Sine-Square Deformation (SSD) and its Relevance to String Theory

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Based on work with N. Ishibashi and [arXiv:1404.6343]
Conformal Field Theory in 2 dim.

(Holomorphic part of)

\[ \mathcal{H}_{\text{Hamiltonian}} \sim L_0 = \frac{1}{2\pi i} \int dz \, z \, T(z) \]

Let us consider a simple (almost trivial) modification to the Hamiltonian

Add \( L_1 \) and \( L_{-1} \)

\[ \frac{1}{2\pi i} \int dz \, T(z) \]

\[ \frac{1}{2\pi i} \int dz \, z^2 T(z) \]
\( L_0, L_1, L_{-1} \) form \( SL(2, \mathbb{R}) \) subalgebra of Virasoro algebra

Introduce

\[
L_+ \equiv \frac{L_1 + L_{-1}}{2} \quad L_- \equiv \frac{L_1 - L_{-1}}{2i}
\]

Casimir Operator

\[
(L_0)^2 - (L_+)^2 - (L_-)^2
\]
Now the modification

\[ e^{-it_0L_0-\frac{it}{2}L_+\frac{-it}{2}L_-} \left( x_0 L_0 + x_+ L_+ + x_- L_- \right) e^{it_0L_0+\frac{it}{2}L_+\frac{-it}{2}L_-} \]

\[ \downarrow \quad SL(2, \mathbb{R}) \]

\[ = x'_0 L_0 + x'_1 L_+ + x'_2 L_- \]

\[ = L_0 \quad x'_0 = 1, x'_1 = x'_2 = 0 \]

\[ (x_0)^2 - (x_1)^2 - (x_2)^2 = (x'_0)^2 - (x'_1)^2 - (x'_2)^2 \]
Non-trivial modification

\[
L_0 - L_+ \rightarrow \text{SL}(2, \mathbb{R})
\]

\[
x_0 L_0 + x_+ L_+ + x_- L_-
\]

with \( (x_0)^2 - (x_+)^2 - (x_-)^2 = 0 \)

No way to realize \( x_0 = 1, \ x_+ = x_- = 0 \)

\[
(x_0)^2 - (x_+)^2 - (x_-)^2 = 0
\]
What does $\mathcal{H} \sim L_0 - L_+$ suggest?

Gap or “Mass”

“Continuous Spectrum”

c.f. “Level” structure of excited states in CFT
To motivate further, let me introduce an interesting work by A. Gendiar, R. Krcmar and T. Nishino
They Started With

Gendiar, Krcmar, Nishino (2009)

1d systems w/ nearest neighbor coupling

\[ \mathcal{H} = - \sum J_{n,n+1} (\sigma_n \cdot \sigma_{n+1}) \]

and

Open Boundary Condition
\[ H = - \sum J_{n,n+1} (\sigma_n \cdot \sigma_{n+1}) \]

\[ J_{1,2} = J_{2,3} = \cdots = J_{N-1,N} \equiv J \]

\[ J_{0,1} = J_{N,N+1} = 0 \]
$$\mathcal{H} = - \sum J_{n,n+1} (\sigma_n \cdot \sigma_{n+1})$$
\[ \mathcal{H} = - \sum J_{n,n+1} (\sigma_n \cdot \sigma_{n+1}) \]

\[ J_{i,i+1} \equiv J \sin^2 \left( \frac{n}{N} \pi \right) \]

Sine Square Deformation
Sine Square Deformation

Closed

Same Ground State

A. Gendiar, R. Krcmar and T. Nishino

The mechanism behind this deformation was clarified by H. Katsura and his collaborators.

**Closed Hamiltonian**

\[ \mathcal{H}_c = \sum_{n=1}^{N} h_{n,n+1} \]

\[ h_{N,1} \neq 0 \]

\[ \mathcal{H}_{\pm 1} = \sum_{n=1}^{N} e^{\pm 2\pi i \frac{n}{N}} h_{n,n+1} \]
\[ \mathcal{H}_c = \sum_{n=1}^{N} h_{n,n+1} \]

\[ \mathcal{H}_{\pm 1} = \sum_{n=1}^{N} e^{\pm 2\pi i \frac{n}{N}} h_{n,n+1} \]

\[ \mathcal{H}_{SSD} \equiv \frac{1}{2} \mathcal{H}_c - \frac{1}{4} (\mathcal{H}_{+1} + \mathcal{H}_{-1}) \]

\[ \frac{1}{2} - \frac{1}{4} \left( e^{2\pi i \frac{n}{N}} + e^{-2\pi i \frac{n}{N}} \right) = \frac{1}{2} \left( 1 - \cos 2\pi \frac{n}{N} \right) \]

\[ = \sin^2 \pi \frac{n}{N} \]

\[ \mathcal{H}_{SSD} = \sum_{n=1}^{N} \sin^2 \left( \pi \frac{n}{N} \right) h_{n,n+1} \]
$H_c = \sum_{n=1}^{N} h_{n,n+1}$

$H_{SSD} = \sum_{n=1}^{N} \sin^2\left(\pi \frac{n}{N}\right) h_{n,n+1}$
$\mathcal{H}_c = \sum_{n=1}^{N} h_{n,n+1}$

$\mathcal{H}_{SSD} = \sum_{n=1}^{N} \sin^2(\pi \frac{n}{N}) h_{n,n+1}$
Provided

- $\mathcal{H}_{\pm 1}$ annihilates $\mathcal{H}_c$'s vacuum $|\text{vac}\rangle$

- $\mathcal{H}_{\pm 1}|\text{vac}\rangle = 0$

- Either $\mathcal{H}_{\text{SSD}}$'s vacuum is unique
  or $\mathcal{H}_{\text{SSD}}$ is bounded below

$|\text{vac}\rangle$ is also $\mathcal{H}_{\text{SSD}}$'s vacuum
\( \mathcal{H}_{\pm 1} |\text{vac}\rangle = 0 \)

\( \mathcal{H}_{SSD} \equiv \frac{1}{2} \mathcal{H}_c - \frac{1}{4} (\mathcal{H}_{+1} + \mathcal{H}_{-1}) \)

\( \mathcal{H}_c |\text{vac}\rangle = E_0 |\text{vac}\rangle \)

\( \mathcal{H}_{SSD} |\text{vac}\rangle = \frac{E_0}{2} |\text{vac}\rangle \)
2D Cft On A Cylinder

\[ \mathcal{H}_c = \frac{2\pi}{\ell} (L_0 + \bar{L}_0) - \frac{\pi c}{6\ell} \]

\[ \mathcal{H}_{\pm 1} = \frac{2\pi}{\ell} (L_{\pm 1} + \bar{L}_{\mp 1}) \]

\[ L_0 |0\rangle = \bar{L}_0 |0\rangle = 0 \quad \leftrightarrow \quad \mathcal{H}_c \text{'s vacuum } |0\rangle \]

sl(2,c) invariance \[ L_{\pm 1} |0\rangle = \bar{L}_{\pm 1} |0\rangle = 0 \]

\[ \mathcal{H}_{\pm 1} |\text{vac}\rangle = 0 \]
\[ H_c = \frac{2\pi}{\ell} (L_0 + \bar{L}_0) - \frac{\pi c}{6\ell} \quad H_{\pm 1} = \frac{2\pi}{\ell} (L_{\pm 1} + \bar{L}_{\mp 1}) \]

\[ H_{SSD} = \frac{1}{2} H_c - \frac{1}{4} (H_{+1} + H_{-1}) \]

\[ \sim \frac{1}{2} \left( L_0 - \frac{L_1 + L_{-1}}{2} \right) + \text{(anti-holomorphic)} \]

\[ H_{SSD} |0\rangle = \frac{E_0}{2} |0\rangle \quad H_c |0\rangle = E_0 |0\rangle \]

\[ E_0 = -\frac{\pi c}{6\ell} \]

$$\mathcal{H}_c = \frac{2\pi}{\ell} (L_0 + \bar{L}_0) - \frac{\pi c}{6\ell} \quad \mathcal{H}_{\pm 1} = \frac{2\pi}{\ell} (L_{\pm 1} + \bar{L}_{\mp 1})$$

$$\mathcal{H}_{SSD} = \frac{1}{2} \mathcal{H}_c - \frac{1}{4} (\mathcal{H}_{+1} + \mathcal{H}_{-1})$$

$$\sim \frac{1}{2} \left( L_0 - \frac{L_1 + L_{-1}}{2} \right) + \text{(anti-holomorphic)}$$

$$\mathcal{H}_{SSD}|0\rangle = \frac{E_0}{2} |0\rangle \quad \mathcal{H}_c |0\rangle = E_0 |0\rangle$$

$$E_0 = -\frac{\pi c}{6l}$$

Implication For String Theory?

Non-Trivial Modification (Deformation)

Affects Boundary Condition

World Sheet Dynamics Of D-Brane
Open/Closed Duality
Implication For String Theory?

Non-Trivial Modification (Deformation)

Affects Boundary Condition

Modification Of World Sheet Metric

World Sheet Dynamics Of D-Brane

Open/Closed Duality

Worth Further Exploration
Let Me Elaborate

Boundary condition — set by hand

Compartmentalize characteristic physics

Useful to concentrate each idiosyncrasy

Often non-perturbative effects involve different boundary conditions

D-brane, open closed duality

Understanding Non-perturbative dynamics

in terms of the world sheet gravity
It follows from the above equation that the canonical conjugate momentum is now
and see if the deformed Lagrangean generates
the reminder of the parameter, can be expressed in terms of the Fourier modes
Then the Hamiltonian can be expressed in terms of \( \mathcal{L} \)
It is easy to see that the following relation holds:
We might, then, expect that the corresponding deformed Lagrangean may take more
The last equality follows from the explicit definition of
For the claim (30) or (31) to be validated, the expression
\( g \)
\( N \)
\( n \)
\( k \)
\( N, m \)
\( g_{m n} = \{ \partial_t \varphi \} \quad \varphi 
\)
\[ \mathcal{L}_\alpha = \frac{g \ell}{2} \sum_{n,k} \dot{\phi}_n \phi_{-n-k} N r^{|k|} \]

\[ -\frac{2\pi^2 g}{\ell} \left\{ n^2 \phi_n \phi_{-n} - \frac{\alpha}{2} (n(n+1) \phi_n \phi_{-n-1} + n(n-1) \phi_n \phi_{-n+1}) \right\} \]

Now conjugate momenta are

\[ \pi_n = g \ell \sum_k N r^{|k|} \phi_{-n-k} \]

\[ \mathcal{H}_\alpha \quad = \quad \sum_n \pi_n \dot{\phi}_n - \mathcal{L}_\alpha \quad \text{Provided} \quad r = \frac{1 - \sqrt{1 - \alpha^2}}{\alpha}, \quad N = \frac{1}{\sqrt{1 - \alpha^2}} \]

\[ = \frac{1}{2g\ell} \left[ \pi_n \pi_{-n} - \frac{\alpha}{2} \pi_n \pi_{-n+1} - \frac{\alpha}{2} \pi_n \pi_{-n-1} \right. \]

\[ + (2\pi g)^2 n^2 \phi_n \phi_{-n} - \frac{\alpha}{2} (2\pi g)^2 n(n+1) \phi_n \phi_{-n-1} \]

\[ \left. - \frac{\alpha}{2} (2\pi g)^2 n(n-1) \phi_n \phi_{-n+1} \right] \]
\[ \mathcal{H}_\alpha = \frac{1}{2g\ell} \left[ \pi_n \pi_{-n} - \frac{\alpha}{2} \pi_n \pi_{-n+1} - \frac{\alpha}{2} \pi_n \pi_{-n-1} 
\right]
\]
\[ \quad + (2\pi g)^2 n^2 \phi_n \phi_{-n} - \frac{\alpha}{2} (2\pi g)^2 n (n + 1) \phi_n \phi_{-n-1} 
\]
\[ \quad - \frac{\alpha}{2} (2\pi g)^2 n (n - 1) \phi_n \phi_{-n+1} \right] 
\]
\[ = \frac{2\pi}{\ell} \left( L_0 + \bar{L}_0 - \frac{\alpha}{2} (L_1 + \bar{L}_1 + L_{-1} + \bar{L}_{-1}) \right) 
\]
\[ \mathcal{H}_{+1} + \mathcal{H}_{-1} \]
\[ = \frac{2\pi}{\ell} (L_1 + \bar{L}_1 + L_{-1} + \bar{L}_{-1}) = \frac{1}{2g\ell} \sum_{n \in \mathbb{Z}} \left\{ \pi_n \pi_{-(n+1)} + \pi_n \pi_{-(n-1)} \right\} 
\]
\[ + (2\pi g)^2 n (n + 1) \phi_n \phi_{-(n+1)} + (2\pi g)^2 n (n - 1) \phi_n \phi_{-(n-1)} \} \]
\[ \mathcal{L}_\alpha = \frac{1}{2} \int_0^\ell dx \left\{ (\partial_t \varphi) F(x) (\partial_t \varphi) - (\partial_x \varphi) G(x) (\partial_x \varphi) \right\} \]

\[ F(x) = N \sum_{k \in \mathbb{Z}} r^{|k|} e^{2\pi i kx / \ell} = N \delta(x) \]

\[ \mathcal{H}_\alpha = \frac{2\pi}{\ell} \left( L_0 + \bar{L}_0 - \frac{1}{2} (L_1 + \bar{L}_1 + L_{-1} + \bar{L}_{-1}) \right) \]

\[ G(x) = 1 - \alpha \cos \frac{2\pi x}{\ell} = 2 \sin^2 \frac{\pi x}{\ell} \]

\[ N \equiv \frac{1}{\sqrt{1 - \alpha^2}} \]

\[ r \equiv \frac{1 - \sqrt{1 - \alpha^2}}{\alpha} \]

\[ \mathcal{H}_{SSD} = \frac{1}{2} \mathcal{H}_c - \frac{1}{4} (\mathcal{H}_{+1} + \mathcal{H}_{-1}) \]

\[ = \frac{\pi}{\ell} \left( L_0 + \bar{L}_0 - \frac{L_1 + L_{-1} + \bar{L}_1 + \bar{L}_{-1}}{2} \right) - \frac{\pi c}{12 \ell} \]

\[ \alpha = 1 \]
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\[ (m, n) = (m + n + c) \]

\[ (1) \]

\[ (m, G) = (m + r, G) \]

\[ (2) \]

\[ \{G, G\} = 2Lr + s + c \]

\[ (3) \]

\[ H_{SSD}(Left) = \frac{\pi}{2} \]

\[ \frac{N}{2} \]
Non-Trivial Divergence Confirmed

Difficult To Tackle Directly

Explore States Other Than \(|0\rangle\)
“Excited” states

A candidate for the implied “continuous” states

\[ \sum_{r=2}^{\infty} \sum_{k=3}^{r+1} \frac{\mu^k}{k! \frac{(r-2)!}{(k-3)!(r-k+1)!}} L_{-r} |0\rangle \]

\( \mu \) : continuous parameter

Exotic states
Exotic states

\[
(L_0 - \frac{L_1 + L_{-1}}{2}) \sum_{r=2}^{\infty} L_{-r} |0\rangle = 0
\]

The lowest energy state for $\mathcal{H}_{SSD}$
Other Than $|0\rangle$

- Exotic states

\[
\left( L_0 - \frac{L_1 + L_{-1}}{2} \right) = 0
\]

\[
\left| \sum_{r=2}^{\infty} L_{-r} |0\rangle \right| \rightarrow \infty
\]

by H. Katsura

The lowest energy state for $\mathcal{H}_{SSD}$
Other Than

- Exotic states

\[ L_0 - \frac{L_1 + L_{-1}}{2} \]

\[ \sum_{r=2}^{\infty} \sum_{k=3}^{r+1} \frac{\mu^k}{k!} \frac{(r - 2)!}{(k - 3)!(r - k + 1)!} L_{-r} |0\rangle \]

\[ \sum_{r=2}^{\infty} L_{-r} |0\rangle \rightarrow \infty \]

So as the previously mentioned candidate states
Other Than

- Exotic states

\[
\left( L_0 - \frac{L_1 + L_{-1}}{2} \right) e^{L_{-1}} |h\rangle = 0
\]

\[
L_0 |h\rangle = h |h\rangle
\]

\[
L_n |h\rangle = 0 \quad (n > 0)
\]

The lowest energy state for \( \mathcal{H}_{\text{SSD}} \)
by the use of (23), (15), (21).

Thus $H$ varies from the original free Hamiltonian to sine square deformed one up to the overall $1/2$ factor as we change $\varpi$ from 0 to 1. Therefore, the case of interest is $\varpi = 1$ and then (eq:rNsol) yields $r = 1$. From (25), this results and the brief description of the results are in order. First, we found the lagrangean corresponds to $H_{SSD}$ is obtained by taking $\varpi$ to 1 in the following expression:

$$L_{\varpi} = \frac{1}{2} \int dx \left\{ \left( \partial_t f \right) \partial_t f + \left( \partial_x f \right) \partial_x f \right\},$$

where $f(x) = 1 + \cos \lambda x$, $f(t) = N \sum_{k=2}^{\infty} |k| e^{2\pi ikx/\lambda}$.

One can readily see the $g_{00}$ component of the world sheet metric in the lagrangean, namely $f_t$ diverges severely as we apply sine square deformation. This is in some sense expected because at SSD point there occurs an event as singular as the change of the boundary condition.

One can apply the following sl(2, C) transformation to (the holonomic part of) $H_0$ obtaining

$$e^a L_0 e^{-a} L_1 - L_1 e^{-a} L_0 = \cosh a L_0 \sinh a L_1 + L_1,$$

The righthand side of the above would have correspond to $H_{SSD}$ if $\cosh a = \sinh a$, which is a direct contradiction with the identity $\cosh^2 a - \sinh^2 a = 1$. One, therefore, need to take $a \to 1$ and suitably rescale. Hence $H_{SSD}$ is not connected with $H_0$ through the ordinary sl(2,c) transformation, but with certain limiting procedure.

$H_{SSD}$ have the following different vacua other than $|h\rangle$, (39),

where $|h\rangle$ is the state corresponds to the primary fields of CFT. However the norm of (39) is divergent. One also need to certain limiting process to properly define (39).

Other Than

- Exotic states

\[ \left( L_0 - \frac{L_1 + L_{-1}}{2} \right) \]

\[ = 0 \]

The lowest energy state for $H_{SSD}$

\[ \left| e^{L_{-1}} h \right| \to \infty \]

Need More Work To Understand The Whole Structure
Summary

Sine Square Deformation

String Theory

Duality

Divergence In Worldsheet

Dynamics

Condensation of world sheet metric
Thank You For Your Attention