Dirac equation in five-dimensional spherical AdS space-time

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Ⅰ. Introduction

✓ We provide a detailed discussion on the representation of the angular sector of Dirac field with SU(2)×U(1) symmetry.

✓ Our analysis applies to asymptotically flat as well as asymptotically AdS and potentially has many application in AdS/CFT context, higher dimensional black holes.

Ⅱ. Background space time

■ 4+1 dimensional spherically symmetric metric of polar coordinates

$$ds^2=-f(r)a^2(r)dt^2+\frac{1}{f(r)}dr^2+\frac{r^2}{4}(d\theta^2+d\varphi^2+d\psi^2+2\cos\theta d\varphi d\psi)$$

where
$0\le\theta\le\pi,$
$0\le\varphi\le 2\pi,$
$0\le\psi\le 4\pi$

This metric has symmetry of SO(4)≅SU(2)$_L$ × SU(2)$_R$. Then, we can define two invariant one-forms σ^L_a,σ^R_a of SU(2) which satisfy $d\sigma^R_a=1/2\epsilon^{abc}\sigma^R_b\wedge\sigma^R_c$ and $d\sigma^L_a=-1/2\epsilon^{abc}\sigma^L_b\wedge\sigma^L_c$.

The invariant one-form of SU(2)

$$\begin{aligned}\sigma^R_1&=-\sin\psi d\theta+\cos\psi\sin\theta d\phi\\ \sigma^R_2&=\cos\psi d\theta+\sin\psi\sin\theta d\phi\\ \sigma^R_3&=d\psi+\cos\theta d\phi\\ \sigma^L_1&=\sin\phi d\theta-\cos\phi\sin\theta d\psi\\ \sigma^L_2&=\cos\phi d\theta+\sin\phi\sin\theta d\psi\\ \sigma^L_3&=d\phi+\cos\theta d\psi\end{aligned}$$

The metric is given by

$$ds^2=-f(r)a^2(r)dt^2+\frac{1}{f(r)}dr^2+\frac{r^2}{4}((\sigma_1^{R,L})^2+(\sigma_2^{R,L})^2+(\sigma_3^{R,L})^2)$$

Also, we can define the following Killing vectors $\xi^{R,L}_\alpha$ which satisfy $\langle\xi^{R,L}_\alpha,\sigma^{R,L}_a\rangle=\delta_{\alpha a}$

$$\begin{aligned}\xi^R_x&=-\sin\psi\partial_\theta+\frac{\cos\psi}{\sin\theta}\partial_\varphi-\cot\theta\cos\psi\partial_\psi & \xi^L_x&=\cos\varphi\partial_\theta+\frac{\sin\varphi}{\sin\theta}\partial_\psi-\cot\theta\sin\varphi\partial_\varphi\\ \xi^R_y&=\cos\psi\partial_\theta+\frac{\sin\psi}{\sin\theta}\partial_\varphi-\cot\theta\sin\psi\partial_\psi & \xi^L_y&=-\sin\varphi\partial_\theta+\frac{\cos\varphi}{\sin\theta}\partial_\psi-\cot\theta\cos\varphi\partial_\varphi\\ \xi^R_z&=\partial_\psi & \xi^L_z&=\partial_\varphi\end{aligned}$$

Let us define two kinds of angular momenta :

$$L^R_\alpha=i\xi^R_\alpha,\ L^L_\alpha=i\xi^L_\alpha$$

Commutation relation :

$$\begin{aligned}[L^L_\alpha,L^L_\beta]&=i\epsilon_{\alpha\beta\gamma}L^L_\gamma\\ [L^R_\alpha,L^R_\beta]&=-i\epsilon_{\alpha\beta\gamma}L^R_\gamma\\ [L^L_\alpha,L^R_\beta]&=0.\end{aligned}$$

In a special case $L^2=(L^R)^2=(L^L)^2$, L^2,L^R_z,L^L_z have the common eigenfunction called Wigner D function

$$\begin{aligned}L^2D^l_{M,K}&=l(l+1)D^l_{M,K}\\ L^L_zD^l_{M,K}&=MD^l_{M,K}\\ L^R_zD^l_{M,K}&=KD^l_{K,M}\end{aligned}$$

Ⅲ. Dirac Hamiltonian

Symmetries of the Dirac Hamiltonian

The Dirac equation in curved space-time : $(e^M_{\hat c}\gamma^{\hat c}D_M-m)\Psi=0$.

$\hat a$: Tangent space indices
μ : Curved space indices

The covariant derivative :

$$D_M=\partial_M+\frac{1}{8}\omega_{M\hat a\hat b}[\gamma^{\hat a},\gamma^{\hat b}]$$

$\omega_{M\hat a\hat b}$: Spin connection defined by $\omega_{\mu\hat a\hat b}=\frac{1}{2}e^\nu_{\hat a}\nabla_\mu e_{\nu\hat b}$

Gamma matrices :

$$\gamma^{\hat 0}=i\begin{pmatrix}I&0\\0&-I\end{pmatrix},\quad \gamma^{\hat i}=i\begin{pmatrix}0&\sigma^i\\-\sigma^i&0\end{pmatrix},\quad \gamma^{\hat 5}=\begin{pmatrix}0&I\\I&0\end{pmatrix}$$

Vielbein :

Difinition : $g_{MN}=e^{\hat a}_Me_{\hat aN}=\eta_{\hat a\hat b}e^{\hat a}_Me^{\hat b}_N$

$$\{e^{\hat a}_\mu\}=\begin{pmatrix}a\sqrt f&0&0&0&0\\ 0&\frac{1}{\sqrt f}\sin\frac{\theta}{2}\cos\frac{\psi-\varphi}{2}&\frac{r}{2}\cos\frac{\theta}{2}\cos\frac{\psi-\varphi}{2}&-\frac{r}{2}\sin\frac{\theta}{2}\sin\frac{\psi-\varphi}{2}&\frac{r}{2}\sin\frac{\theta}{2}\sin\frac{\psi-\varphi}{2}\\ 0&\frac{1}{\sqrt f}\sin\frac{\theta}{2}\sin\frac{\psi-\varphi}{2}&\frac{r}{2}\cos\frac{\theta}{2}\sin\frac{\psi-\varphi}{2}&\frac{r}{2}\sin\frac{\theta}{2}\cos\frac{\psi-\varphi}{2}&-\frac{r}{2}\sin\frac{\theta}{2}\cos\frac{\psi-\varphi}{2}\\ 0&\frac{1}{\sqrt f}\cos\frac{\theta}{2}\cos\frac{\psi+\varphi}{2}&-\frac{r}{2}\sin\frac{\theta}{2}\cos\frac{\psi+\varphi}{2}&-\frac{r}{2}\cos\frac{\theta}{2}\sin\frac{\psi+\varphi}{2}&-\frac{r}{2}\cos\frac{\theta}{2}\sin\frac{\psi+\varphi}{2}\\ 0&\frac{1}{\sqrt f}\cos\frac{\theta}{2}\sin\frac{\psi+\varphi}{2}&-\frac{r}{2}\sin\frac{\theta}{2}\sin\frac{\psi+\varphi}{2}&\frac{r}{2}\cos\frac{\theta}{2}\cos\frac{\psi+\varphi}{2}&\frac{r}{2}\cos\frac{\theta}{2}\cos\frac{\psi+\varphi}{2}\end{pmatrix}$$

The relation between Cartesian coordinate (x,y,z,w) and polar coordinate (r,θ,φ,ψ)

$$x_1=r\sin\frac{\theta}{2}\cos\frac{\psi-\varphi}{2},\quad x_2=r\sin\frac{\theta}{2}\sin\frac{\psi-\varphi}{2}$$
$$x_3=r\cos\frac{\theta}{2}\cos\frac{\psi+\varphi}{2},\quad x_4=r\cos\frac{\theta}{2}\sin\frac{\psi+\varphi}{2}.$$

The Dirac equation can be written as $i\partial_t\Psi=\mathcal{H}\Psi$. Further, we assume that the spinor can be decomposed as an energy eigenequation

$$\Psi(t,\boldsymbol{x})=e^{iEt}\begin{pmatrix}\chi_1(\boldsymbol{x})\\ \chi_2(\boldsymbol{x})\end{pmatrix}$$

$$\mathcal{H}\begin{pmatrix}\chi_1\\ \chi_2\end{pmatrix}=E\begin{pmatrix}\chi_1\\ \chi_2\end{pmatrix}\quad \text{where}\quad \mathcal{H}=\begin{pmatrix}\sqrt f am&-i\tau_\mu p_\mu\\ i\tau^\dagger_\mu p_\mu&-\sqrt f am\end{pmatrix}$$

This Dirac Hamiltonian does not commute with the space time symmetry generators L^2,L^R_z,L^L_z . The reason is that these operators are the generators of the angular momenta, instead Dirac fields carry a spin. We define following total angular momenta :

$$G^R_a=L^R_a-\frac{1}{2}\begin{pmatrix}\tau_a&0\\0&0\end{pmatrix},\quad G^L_\alpha=L^L_\alpha+\frac{1}{2}\begin{pmatrix}0&0\\0&\tau_\alpha\end{pmatrix}\quad \text{where}\quad [G^R_a,\mathcal{H}]=[G^L_\alpha,\mathcal{H}]=0.$$

Angular operator of the Dirac Hamiltonian

The Dirac Hamiltonian has angular operators $i\mathcal{D}_\phi,i\mathcal{D}_\psi$ in the non-diagonal part. They are written as :

$$i\mathcal{D}_\phi=\begin{pmatrix}L^L_z&\sqrt2L^L_-\\-\sqrt2L^L_+&-L^L_z\end{pmatrix},\quad i\mathcal{D}_\psi=\begin{pmatrix}-L^R_z&\sqrt2L^R_+\\-\sqrt2L^R_-&L^R_z\end{pmatrix}\quad \text{where}\quad \begin{aligned}L^R_\pm&=(iL^R_x\pm L^R_y)/\sqrt2\\ L^L_\pm&=(iL^L_x\pm L^L_y)/\sqrt2.\end{aligned}$$

The spinorial harmonics can be obtained by using SU(2) Clebsch-Gordan coefficients $C^{GM_{G_1}}_{lM\frac12\sigma^3},C^{GM_{G_2}}_{lK\frac12\sigma^3}$.

$$|i\rangle_l=\sum_{M\sigma^3}C^{GM_{G_1}}_{lM\frac12\sigma^3}|lMK\rangle|\sigma^3\rangle,\quad |i\rangle_r=\sum_{K\sigma^3}C^{GM_{G_2}}_{lK\frac12\sigma^3}|lMK\rangle|\sigma^3\rangle$$

$$\begin{aligned}i\mathcal{D}_{(\psi,\phi)}|0\rangle_{(r,l)}&=\left(G-\frac12\right)|0\rangle_{(r,l)},\\ i\mathcal{D}_{(\psi,\phi)}|1\rangle_{(r,l)}&=-\left(G+\frac32\right)|1\rangle_{(r,l)},\\ i\mathcal{D}_{(\psi,\phi)}|2\rangle_{(r,l)}&=-(G+1)|2\rangle_{(r,l)},\\ i\mathcal{D}_{(\psi,\phi)}|3\rangle_{(r,l)}&=-(G+2)|3\rangle_{(r,l)}\end{aligned}$$

The explicit form of the angular basis

$$|0\rangle_l=\sqrt{\frac{G+K}{2G}}D^{G-\frac12}_{K-\frac12,M}\begin{pmatrix}1\\0\end{pmatrix}+\sqrt{\frac{G-K}{2G}}D^{G-\frac12}_{K+\frac12,M}\begin{pmatrix}0\\1\end{pmatrix}\qquad |2\rangle_l=\sqrt{\frac{G+M+\frac12}{2G}}D^{G}_{K,M+\frac12}\begin{pmatrix}1\\0\end{pmatrix}-\sqrt{\frac{G-M+\frac12}{2G}}D^{G}_{K,M-\frac12}\begin{pmatrix}0\\1\end{pmatrix}$$
$$|1\rangle_l=-\sqrt{\frac{G-K+1}{2G+2}}D^{G+\frac12}_{K-\frac12,M}\begin{pmatrix}1\\0\end{pmatrix}+\sqrt{\frac{G+K+1}{2G+2}}D^{G+\frac12}_{K+\frac12,M}\begin{pmatrix}0\\1\end{pmatrix}\qquad |3\rangle_l=-\sqrt{\frac{G-M+\frac12}{2G+2}}D^{G}_{K,M+\frac12}\begin{pmatrix}1\\0\end{pmatrix}-\sqrt{\frac{G+M+\frac12}{2G+2}}D^{G}_{K,M-\frac12}\begin{pmatrix}0\\1\end{pmatrix}.$$
$$|0\rangle_r=\sqrt{\frac{G-M}{2G}}D^{G-\frac12}_{K,M+\frac12}\begin{pmatrix}1\\0\end{pmatrix}+\sqrt{\frac{G+M}{2G}}D^{G-\frac12}_{K,M-\frac12}\begin{pmatrix}0\\1\end{pmatrix}\qquad |2\rangle_r=-\sqrt{\frac{G-K+\frac12}{2G}}D^{G}_{K-\frac12,M}\begin{pmatrix}1\\0\end{pmatrix}+\sqrt{\frac{G+K+\frac12}{2G}}D^{G}_{K+\frac12,M}\begin{pmatrix}0\\1\end{pmatrix}$$
$$|1\rangle_r=\sqrt{\frac{G+M+1}{2G+2}}D^{G+\frac12}_{K,M+\frac12}\begin{pmatrix}1\\0\end{pmatrix}-\sqrt{\frac{G-M+1}{2G+2}}D^{G+\frac12}_{K,M+\frac12}\begin{pmatrix}0\\1\end{pmatrix}\qquad |3\rangle_r=-\sqrt{\frac{G-K+\frac12}{2G+2}}D^{G}_{K-\frac12,M}\begin{pmatrix}1\\0\end{pmatrix}-\sqrt{\frac{G+K+\frac12}{2G+2}}D^{G}_{K-\frac12,M}\begin{pmatrix}0\\1\end{pmatrix}.$$

These angular basis satisfy $G^L_3\begin{pmatrix}|i\rangle_r\\|i'\rangle_l\end{pmatrix}=K\begin{pmatrix}|i\rangle_r\\|i'\rangle_l\end{pmatrix},\ G^R_3\begin{pmatrix}|i\rangle_r\\|i'\rangle_l\end{pmatrix}=M\begin{pmatrix}|i\rangle_r\\|i'\rangle_l\end{pmatrix},\boldsymbol{G}^2\begin{pmatrix}|i\rangle_r\\|i'\rangle_l\end{pmatrix}=-G(G+1)\begin{pmatrix}|i\rangle_r\\|i'\rangle_l\end{pmatrix}.$

Parity of angular basis

Parity transformation : $(r,\theta,\varphi,\psi)\to(r,2\pi-\theta,\varphi+\pi,\psi-\pi)$

Parity of Wigner D function:

$$\begin{aligned}D^G_{K,M}(\varphi,-\theta,\psi)&=(-1)^{K-M}D^G_{K,M}(\varphi,\theta,\psi)\\ D^G_{K,M}(\varphi,\theta\pm2n\pi,\psi)&=(-1)^{2nG}D^G_{K,M}(\varphi,\theta,\psi)\\ D^G_{K,M}(\varphi\pm n\pi,\theta,\psi)&=(-i)^{\pm2nK}D^G_{K,M}(\varphi,\theta,\psi)\\ D^G_{K,M}(\varphi,\theta,\psi\pm n\pi)&=(-i)^{\pm2nM}D^G_{K,M}(\varphi,\theta,\psi)\end{aligned}$$

$$D^G_{K,M}(\varphi+\pi,2\pi-\theta,\psi-\pi)=(-1)^{2G+2K-2M}D^G_{K,M}(\varphi,\theta,\psi)$$

The parity of the angular basis is defined according to the parity of the upper component $|i\rangle_r$.

parity even spinors : $|0\rangle_r,|1\rangle_r$

parity odd spinors : $|2\rangle_r,|3\rangle_r$

Ⅳ. The radial equation

The eigenfunction Ψ can be separated by the angular basis and radial part such as $\Psi^{(i)}=\begin{pmatrix}\mathcal{F}_i(r)|i\rangle_r\\ \mathcal{G}_{i'}(r)|i'\rangle_\ell\end{pmatrix}$.

It is well-known that by eliminating the lower ("the smaller") component, one can obtain the Schrödinger-like radial equations for the positive eigenvalues

$$a^2f^2\mathcal{F}''_i+P_1\mathcal{F}'_i+\left(P^a_0-P^b_0(G_i-2)-\frac{a^2f}{r^2}G_i(G_i-2)\right)\mathcal{F}_i=0.$$

Similarly, for the negative eigenvalues, the equations are obtained by eliminating the smaller (in this case, upper) component

$$a^2f^2\mathcal{G}''_i+Q_1\mathcal{G}'_i+\left(Q^a_0-Q^b_0(G_i-2)-\frac{a^2f}{r^2}G_i(G_i-2)\right)\mathcal{G}_i=0,$$

Where $G_0=2G+1,G_1=-G_0,G_2=-2G,G_3=2G+2$, $P^a_0,P^b_0,P_1,Q^a_0,Q^b_0,Q_1$ are polynomials of the metric functions.

Ⅴ. The plane wave in flat case

Flat space-time $a(r)=1,\quad f(r)=1$

- The radial equations :

$$r^2\mathcal{F}''_i+r\mathcal{F}'_i+\mathcal{F}_i\left(r^2(E^2-m^2)-G_i^2\right)=0$$

- The solution :

$$\mathcal{F}_i=\frac{J(kr)}{r}$$

The complete basis sets $\{\Xi_i\}=\{u_i\},\{v_i\}$

$$u_a=N_k\begin{pmatrix}i\omega^+_{E_k}\frac{J_{2G}(kr)}{r}|0\rangle_r\\ \omega^-_{E_k}\frac{J_{2G+1}(kr)}{r}|2\rangle_l\end{pmatrix}\quad v_a=N_k\begin{pmatrix}i\omega^+_{E_k}\frac{J_{2G+1}(kr)}{r}|2\rangle_r\\ -\omega^-_{E_k}\frac{J_{2G}(kr)}{r}|0\rangle_l\end{pmatrix}$$
$$u_b=N_k\begin{pmatrix}i\omega^+_{E_k}\frac{J_{2G+2}(kr)}{r}|1\rangle_r\\ -\omega^-_{E_k}\frac{J_{2G+1}(kr)}{r}|3\rangle_l\end{pmatrix}\quad v_b=N_k\begin{pmatrix}i\omega^+_{E_k}\frac{J_{2G+1}(kr)}{r}|3\rangle_r\\ \omega^-_{E_k}\frac{J_{2G+2}(kr)}{r}|1\rangle_l\end{pmatrix}$$

$|0\rangle_{r,l}\sim|3\rangle_{r,l}$: Angular basis
$J_i(r)$: Bessel function
k : Eigenvalue of momentum

N_k : Normalization constant
$\omega^+_{E_k>0},\omega^-_{E_k<0}=\mathrm{sgn}(E_k),$
$\omega^-_{E_k>0},\omega^+_{E_k<0}=\frac{k}{E_k+m}$

Ⅵ. The normal mode in AdS

A. The analytical study

Massless case

AdS vacuum $a(r)=1,\quad f(r)=1+\frac{r^2}{l^2}$

- The asymptotic behaviors
  - origin $F_i=a_1r^{G_i-2}+a_2r^{-G_i}$
  - infinity $F_i=\frac{c_1}{r^3}+\frac{c_2}{r^2}$
- Change the variable and the function :

$$\frac{r}{l}=\sqrt{\frac{1}{x^2}-1},\quad \mathcal{F}_i(x)=x^3(1-x)^{\frac{G_i-2}{2}}(1+x)^{-\frac{G_i+1}{2}}H(x)$$

- The equation :

$$(1-x^2)H''+(1-2G_i-x)H'+\epsilon^2H=0$$

The normalizability condition $c_2=0$

$$\int r^3dr\mathcal{F}_i^2<\infty$$
$\Leftrightarrow\ r^3F_i^2$ is faster than r^{-1}
$\Leftrightarrow\ F_i$ is faster than r^{-2}

The regularity conditions

$$\begin{cases}a_1=0 & for\ \ G_i>2\\ a_2=0 & for\ \ G_i<0\end{cases}$$

The solution

$$H(x)={}_2F_1(E,-E,G_i,\frac{1-x}{2})$$

for $G_i=3$
$E=3.725,5.65,7.62,9.6$

Massive case

- The asymptotic behavior
  - origin $F_\infty=A_\infty r^{-2-m}+B_\infty r^{-2+m}$
  - infinity $F_i=A_0r^{G_i-2}+B_0r^{-G_i}$
- Change the variable and the function

$$\mathcal{F}_i(x)=x^{m+2}(1-x)^{\frac{G_i-2}{2}}(1+x)^{\frac{G_i-1}{2}}H(x)$$

- The equation

$$x(1-x^2)(Ex+m)H''+P_3H'+P_2H=0$$

P_3,P_2 are polynomial of x .

The normalizability condition $B_\infty,m\ge1$

The regularity conditions $\begin{cases}A_0=0 & for\ \ G_i>2\\ B_0=0 & for\ \ G_i<0\end{cases}$

The solution

$$C^{m+G_i+p-\frac12}_pH(x)=P^{m-\frac12,G_i}_p(1-2x^2)+xP^{m+1/2,G_i}_{p-1}(1-2x^2)$$
$E_{2p}=G_i+m+2p\qquad p$: integer

B. The numerical study

■ A reason of performing numerical study

If one obtain the smaller component of the solutions, we should use following relations,

$$\mathcal{G}_{i'}|i'\rangle_\ell=\frac{i\tau^\dagger_\mu p_\mu}{E+a\sqrt fm}\mathcal{F}_i|i\rangle_r\quad \text{for}\ \ E>0,\qquad \mathcal{F}_{i'}|i'\rangle_r=\frac{i\tau_\mu p_\mu}{|E|+a\sqrt fm}\mathcal{G}_i|i\rangle_\ell\quad \text{for}\ \ E<0.$$

But, it is tedious task to compute analytically. So, we numerically solve the equations and obtain the smaller component. Also, we check the validity of our obtained analytical results.

■ The numerical method

(Naoki Watanabe, http://www-cms.phys.s.u-tokyo.ac.jp/~naoki/CIPINTRO/CIP/index.html)

For the numerical analysis, we employ a scheme based on simple first order perturbation, which is quite efficient for the present eigenvalue problem. The method is summerized:

1. We employ the rescaling of the coordinate $r\to\frac{y}{1-y}$.

2. We assume an eigenvalue E_0, and solve the equation for $\mathcal{F}_i(y)$ from $y=0$ to an intermediate matching point $y=y_{\rm fit}$ by using the standard Runge Kutta method.

3. We match the asymptotic solution F_∞ with the value of the solution $F_i(y)$ at $y_{\rm fit}$ by multiplicating a factor α :

$$\alpha F_\infty(y_{\rm fit})\equiv F_i(y_{\rm fit}).$$

4. We introduce an arbitrary δ-functional potential at an intermediate value $y_{\rm fit}$:

$$V_\delta(y):=-\frac{[F'_i(y_{\rm fit})]^{y_{\rm fit}+0}_{y_{\rm fit}-0}}{F_i(y_{\rm fit})}\delta(y-y_{\rm fit}).$$

Because, the eigenfunction is continuous at the matching point if the δ-functional potential exists, but its derivative is not. Therefore the correction in terms of the first order perturbation

$$\Delta E=\int\frac{y^3dy}{(1-y)^5}F_i^*(y)V_\delta(y)F_i(y)=-\frac{y^3_{\rm fit}}{(1-y_{\rm fit})^5}[F'_i(y_{\rm fit})]^{y_{\rm fit}+0}_{y_{\rm fit}-0}F_i(y_{\rm fit})$$

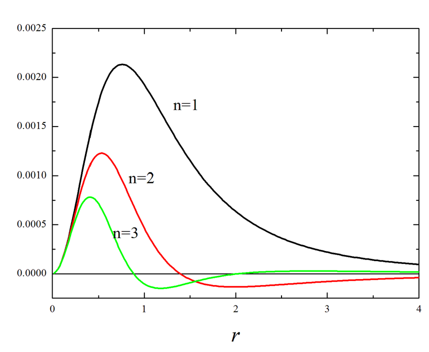
efficiently improves the eigenvalue, i.e., the eigenfunction.

5. We repeat the process ***2-4***. If the analysis reaches the correct eigenfunction, it no longer has discontinuity at all and the computation is successfully terminated.

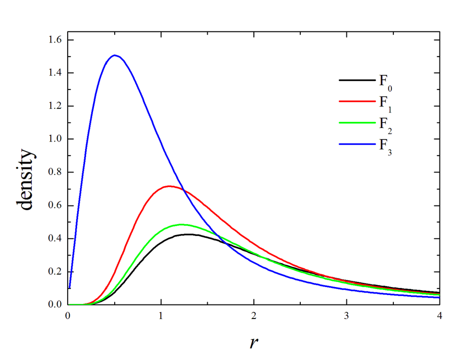
■ The numerical results

- The Dirac spinor :

$m=1,G=1$

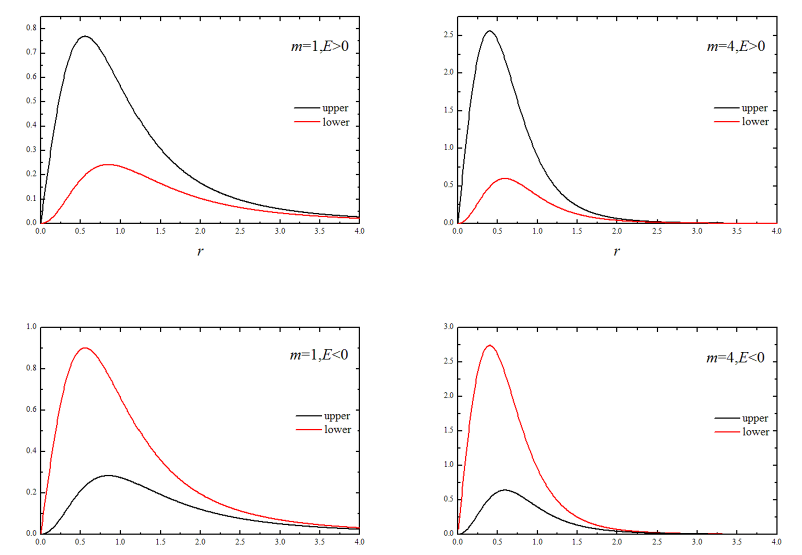


The four first eigenstates for $|0\rangle_r$.



The densities of ground state of $|0\rangle_r,|1\rangle_r,|2\rangle_r,|3\rangle_r$.

$G=1$

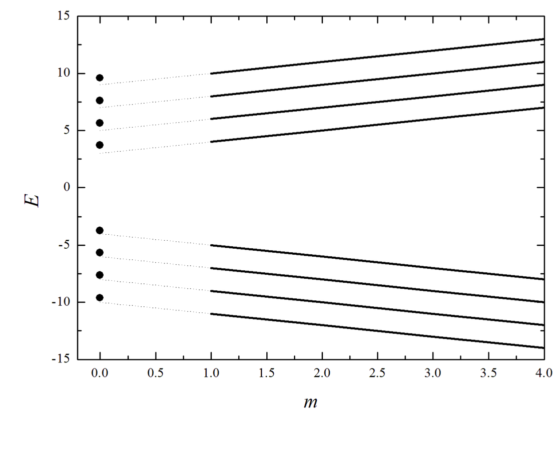


The upper and lower component of the eigenstates

- The energy spectrum :

$m=1,G=1$

	$n=0$	$n=1$	$n=2$
$\mathcal{F}_0$	4.00000003	6.00000008	8.00000015
$\mathcal{G}_0$	-5.00000024	-7.00000069	-9.00000107
$\mathcal{F}_1$	5.00000003	7.00000038	9.00000095
$\mathcal{G}_1$	-4.00000004	-6.00000009	-8.00000017
$\mathcal{F}_2$	2.00000002	4.00000004	6.0000010
$\mathcal{G}_2$	-3.00000003	-5.00000006	-7.00000011
$\mathcal{F}_3$	3.00000004	5.00000008	7.00000015
$\mathcal{G}_3$	-2.00000001	-4.00000004	-6.00000007



Ⅶ. Summary and further outlook

✓ We discuss in details the separation of the radial and angular variables for the Dirac equation in a five dimensional space time.

➢ We apply our angular basis to a five dimensional rotating space time with equal angular momenta.