta}}\bar{\pi}^k_\beta\pi_{k\dot{\beta}}\right)$$

Using the mass-shell condition, we can show that

$$i\hbar\sqrt{2}\partial^{\alpha\dot{\alpha}}\Phi^i_\alpha-m\epsilon^{ij}\Phi^{\dot{\alpha}}_j=0\,,\qquad i\hbar\sqrt{2}\partial_{\alpha\dot{\alpha}}\epsilon^{ij}\Phi^{\dot{\alpha}}_j-m\Phi^i_\alpha=0\,.$$

These equation can be written as

$$(i\gamma^\mu\partial_\mu-m)\psi(x)=0\,,\quad(i\gamma^\mu\partial'_\mu-m)\psi^C(x')=0\quad(\partial'_\mu:=-\partial_\mu)$$

with

$$\psi(x):=\begin{pmatrix}\Phi^{\dot{\alpha}}_2(x)\\ \Phi^1_\alpha(x)\end{pmatrix},\quad \psi^C(x'):=\mathcal{C}\psi(x')=\begin{pmatrix}\Phi^{\dot{\alpha}}_1(-x')\\ \Phi^2_\alpha(-x')\end{pmatrix}=\begin{pmatrix}\Phi^{\dot{\alpha}}_1(x)\\ \Phi^2_\alpha(x)\end{pmatrix},\quad(x':=-x).$$

- The result is summarized as follows:

	Particle / t = 1/2	Anti-Particle / t = - 1/2
Right-handed / s = 1/2	$\Phi^{\dot{\alpha}}_2(x)$	$\Phi^{\dot{\alpha}}_1(x)$
	SU(2) transformation	
Left-handed / s = - 1/2	$\Phi^1_\alpha(x)$	$\Phi^2_\alpha(x)$

Thus, the SU(2) symmetry is regarded as symmetry between the particle and the anti-particle.

4. The spinor wave functions and the generating function

We consider linear combinations

$$\Psi^{+i_1\ldots i_m}_{\alpha_1\ldots\alpha_m;\,j_1\ldots j_n\,\dot{\alpha}_1\ldots\dot{\alpha}_n}(z):=\frac{(-1)^m}{(2\pi i)^8}\int\tilde{f}^+(\bar{\pi},\pi)\,\Phi^{i_1\cdots i_m}_{\alpha_1\cdots\alpha_m;\,j_1\cdots j_n\dot{\alpha}_1\cdots\dot{\alpha}_n}(z,\bar{\pi},\pi)$$
$$\times d^2\bar{\pi}^1\wedge d^2\bar{\pi}^2\wedge d^2\pi_1\wedge d^2\pi_2$$

and

$$\Psi^{-i_1\ldots i_m}_{\alpha_1\ldots\alpha_m;\,j_1\ldots j_n\,\dot{\alpha}_1\ldots\dot{\alpha}_n}(z):=\frac{1}{(2\pi i)^8}\int\tilde{f}^-(\bar{\pi},\pi)\,\bar{\Phi}^{i_1\cdots i_m}_{\alpha_1\cdots\alpha_m;\,j_1\cdots j_n\dot{\alpha}_1\cdots\dot{\alpha}_n}(z,\bar{\pi},\pi)$$
$$\times d^2\bar{\pi}^1\wedge d^2\bar{\pi}^2\wedge d^2\pi_1\wedge d^2\pi_2$$

The generating function for $\Psi^{\pm i_1\ldots i_m}_{\alpha_1\ldots\alpha_m;\,j_1\ldots j_n\,\dot{\alpha}_1\ldots\dot{\alpha}_n}(z)$ is defined as

$$\Psi^\pm(z,\iota,\kappa):=\sum_{m=0}^\infty\sum_{n=0}^\infty\frac{1}{m!n!}\Psi^{\pm i_1\ldots i_m}_{\alpha_1\ldots\alpha_m;\,j_1\ldots j_n\,\dot{\alpha}_1\ldots\dot{\alpha}_n}(z)\iota^{\alpha_1}_{i_1}\cdots\iota^{\alpha_m}_{i_m}\kappa^{j_1\dot{\alpha}_1}\cdots\kappa^{j_n\dot{\alpha}_n}\,,$$

where ι^α_i and $\kappa^{j\dot{\alpha}}$ arbitrary undotted and dotted spinors, respectively.
$\Psi^\pm(z,\iota,\kappa)$ satisfy the *(complexified)* *unfolded equation:*

$$\left(-i\frac{\partial}{\partial z^{\alpha\dot{\alpha}}}-\frac{\partial^2}{\partial\iota^\alpha_i\partial\kappa^{i\dot{\alpha}}}\right)\Psi^\pm(z,\iota,\kappa)=0\,.$$

5. Summary and future issues

■ Summary

✓ We study a model for massive spinning particles.
✓ We investigated the physical meaning of the SU(2) symmetry and the result is not agree with Penrose's attempt.
✓ We obtained the unfolded equation within the model.

■ Future issues

□ Consider the general physical meaning of s and t, in the model.
□ Incorporate interactions with gauge fields lying in spacetime.