Non-renormalization Theorem and Cyclic Leibniz Rule in Lattice Supersymmetry

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in collaboration with M. Kato and H. So
based on JHEP 1305(2013)089; arXiv:1311.4962;
and in progress

Supersymmetry on lattice

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Our results suggest that the answer is possibly ***negative***.

Leibniz rule and SUSY algebra

We want to find lattice SUSY transf. $\delta_Q , \delta_{Q'}$ such that

lattice SUSY transf. lattice action

$$\delta_Q S[\phi, \chi, F] = \delta_{Q'} S[\phi, \chi, F] = 0$$

with the SUSY algebra

"translation" on lattice

$$\{ \delta_Q , \delta_{Q'} \} = \delta_P$$

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with the SUSY algebra

$$\{\,\delta_Q\,,\,\delta_{Q'}\,\} = \underbrace{\delta_P}_{\text{“translation” on lattice}}$$

One might replace δ_P by a difference operator ∇.
Then, we need to find ∇ which satisfies the ***Leibniz rule***.

$$\delta_P(\phi\psi) = (\delta_P\phi)\,\psi + \phi\,(\delta_P\psi)$$

$$\xrightarrow{\delta_P \to \nabla} \boxed{\nabla(\phi\psi) = (\nabla\psi)\,\psi + \phi\,(\nabla\psi)}$$ Leibniz rule

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However, we can show that it is hard to realize the Leibniz rule on lattice!!

No-Go theorem

To answer the question whether the Leibniz rule can be realized on lattice or not, let us consider general forms of difference operators and field products such as

difference operator: $$(\nabla\phi)_n \equiv \sum_m \nabla_{nm}\phi_m$$

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For example,

$$(\nabla\phi)_n = \phi_{n+1} - \phi_n \quad \Longrightarrow \quad \nabla_{nm} = \delta_{n+1,m} - \delta_{n,m}$$

$$(\phi * \psi)_n = \phi_n\psi_n \quad \Longrightarrow \quad M_{nlm} = \delta_{n,l}\delta_{n,m}$$

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No-Go Theorem

There is no difference operator ∇ satisfying the following three properties:

i) translation invariance

ii) locality

iii) Leibniz rule $\nabla(\phi * \psi) = (\nabla\phi) * \psi + \phi * (\nabla\psi)$

Our approach to construct lattice SUSY models

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Our strategy to construct lattice SUSY models is

full SUSY algebra $\longrightarrow$ **Nilpotent SUSY algebra**

$$(\delta_Q)^2 = (\delta_{Q'})^2 = \{\delta_Q, \delta_{Q'}\} = 0$$

Leibniz rule $\longrightarrow$ **Cyclic Leibniz rule**

Complex SUSY quantum mechanics on lattice

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Lattice action

$$
\begin{aligned}
S = (\nabla\phi_-, \nabla\phi_+) - (F_-, F_+) - i(\chi_-, \nabla\bar{\chi}_+) + i(\nabla\bar{\chi}_-, \chi_+) \\
-\lambda_+(F_+, \phi_+ * \phi_+) + 2\lambda_+(\chi_+, \bar{\chi}_+ * \phi_+) \\
-\lambda_-(F_-, \phi_- * \phi_-) - 2\lambda_-(\chi_-, \bar{\chi}_- * \phi_-)
\end{aligned}
$$

difference operator: $(\nabla\phi)_n \equiv \sum_m \nabla_{nm}\phi_m$

field product: $(\phi * \psi)_n \equiv \sum_{lm} M_{nlm}\phi_l\psi_m$

inner product: $(\phi, \psi) \equiv \sum_n \phi_n\psi_n$

- ❑ To make our discussions simple, we here put $m=0$.
- ❑ We can add mass terms as well as ***supersymmetric Wilson terms*** to prevent the doubling.

N=2 nilpotent SUSYs

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$$\begin{cases} \delta_+\phi_+ = \bar{\chi}_+ \\ \delta_+\chi_+ = F_+ \\ \delta_+\chi_- = -i\nabla\phi_- \\ \delta_+F_- = -i\nabla\bar{\chi}_- \\ \text{others} = 0 \end{cases} \qquad \begin{cases} \delta_-\chi_+ = i\nabla\phi_+ \\ \delta_-F_+ = -i\nabla\bar{\chi}_+ \\ \delta_-\phi_- = -\bar{\chi}_- \\ \delta_-\chi_- = F_- \\ \text{others} = 0 \end{cases}$$

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$$\delta_\pm S = 0$$

$$\Downarrow$$

$$\boxed{(\nabla\bar{\chi}_\pm, \phi_\pm * \phi_\pm) + (\nabla\phi_\pm, \phi_\pm * \bar{\chi}_\pm) + (\nabla\phi_\pm, \bar{\chi}_\pm * \phi_\pm) = 0}$$

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We call this ***Cyclic Leibniz rule***.

Cyclic Leibniz rule vs. Leibniz rule

We have found that the ***Cyclic Leibniz Rule*** guarantees the N=2 nilpotent SUSYs.

Cyclic Leibniz Rule (CLR)

$$(\nabla A,\ B * C) + (\nabla B,\ C * A) + (\nabla C,\ A * B) = 0$$

vs.

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No-Go theorem

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The cyclic Leibniz rule ensures ***a lattice analog of vanishing surface terms!***

$$\underbrace{(\nabla\phi,\, \phi * \phi) = 0}_{\text{on lattice}} \quad \longleftarrow \quad \underbrace{\int dx\, \partial_x\big(\phi(x)\big)^3 = 0}_{\text{in continuum}}$$

CLR

An example of CLR

An explicit example of the ***Cyclic Leibniz Rule*** :

$$(\nabla\phi)_n = \frac{1}{2}\left(\phi_{n+1} - \phi_{n-1}\right)$$

$$(\phi * \psi)_n = \frac{1}{6}\Big(2\phi_{n+1}\psi_{n+1} + 2\phi_{n-1}\psi_{n-1} + \phi_{n+1}\psi_{n-1} + \phi_{n-1}\psi_{n+1}\Big)$$

lattice spacing $a=1$

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which satisfy i) *translation invariance*, ii) *locality* and iii) *Cyclic Leibniz Rule.*

! *The field product* $(\phi * \psi)_n$ *should be non-trivial!*

Advantages of CLR

Advantages of our lattice model with ***CLR*** are given by

	CLR	no CLR
nilpotent SUSYs		
Nicolai maps		
“surface” terms		
non-renormalization theorem		
cohomology		

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❑ 4d N=1 Wess-Zumino model in continuum

chiral superfield *superpotential*

$$S = \int d^4x \Big\{ \underbrace{\int d^2\theta d^2\bar\theta\ \Phi^\dagger(\bar\theta)\Phi(\theta)}_{\text{D term (kinetic terms)}} + \underbrace{\int d^2\theta\, W(\Phi) + c.c.}_{\text{F term (potential terms)}} \Big\}$$

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D term
(kinetic terms)

F term
(potential terms)

Non-renormalization Theorem

There is ***no quantum correction to the F-terms*** in any order of perturbation theory.

Essence of non-renormalization theorem

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Holomorphy plays an important role in the non-renormalization theorem.

tree superpotential — *chiral superfield* — *coupling constant* — *anti-chiral superfield*

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quantum corrections $\Downarrow$

effective superpotential

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holomorphy

anti-holomorphy

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holomorphy *anti-holomorphy*

$\Downarrow$

No quantum correction!! $W_{\rm eff} = W_{\rm tree}$

N.Seiberg, Phys. Lett. B318 (1993) 469

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$$\Downarrow$$

anti-chiral superfield *chiral superfield*

$$\boxed{\int d^2\theta\, W_{\rm eff}(\Phi, \lambda; \cancel{\Phi^\dagger}, \cancel{\lambda^*})\quad \textit{holomorphy}} + \int d^2\bar\theta\, \bar W_{\rm eff}(\Phi^\dagger, \lambda^*; \cancel{\Phi}, \cancel{\lambda})$$

anti-holomorphy

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Difficulty in defining chiral superfield on lattice

The holomorphy requires that the F term $W(\Phi)$ depends only on the ***chiral*** superfield $\Phi(x,\theta)$, which is defined by

$$\bar{D}\Phi(x,\theta) \equiv \left(\frac{\partial}{\partial\bar{\theta}} - i\theta\sigma_\mu\partial_\mu\right)\Phi(x,\theta) = 0 \quad \text{in continuum}$$

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$$\bar{D}\Phi(\theta)_n \equiv \left(\frac{\partial}{\partial\bar{\theta}} - i\theta\sigma_\mu\nabla_\mu\right)\Phi(\theta)_n = 0 \quad \text{on lattice}$$

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$$\Big\downarrow \textbf{?}$$

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However, the above definition of the chiral superfield is ***ill-defined*** because any products of chiral superfields are not chiral due to the ***breakdown of LR on lattice!***

$$\bar{D}\Phi_1 = \bar{D}\Phi_2 = 0 \quad \Longrightarrow \quad \bar{D}(\Phi_1\Phi_2) \neq 0$$

the breakdown of the Leibniz rule on lattice

Superfield formulation in our lattice model

❑ Lattice superfields

$$\Psi_\pm(\theta_+, \theta_-) \equiv \chi_\pm + \theta_\pm F_\pm + \theta_\mp i\nabla\phi_\pm + \theta_\pm\theta_\mp i\nabla\bar{\chi}_\pm$$

$$\Lambda_\pm(\theta_\pm) \equiv \phi_\pm + \theta_\pm\chi_\pm$$

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❑ Lattice action in superspace $S = S_{\mathrm{type\,I}} + S_{\mathrm{type\,II}}$

$$S_{\mathrm{type\,I}} = \int d\theta_+ d\theta_-\ \Psi_-\Psi_+ \quad \Longrightarrow$$ *kinetic terms (D-term)*

$$S_{\mathrm{type\,II}} = \int d\theta_+ d\theta_-\ \left\{\theta_-\,\lambda_+(\Psi_+, \Lambda_+ * \Lambda_+) + \theta_+\,\lambda_-(\Psi_-, \Lambda_- * \Lambda_-)\right\}$$

$\Longrightarrow$ *potential terms (F-term)*

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$$S_{\mathrm{type\,I}} = \int d\theta_+ d\theta_- \; K(\Psi_+,\Lambda_+;\Psi_-,\Lambda_-)$$

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!

$S_{\mathrm{type\,II}}$ is SUSY-invariant ***if and only if*** $W(\Psi_+,\Lambda_+)$ depends only on Ψ_+,Λ_+ and is written into the form

$$W(\Psi_+,\Lambda_+) = \sum_n \lambda_+^{(n)}(\Psi_+,\overbrace{\Lambda_+ * \Lambda_+ * \cdots * \Lambda_+}^{n-1})$$

and $(\Psi_+,\Lambda_+ * \Lambda_+ * \cdots * \Lambda_+)$ has to obey ***CLR***.

M.Kato, M.S., H.So, in preparation

Non-renormalization theorem in our lattice model

$$\int d\theta_+ d\theta_- \ \theta_- W_{\mathrm{tree}}(\Psi_+, \Lambda_+, \lambda_+)$$

$$W_{\mathrm{tree}} = \lambda_+(\Psi_+, \Lambda_+ * \Lambda_+)$$

$\big\downarrow$ quantum corrections

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$$\int d\theta_+ d\theta_- \ \theta_- W_{\mathrm{eff}}(\Psi_+, \Lambda_+, \lambda_+; \Psi_-, \Lambda_-, \lambda_-)$$

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with ***CLR*** forbids them!

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The ***holomorphic property***
is realized in our lattice model.

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SUSY-invariance with ***CLR*** forbids them!

The ***holomorphic property*** is realized in our lattice model.

no quamtum corrections: $W_{\rm eff} = W_{\rm tree}$

Non-renormalization theorem in our lattice model 15

$$\int d\theta_+ d\theta_- \; \theta_- W_{\mathrm{tree}}(\Psi_+, \Lambda_+, \lambda_+)$$

$$W_{\mathrm{tree}} = \lambda_+(\Psi_+, \Lambda_+ * \Lambda_+)$$

quantum corrections

$$\int d\theta_+ d\theta_- \; \theta_- W_{\mathrm{eff}}(\Psi_+, \Lambda_+, \lambda_+; \underbrace{\cancel{\Psi_-}, \cancel{\Lambda_-}, \cancel{\lambda_-}})$$

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is realized in our lattice model.

no quamtum corrections: $W_{\mathrm{eff}} = W_{\mathrm{tree}}$

The non-renormalization theorem holds
even for a finte lattice spacing in our lattice model.

Q-exact form and cohomology

$$S = \overbrace{\delta_+\delta_- K(\phi_\pm, F_\pm, \chi_\pm, \bar{\chi}_\pm)}^{S_{\mathrm{type\,I}}} + \overbrace{\delta_+ W(\phi_+, \chi_+) + \delta_- \bar{W}(\phi_-, \chi_-)}^{S_{\mathrm{type\,II}}}$$

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because of the ***nilpotency:***

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$$S_{\mathrm{type\,I}}$$

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but not exact

$$W \sim (\chi_+, \phi_+ * \phi_+ * \cdots * \phi_+)$$

which has the properties:

$$W \neq \delta_- K'$$

$$\delta_- W \sim (\nabla\phi_+, \phi_+ * \phi_+ * \cdots * \phi_+) \overset{\text{CLR}}{=} 0$$

M.Kato, M.S., H.So in preparation

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M.Kato, M.S., H.So in preparation

The type II terms are ***cohomologically non-trivial !***

Summary

- We have proved the ***No-Go theorem*** that the Leibniz rule cannot be realized on lattice under reasonable assumptions.
- We proposed a lattice SUSY model equipped with the ***cyclic Leibniz rule*** as a modified Leibniz rule.
- A striking feature of our lattice SUSY model is that the ***non-renormalization theorem*** holds for a finite lattice spacing.
- Our results suggest that ***the cyclic Leibniz rule grasps important properties of SUSY.***

Remaining tasks

- ❑ Extension to higher dimensions

  We have to extend our analysis to higher dimensions. Especially, we need to find solutions to CLR in more than one dimensions.

- ❑ inclusion of gauge fields

- ❑ Nilpotent SUSYs with CLR $\overset{?}{\longleftrightarrow}$ full SUSYs

  Are nilpotent SUSYs extended by CLR enough to guarantee full SUSYs ?

Appendix

SUSY transformations of superfields

$$\Psi_{\pm}(\theta_{+}, \theta_{-}) \equiv \chi_{\pm} + \theta_{\pm} F_{\pm} + \theta_{\mp} i\nabla\phi_{\pm} + \theta_{\pm}\theta_{\mp} i\nabla\bar{\chi}_{\pm}$$

$$\Lambda_{\pm}(\theta_{\pm}) \equiv \phi_{\pm} + \theta_{\pm}\chi_{\pm}$$

transform under SUSY transformations $\delta_{\pm}$ as

$$\delta_{\pm}\mathcal{O}(\theta_{\pm}) = \frac{\partial}{\partial\theta_{\pm}}\mathcal{O}(\theta_{\pm})$$

Nicolai maps

Two Nicolai maps:

$$\xi_\pm \equiv \nabla\phi_- \pm \phi_+ * \phi_+$$

$$\bar{\xi}_\pm \equiv \nabla\phi_+ \pm \phi_- * \phi_-$$

Action: $S = S_{\mathbf{B}} + S_{\mathbf{F}}$

$$S_{\mathbf{B}} = (\bar{\xi}_+, \xi_+) \underset{\uparrow}{=} (\bar{\xi}_-, \xi_-)$$

$$(\nabla\phi_\pm, \phi_\pm * \phi_\pm) \underset{\substack{\uparrow \\ \text{CLR}}}{=} 0$$

Proof of No-Go Theorem

difference operator: $(\nabla\phi)_n \equiv \sum_m \nabla_{nm}\phi_m$

field product: $(\phi * \psi)_n \equiv \sum_{lm} M_{nlm}\phi_l\psi_m$

Proof of No-Go Theorem

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field product: $(\phi * \psi)_n \equiv \sum_{lm} M_{nlm}\phi_l\psi_m$

i) translation invariance

$$\nabla_{nm} = \nabla(n-m)$$

$$M_{nlm} = M(l-n, m-n)$$

Proof of No-Go Theorem

difference operator: $(\nabla\phi)_n \equiv \sum_m \nabla_{nm}\phi_m$

field product: $(\phi * \psi)_n \equiv \sum_{lm} M_{nlm}\phi_l\psi_m$

ii) locality

$$\nabla(m) \xrightarrow{|m|\to\infty} 0 \quad \text{(exponentially)}$$

$$M(l,m) \xrightarrow{|l|,|m|\to\infty} 0 \quad \text{(exponentially)}$$

holomorphic representation

$$\widetilde{\nabla}(z) \equiv \sum_m \nabla(m)\, z^m$$

$$\widetilde{M}(z,w) \equiv \sum_{lm} M(l,m)\, z^l w^m$$

on $1-\varepsilon < |z|, |w| < 1+\varepsilon$

$\widetilde{\nabla}(z), \widetilde{M}(z,w)$ have to be holomorphic on $1-\varepsilon < |z|, |w| < 1+\varepsilon$

Proof of No-Go Theorem

difference operator: $(\nabla\phi)_n \equiv \sum_m \nabla_{nm}\phi_m$

field product: $(\phi * \psi)_n \equiv \sum_{lm} M_{nlm}\phi_l\psi_m$

iii) Leibniz rule

$$\nabla(\phi * \psi) = (\nabla\phi) * \psi + \phi * (\nabla\psi)$$

$$\Longrightarrow \quad M(z,w)\big(\nabla(zw) - \nabla(z) - \nabla(w)\big) = 0$$

$$\Longrightarrow \quad \nabla(zw) - \nabla(z) - \nabla(w) = 0$$

$$\Longrightarrow \quad \nabla(z) \propto \log z$$

$$\Longrightarrow \quad \log z \text{ is non-holomorphic on } 1 - \varepsilon < |z| < 1 + \varepsilon.$$

$$\Longrightarrow \quad \text{The Leibniz rule cannot be realized on lattice!}$$